# On the problem of tiling the plane with a polyomino

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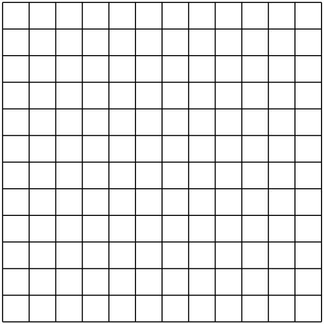
# Outline

# Introduction to polyominoes

- Discrete plane : $\mathbb{Z}^2$



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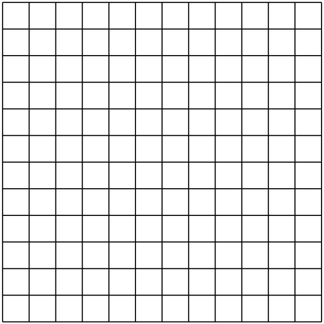
- **Definition** : A *polyomino* is a finite, 4-connected subset of the plane, without holes.



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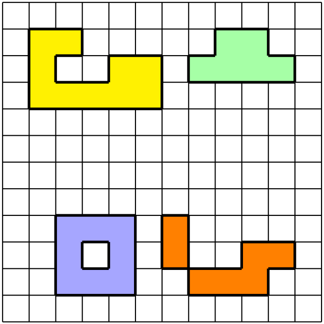

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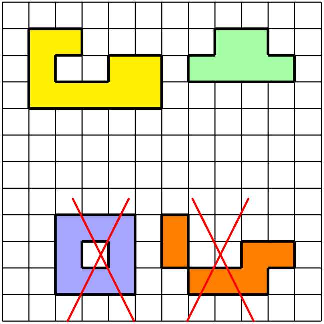
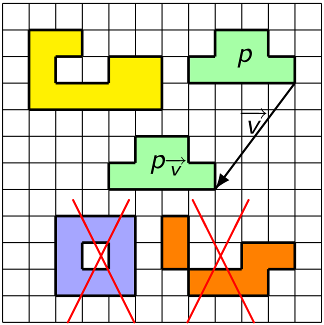


# Introduction to polyominoes

- Discrete plane : $\mathbb{Z}^2$
- **Definition** : A *polyomino* is a finite, 4-connected subset of the plane, without holes.
- **Notation** : Let $p$ be a polyomino and $\overrightarrow{v}$ a vector of $\mathbb{Z}^2$, $p_{\overrightarrow{v}}$ will denote the image of $p$ by de translation $\overrightarrow{v}$.



# General statement of the tiling problem

## Definition (Tiling)

*A* tiling $\mathcal{T}$ *of a subset* $D \subset \mathbb{Z}^2$ *by a set of polyominoes* $\mathcal{P}$ *is a set of couples* $(p, \overrightarrow{u}) \in \mathcal{P} \times \mathbb{Z}^2$ *such that :*

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- *$D$ is the union of the polyominoes $p_{\overrightarrow{u}}$.*
- *For any distinct pair $(p, \overrightarrow{u}), (p', \overrightarrow{v}) \in \mathcal{T}$, $p_{\overrightarrow{u}}$ and $p'_{\overrightarrow{v}}$ are non-overlapping.*

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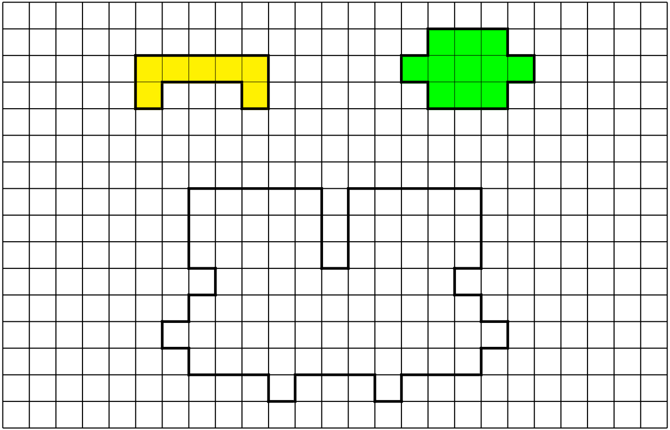
## Definition (The Tiling Problem)

*Given a set of polyominoes* $\mathcal{P}$ *and a subset* $D \subset \mathbb{Z}^2$.
*Does $D$ admits a tiling by* $\mathcal{P}$ *?*

# Example



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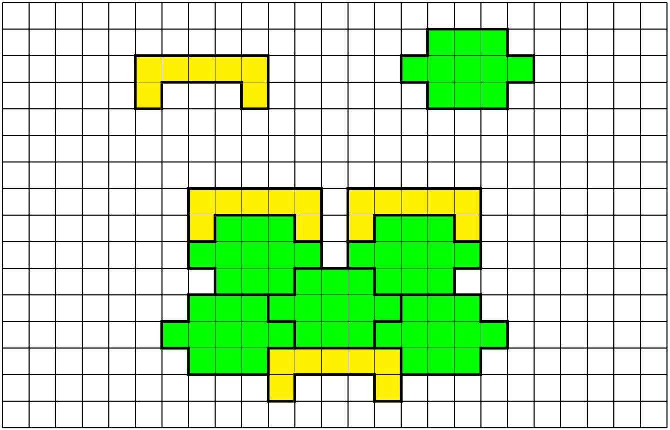

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**Theorem (Garey, Johnson and Papadimitriou)**

*The tiling problem with D finite and* $\mathcal{P} = \{$  ,  $\}$ *is NP-Complete.*

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We concider the case where $D = \mathbb{Z}^2$ and $\mathcal{P}$ is finite.

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**Definition (Periodic Tiling)**

*A tiling $\mathcal{T}$ is* periodic *if there exist two linearly independant vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ such that $\mathcal{T}$ is not changed by the corresponding translations.*

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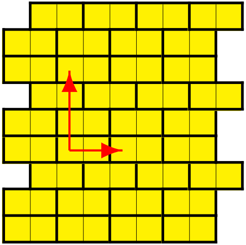
## Definition (Half-Periodic Tiling)

*A tiling $\mathcal{T}$ is* half-periodic *if there exists a vectors $\overrightarrow{u}$ such that $\mathcal{T}$ is not changed by the corresponding translation.*

# Example



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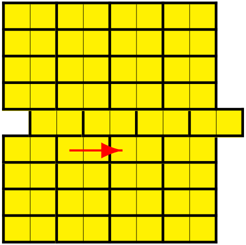

Half-periodic tiling

# Half-Periodic implies periodic

**Remark**

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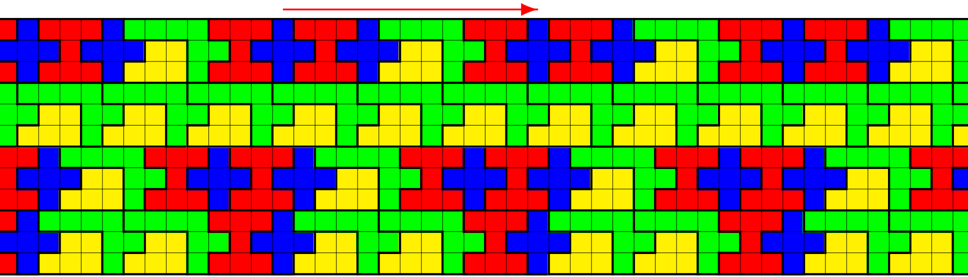

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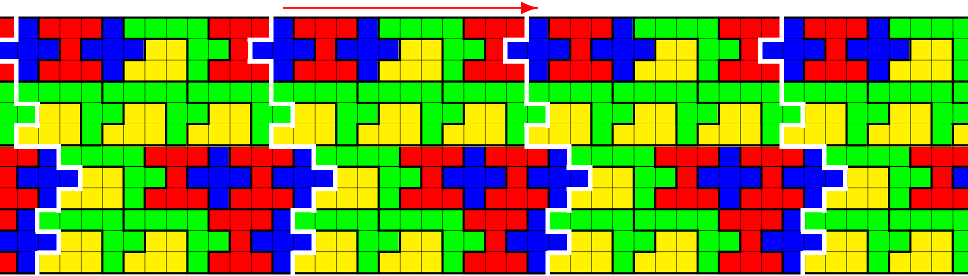

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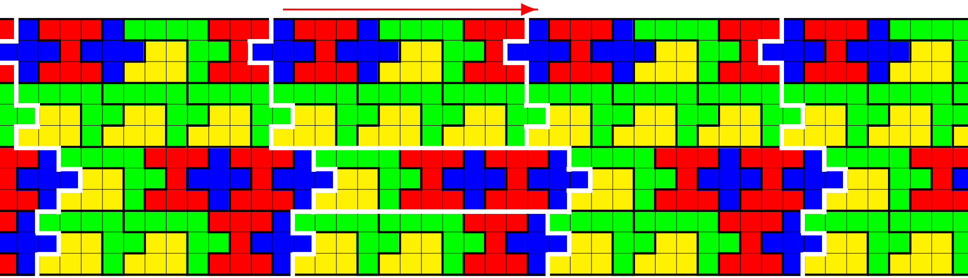

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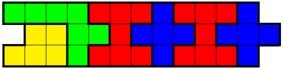

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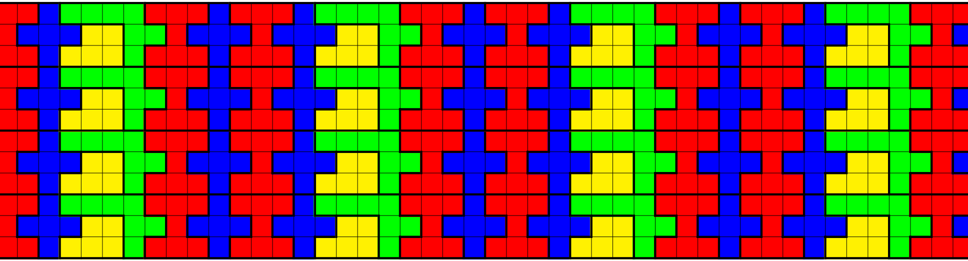

# Nonperiodic tilings

**Theorem (Berger, 1966)**

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*The tiling problem with $\mathcal{P}$ finite and $D = \mathbb{Z}^2$ is undecidable.*

**Corollary**

*There are some finite sets $\mathcal{P}$ such that tilings of the plane by $\mathcal{P}$ do exist and are all nonperiodic.*

# Tilings with one polyomino

**Definition**

*A polyomino p is* exact *if the set* $\mathcal{P} = \{p\}$ *tiles the plane.*

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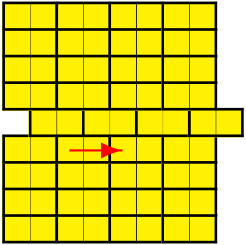
## Definition

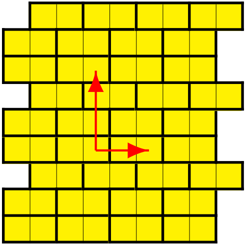
*A tiling of the plane $\mathcal{T}$ by an exact polyomino $p$ is* regular *if there exist two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ such that*

$$\mathcal{T} = \{(p, i\overrightarrow{u} + j\overrightarrow{v}) | i, j \in \mathbb{Z}^2\}$$

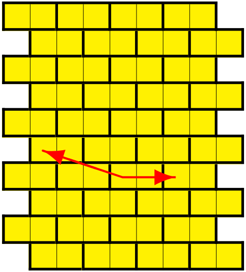
# Examples



Half-periodic tiling



Periodic tiling



Regular tiling

# Tilings with one polyomino

**Theorem (Wijshoff and Van Leeuwen, 1984)**

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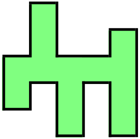
*If a polyomino p is exact, then there exists a regular tiling of the plane by p.*

**Corollary**

*The tiling problem with $|\mathcal{P}| = 1$ and $D = \mathbb{Z}^2$ is decidable in polynomial time.*

# Example



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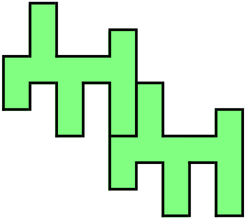

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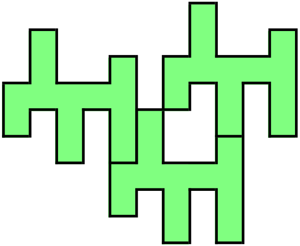

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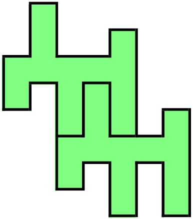

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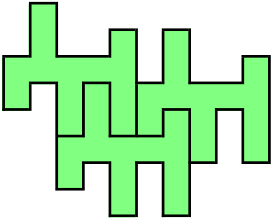

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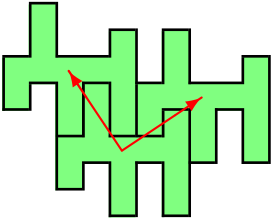

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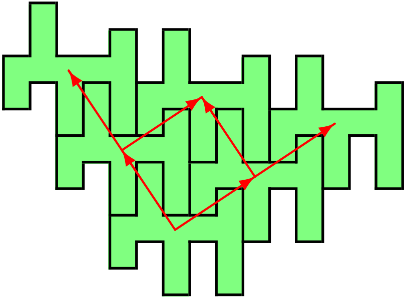

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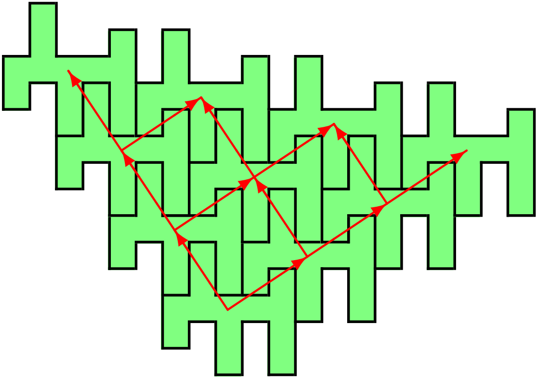
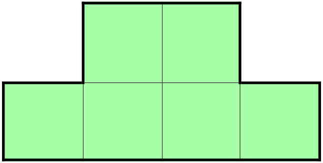


# Outline

# Coding the boundary of a polyomino

$$\Sigma = \{a, \overline{a}, b, \overline{b}\}$$



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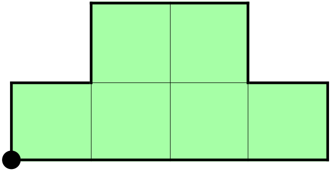

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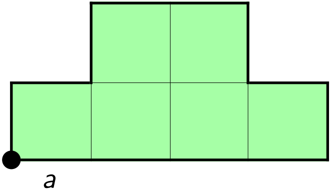


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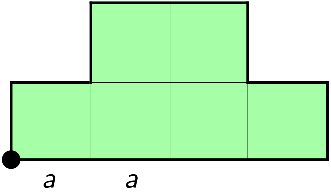


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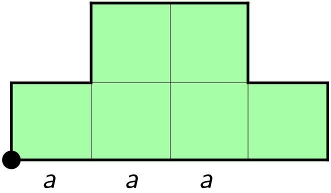


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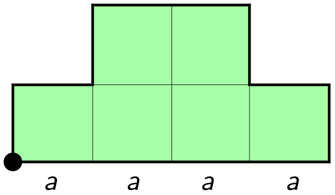


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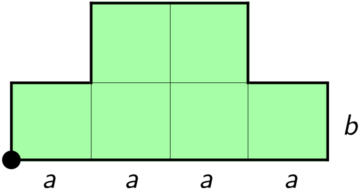


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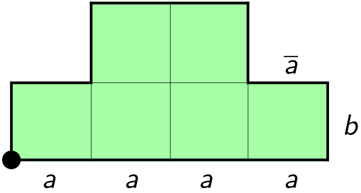


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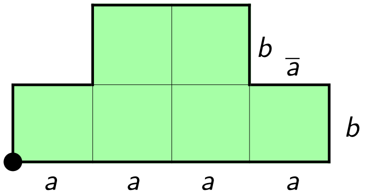


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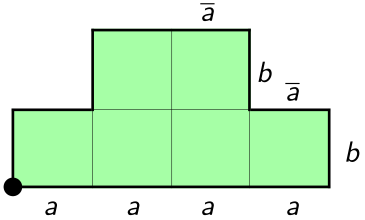


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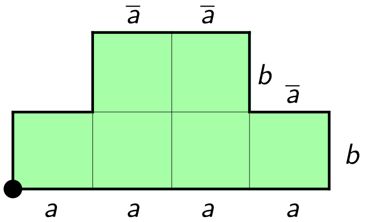


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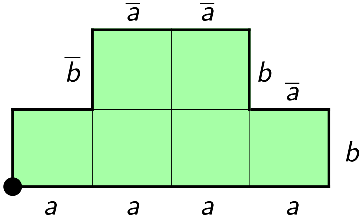


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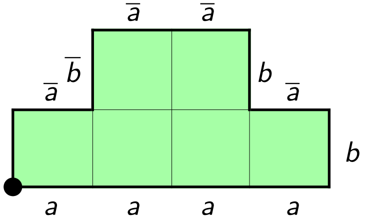


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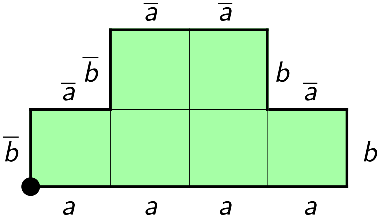


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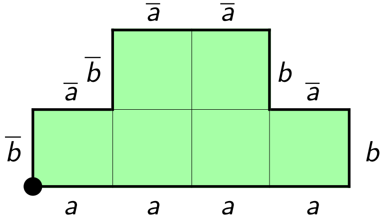


$$w = a\,a\,a\,a\,b\,\overline{a}\,b\,\overline{a}\,\overline{a}\,\overline{b}\,\overline{a}\,\overline{b}.$$

**Notation :**
$w \equiv w'$ notes that $w$ and $w'$ are conjugate.

There exist $u, v \in \Sigma^*$ such that :
$w = uv$ and $w' = vu$.

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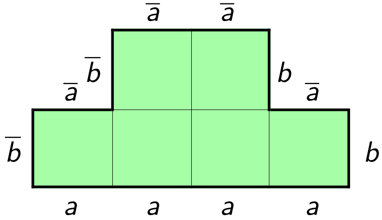


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**Notation :**
$w \equiv w'$ notes that $w$ and $w'$ are conjugate.

There exist $u, v \in \Sigma^*$ such that :
$w = uv$ and $w' = vu$.

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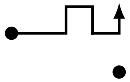

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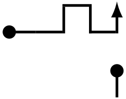

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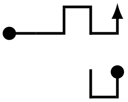

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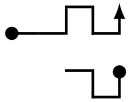

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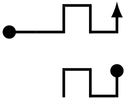

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**Theorem (Beauquier and Nivat, 1991)**

*A polyomino p is exact if and only its boundary word $w \equiv XYZ\widehat{X}\widehat{Y}\widehat{Z}$ for some $X, Y, Z \in \Sigma^*$.*

# Neighbouring

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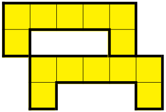

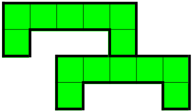

# Triad

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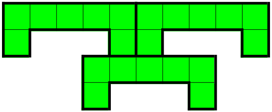

# Surrounding

## Definition

*A* surrounding *of the polyomino $p$ is an ordered sequence of translated copies $(p_0, p_1, \ldots, p_{k-1})$ such that for every $i$ from $0$ to $k-1$, the polyominoes $p$, $p_i$ and $p_{i+1}$ form a triad.*

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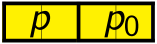

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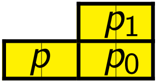

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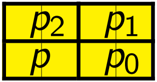

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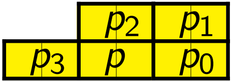

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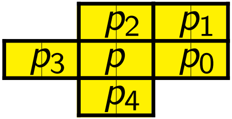

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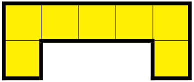

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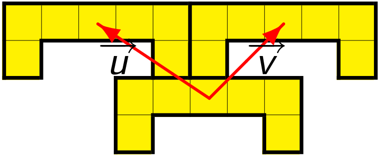

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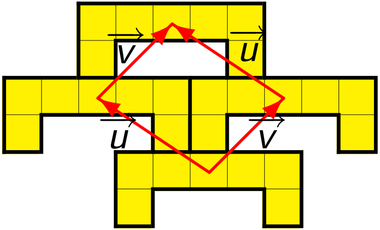

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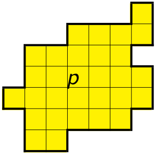

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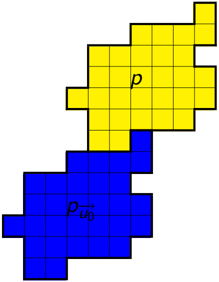

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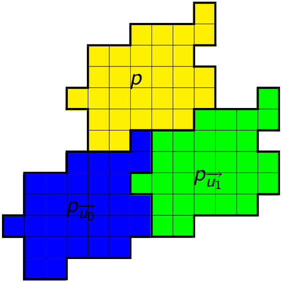

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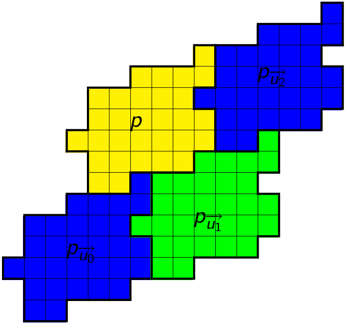

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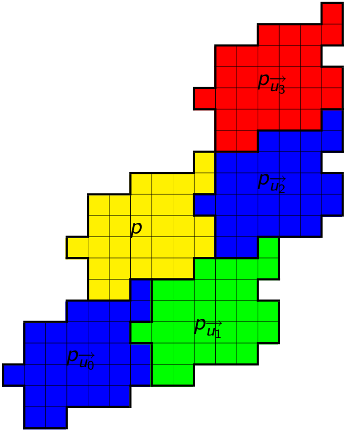
$p_{\overrightarrow{u_3}}$
$p_{\overrightarrow{u_2}}$
$p$
$p_{\overrightarrow{u_1}}$
$p_{\overrightarrow{u_0}}$

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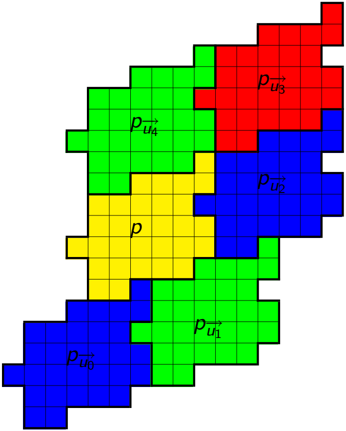

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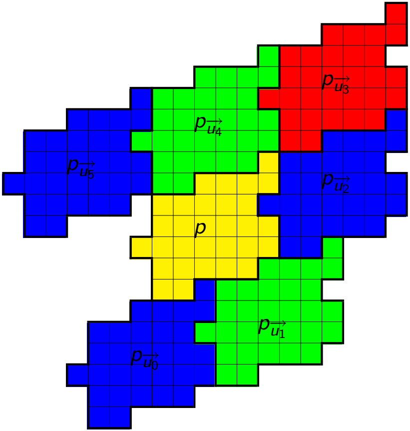
$p_{\overrightarrow{u_3}}$
$p_{\overrightarrow{u_4}}$
$p_{\overrightarrow{u_5}}$
$p_{\overrightarrow{u_2}}$
$p$
$p_{\overrightarrow{u_1}}$
$p_{\overrightarrow{u_0}}$

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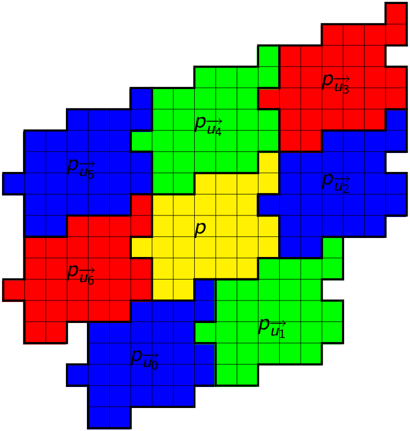
$p_{\overrightarrow{u_3}}$
$p_{\overrightarrow{u_4}}$
$p_{\overrightarrow{u_5}}$
$p_{\overrightarrow{u_2}}$
$p$
$p_{\overrightarrow{u_6}}$
$p_{\overrightarrow{u_1}}$
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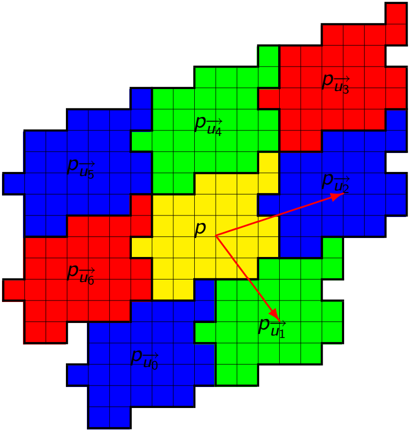

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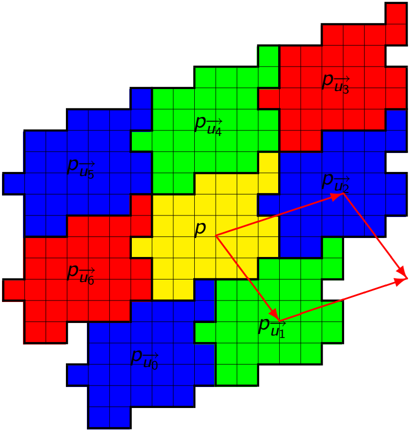

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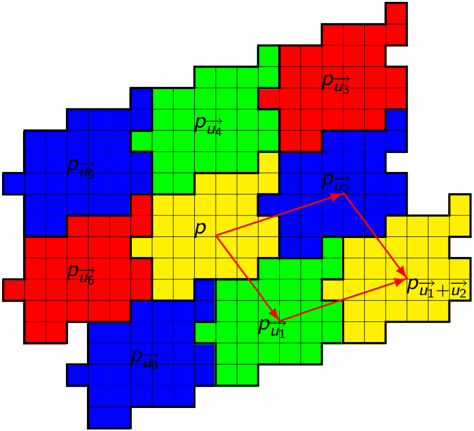
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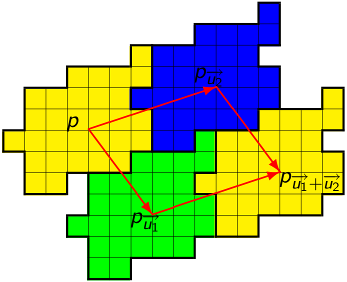
p
$p_{\overrightarrow{u_2}}$
$p_{\overrightarrow{u_1}}$
$p_{\overrightarrow{u_1}+\overrightarrow{u_2}}$

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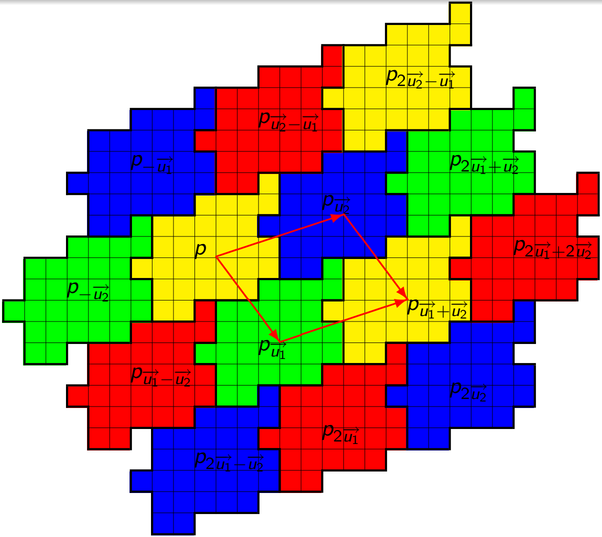
$p_{2\vec{u_2}-\vec{u_1}}$
$p_{\vec{u_2}-\vec{u_1}}$
$p_{-\vec{u_1}}$
$p_{2\vec{u_1}+\vec{u_2}}$
$p_{\vec{u_2}}$
$p$
$p_{2\vec{u_1}+2\vec{u_2}}$
$p_{-\vec{u_2}}$
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$p_{2\vec{u_2}}$
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# Surroundings and the factorization

## Proposition

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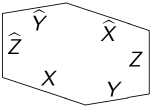

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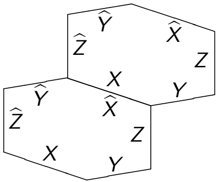

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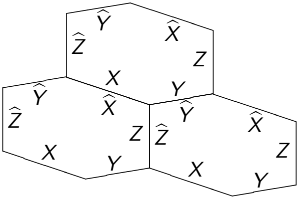

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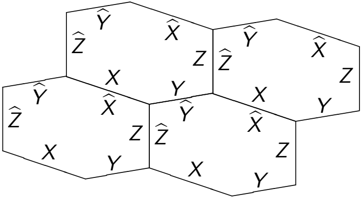

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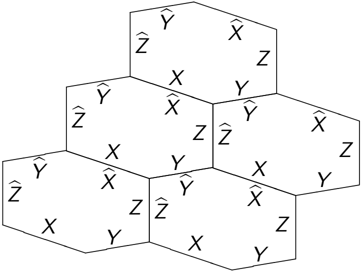

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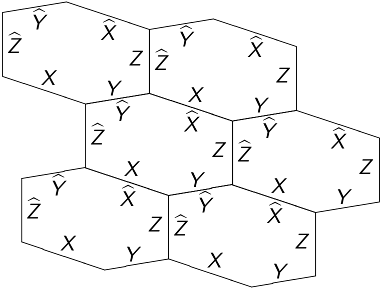

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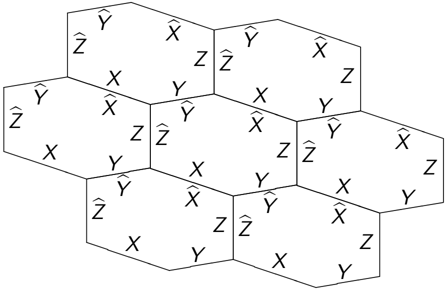

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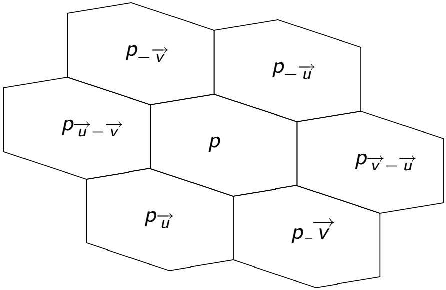

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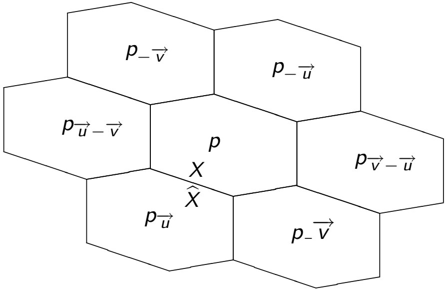

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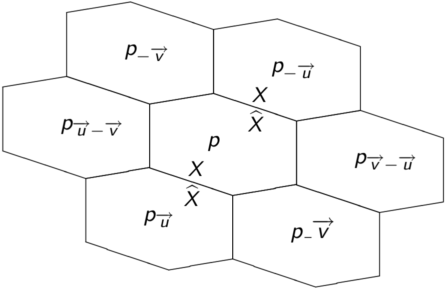

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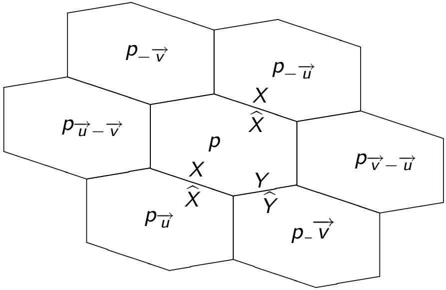

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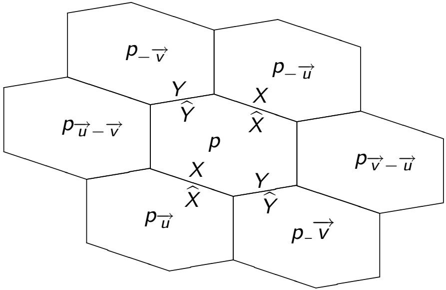

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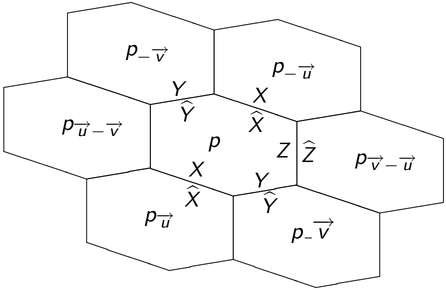

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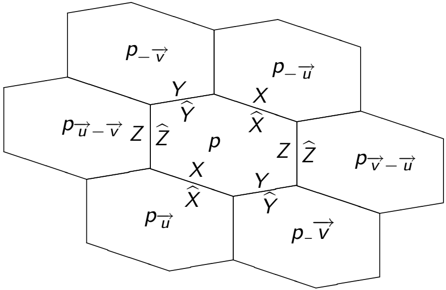

# Pseudo-square and pseudo-hexagons

## Definition

*An exact polyomino p with Beauquier-Nivat factorization* $XYZ\widehat{X}\widehat{Y}\widehat{Z}$ *is called a* pseudo-square *if one of the factors* $X, Y, Z$ *is the empty word. It is called a* pseudo-hexagon *otherwise.*

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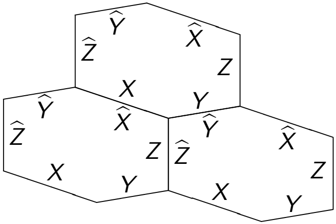
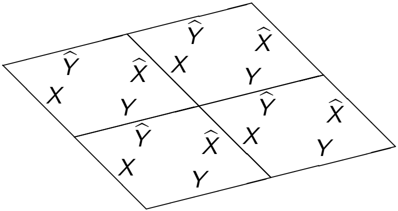



Pseudo-square

$w \equiv XY\widehat{X}\widehat{Y}.$



# Complexity

Let $n$ be the length of the word coding the boundary of a polyomino $p$.

**Remark**

*The Beauquier-Nivat characterization provides a naive algorithm to determine if $p$ is exact in $\mathcal{O}(n^4)$.*

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**Theorem (Gambini and Vuillon, 2003)**

*There is an algorithm to test if a polyomino satisfies the Beauquier-Nivat characterization in* $\mathcal{O}(n^2)$.

# Outline

# Admissible factors

**Definition**

*Let $A$ be a factor of the word $w$ coding a polyomino $p$. $A$ is* admissible *if*

- $w \equiv Ax\widehat{A}y$, *for* $x, y$ *such that* $|x| = |y|$.
- $A$ *is* maximal, *that is,* $\mathrm{first}(x) \neq \overline{\mathrm{last}(x)}$ *and* $\mathrm{first}(y) \neq \overline{\mathrm{last}(y)}$.

# Admissible factors

**Proposition**

*Let $\mathcal{A}$ be the set of all admissible factors overlapping a position $\alpha$ in $w$ and $\widehat{\mathcal{A}}$ be the set of their respective homologous factors. Then, there is at least one position in $w$ that is not covered by any element of $\mathcal{A} \cup \widehat{\mathcal{A}}$.*

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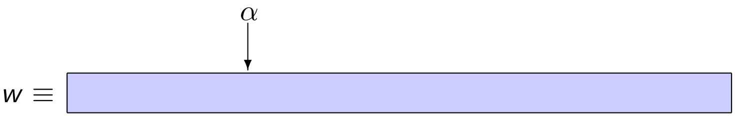

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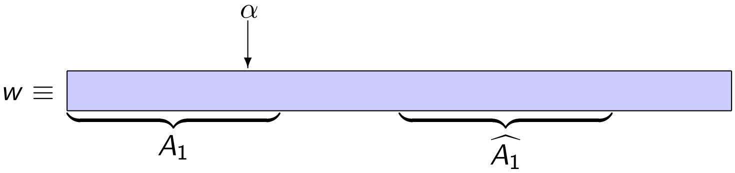

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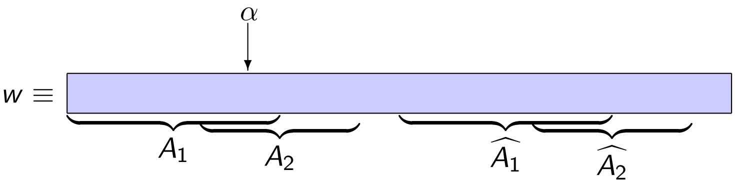

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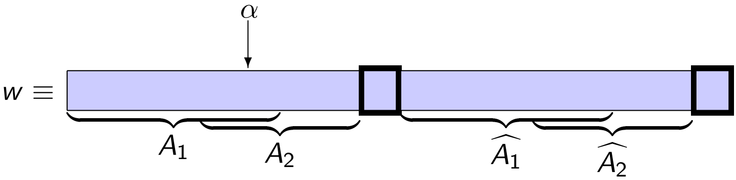

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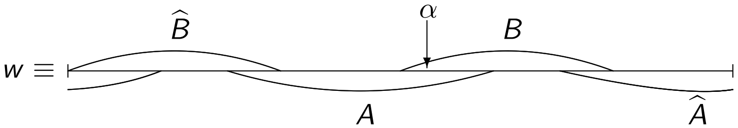

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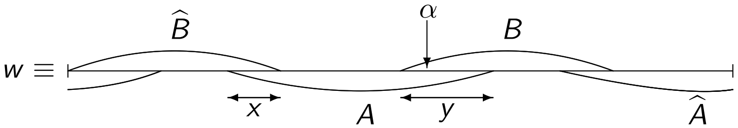

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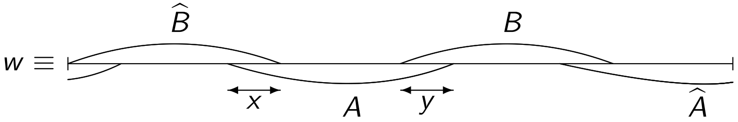

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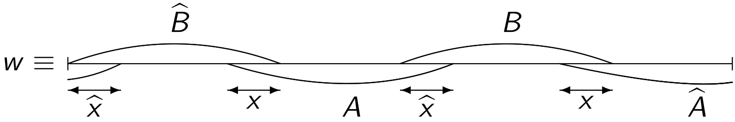

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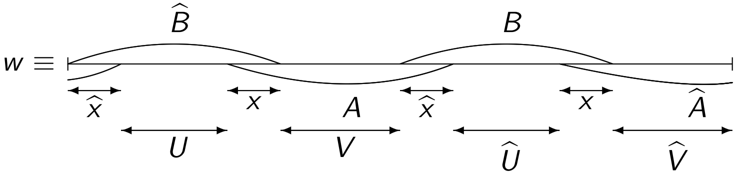

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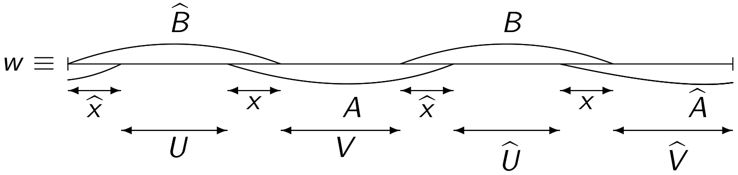


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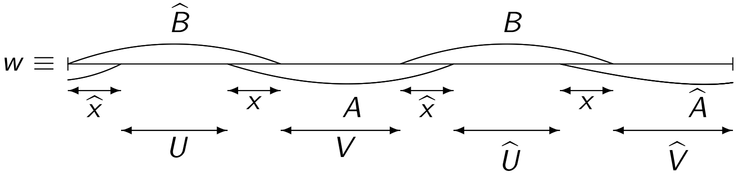


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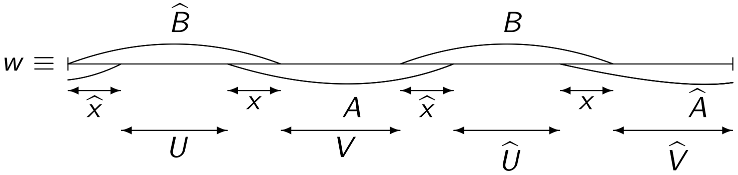


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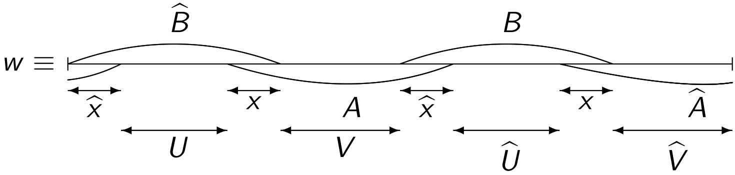


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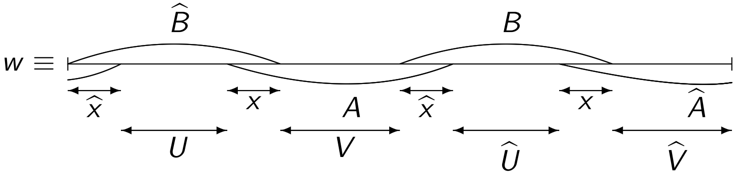


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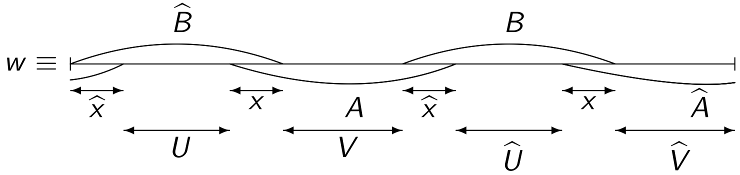


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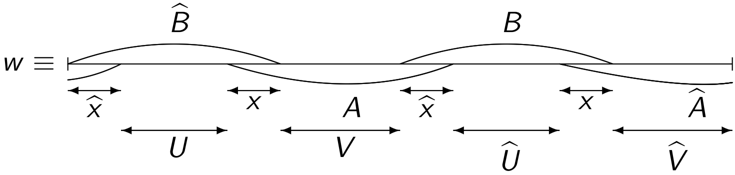


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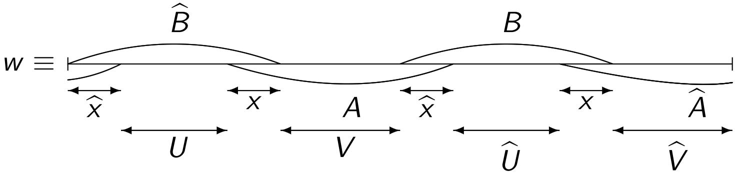


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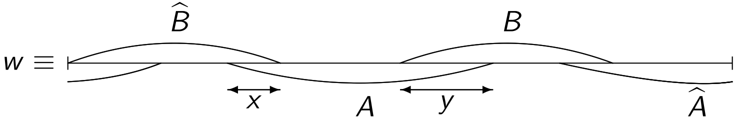

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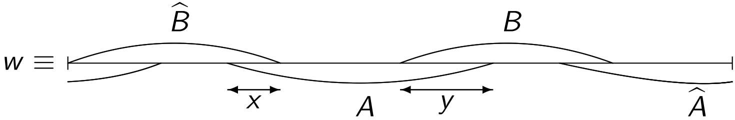


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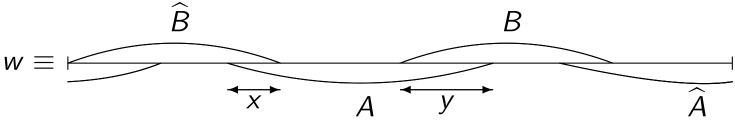


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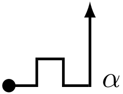

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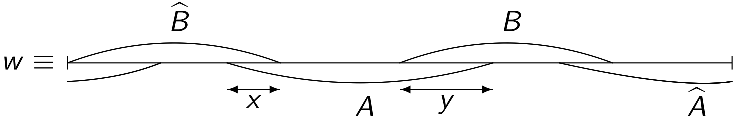


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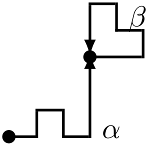

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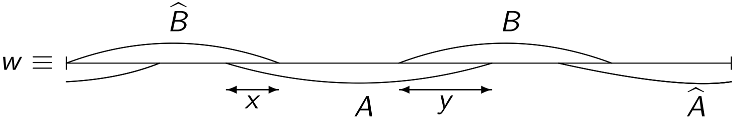


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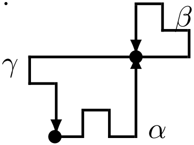

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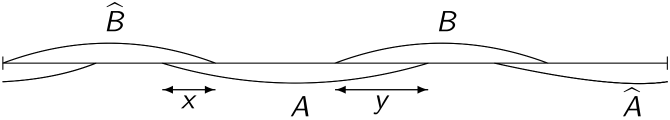
$w \equiv$ 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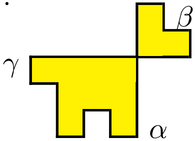

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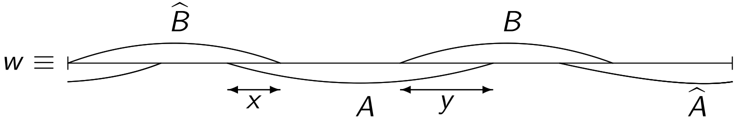


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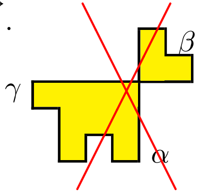
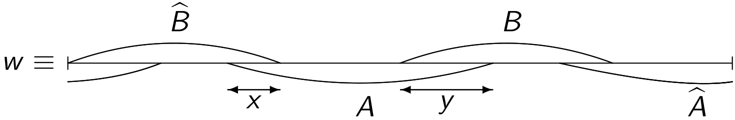


# Admissible factors

2. $|x| < |y|$, $\widehat{y}$ does not overlap $\widehat{A}$ in $\widehat{B}$.



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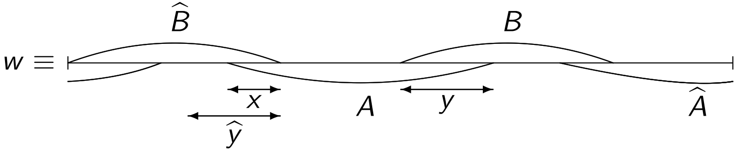

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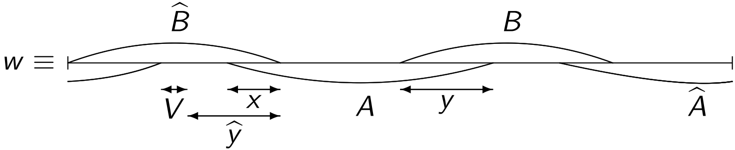

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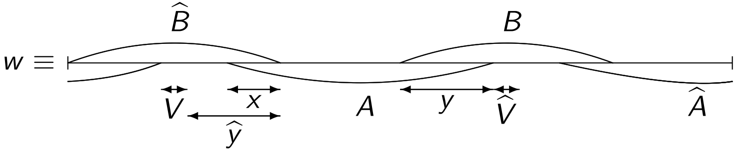

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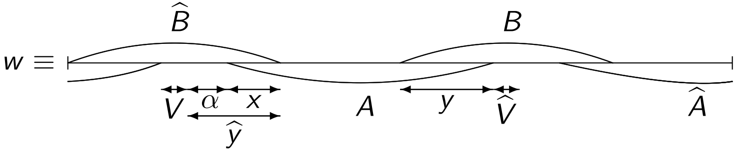

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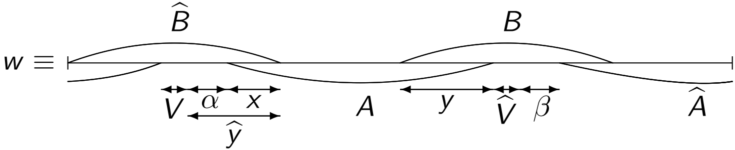

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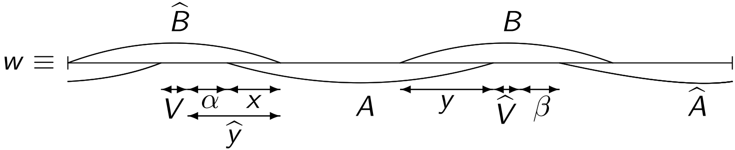


$$w \equiv A \; \widehat{V} \; \beta \; \widehat{A} \; V \; \alpha.$$

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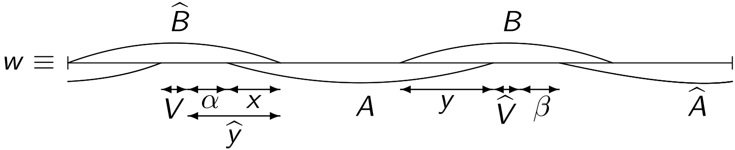


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$$\overrightarrow{w} = \overrightarrow{A} + \overrightarrow{\widehat{V}} + \overrightarrow{\beta} + \overrightarrow{\widehat{A}} + \overrightarrow{V} + \overrightarrow{\alpha}$$

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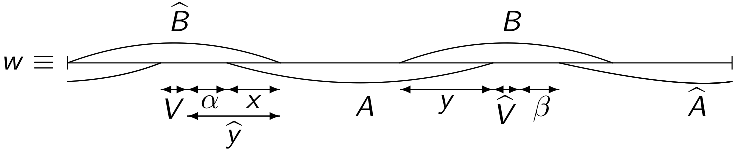


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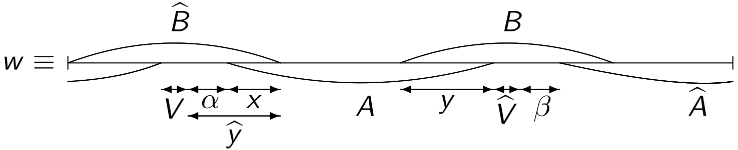


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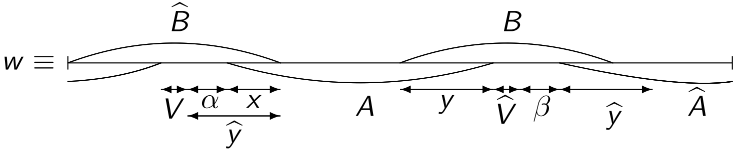


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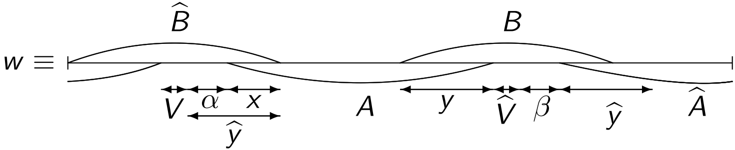


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$$\beta\, \widehat{y} = \beta\, \alpha\, x.$$

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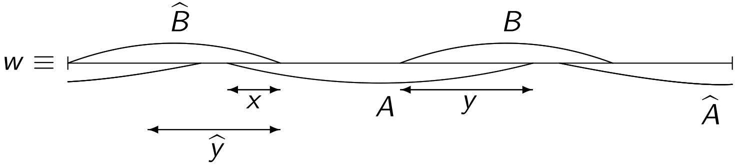

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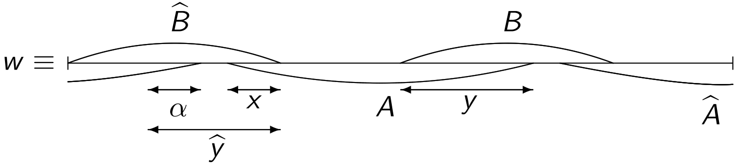

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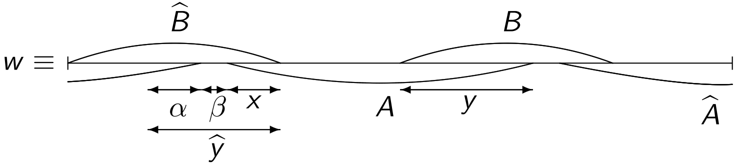

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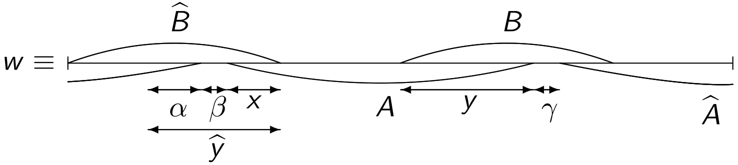

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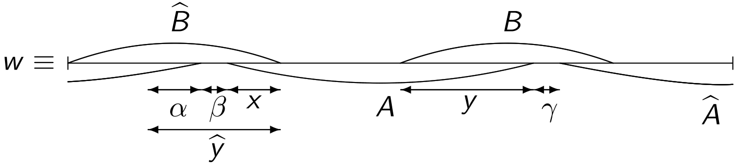


$$w \equiv A \; \gamma \; \widehat{A} \; \beta.$$

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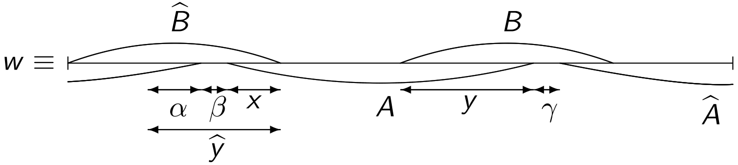


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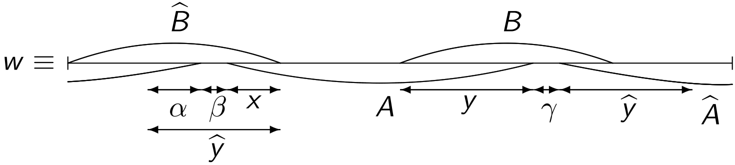


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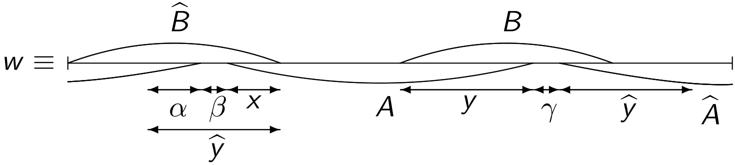


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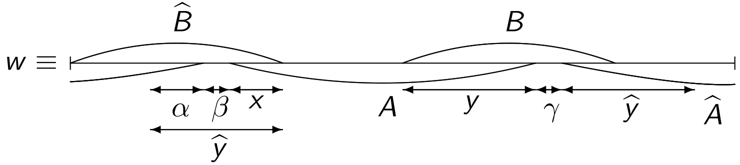


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**Corollary**

*Let $w$ a word coding a polyomino $p$ with Beauquier-Nivat's factorization $w \equiv XYZ\widehat{X}\widehat{Y}\widehat{Z}$. Then, $X, Y$ and $Z$ are admissible.*

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- $w \equiv Ax\widehat{A}y$, for $x, y$ such that $|x| = |y|$.
- $A$ is $\mathrm{maximal}$, that is, $\mathrm{first}(x) \neq \overline{\mathrm{last}(x)}$ and $\mathrm{first}(y) \neq \overline{\mathrm{last}(y)}$.

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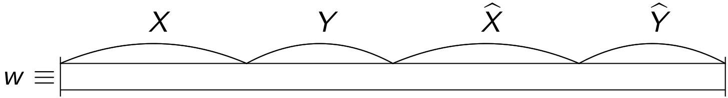

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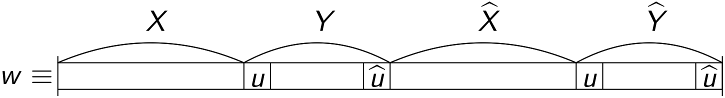

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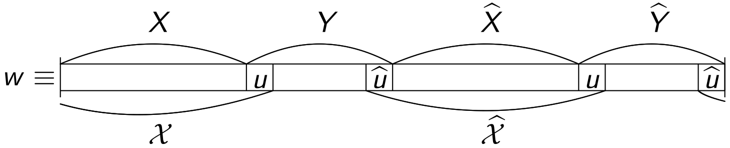

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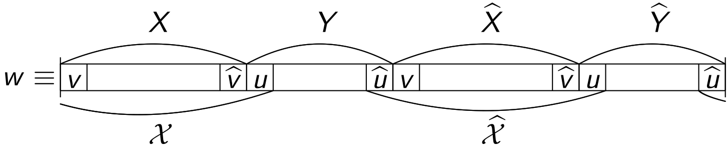

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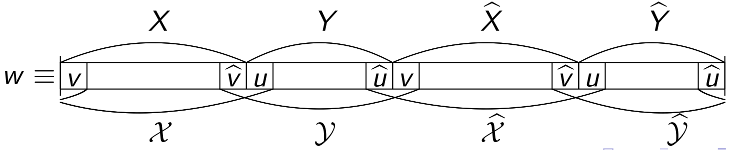

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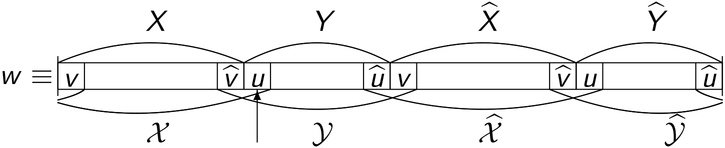

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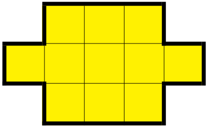
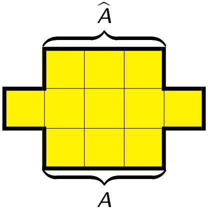
$$w \equiv a\ a\ a\ b\ a\ b\ \overline{a}\ b\ \overline{a}\ \overline{a}\ \overline{a}\ \overline{b}\ \overline{a}\ \overline{b}\ a\ \overline{b}$$



# Admissible factors

$$w \equiv \underbrace{a\ a\ a}_{A}\ \underbrace{b\ a\ b\ \overline{a}\ b}_{x}\ \underbrace{\overline{a}\ \overline{a}\ \overline{a}}_{\widehat{A}}\ \underbrace{\overline{b}\ \overline{a}\ \overline{b}\ a\ \overline{b}}_{y}$$



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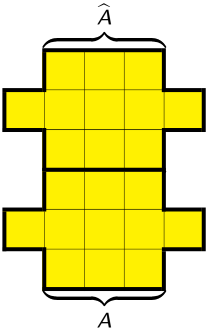

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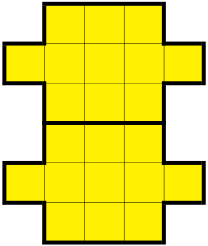

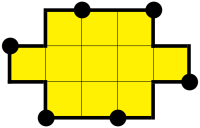

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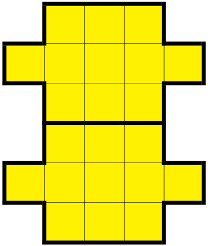

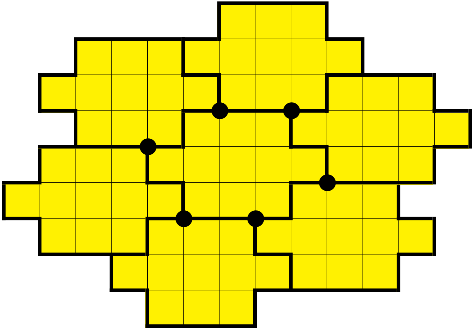

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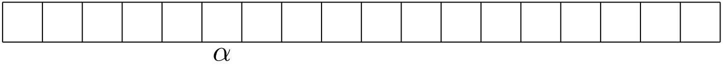
$w \equiv$ 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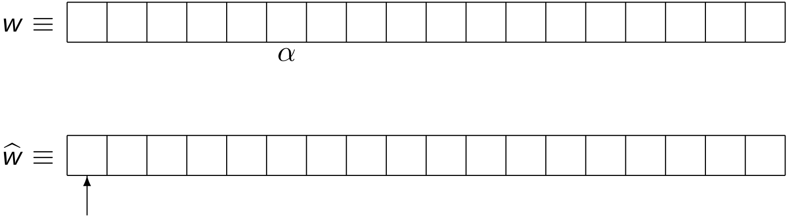

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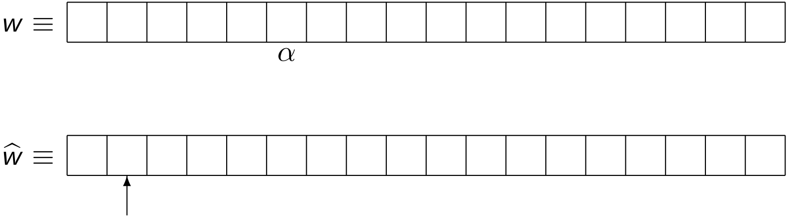

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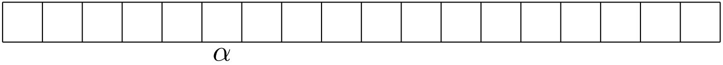
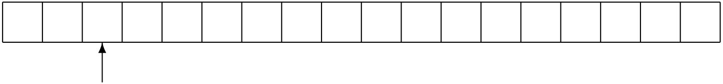
$w \equiv$ 


$\widehat{w} \equiv$ 

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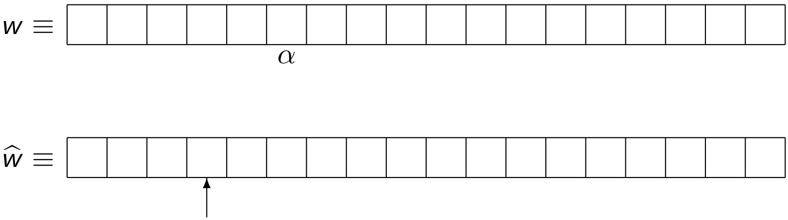

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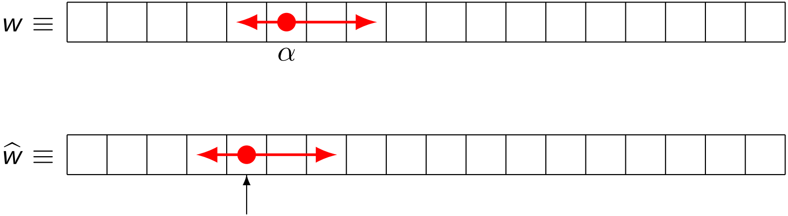

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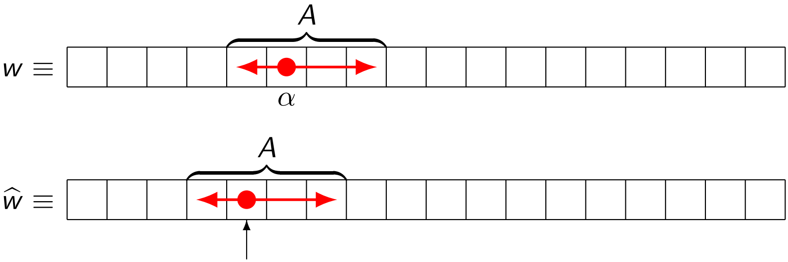

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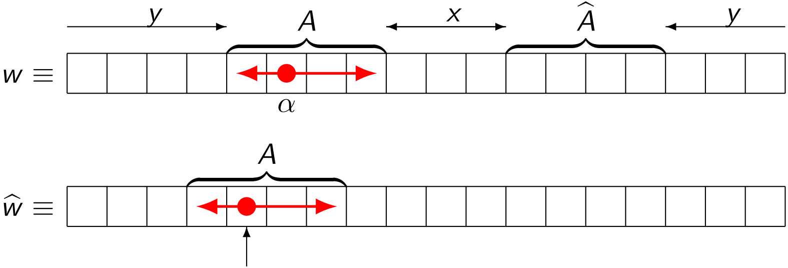

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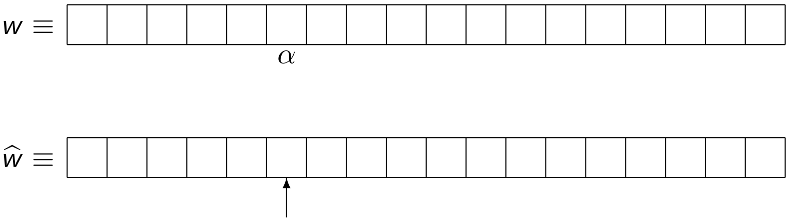

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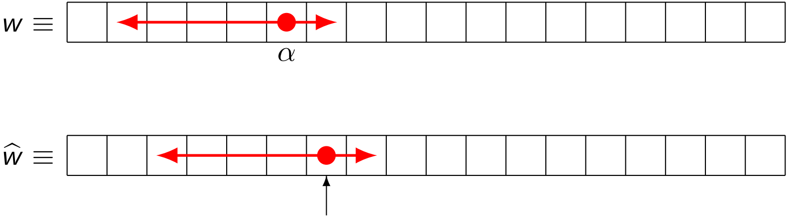

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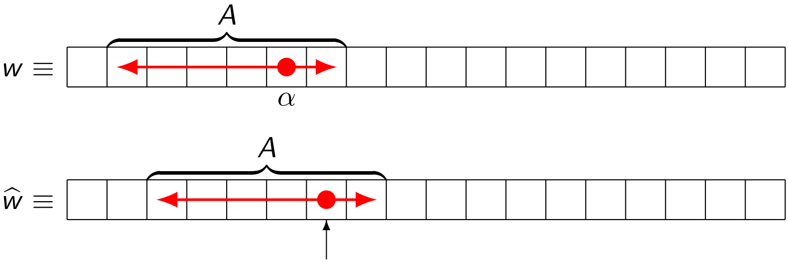

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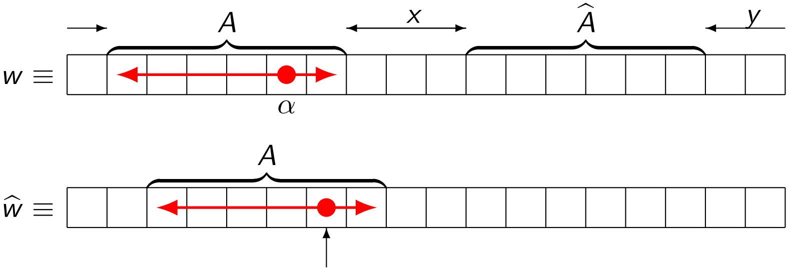

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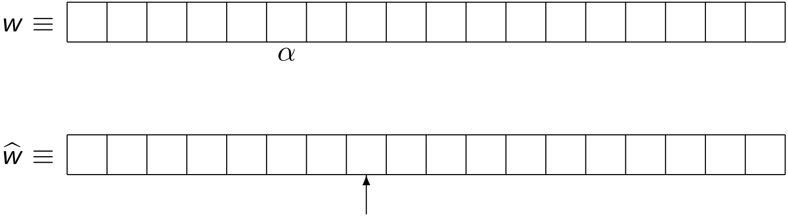

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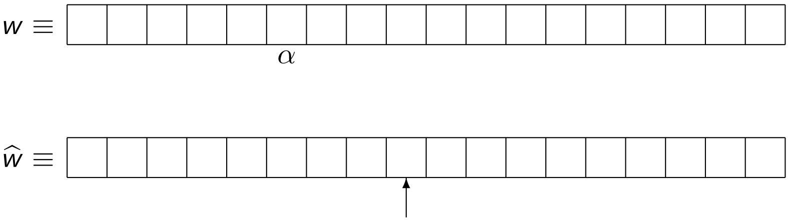

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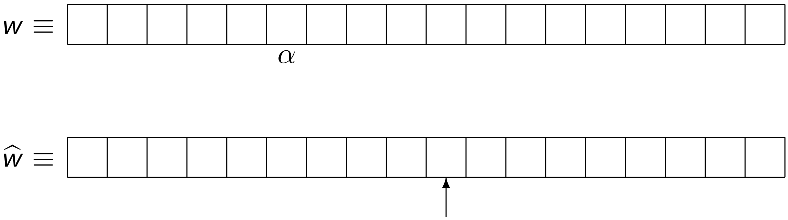

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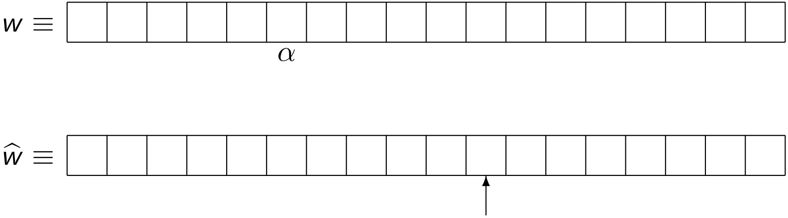

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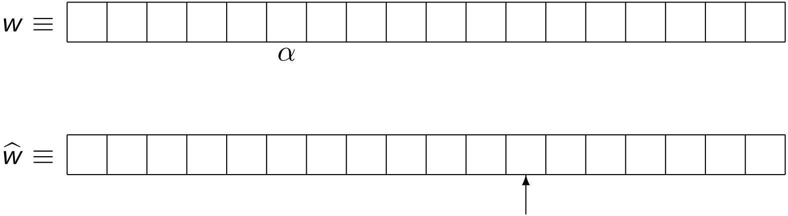

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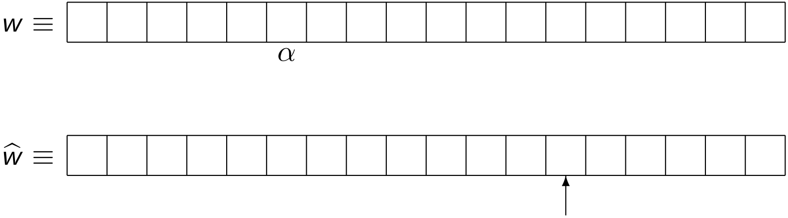

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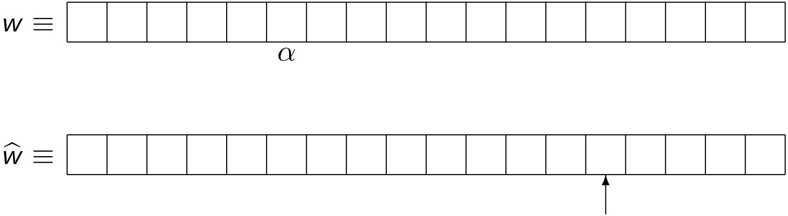

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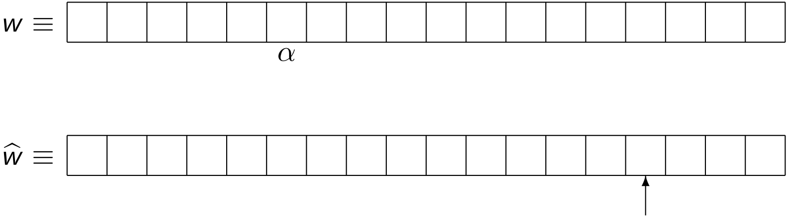

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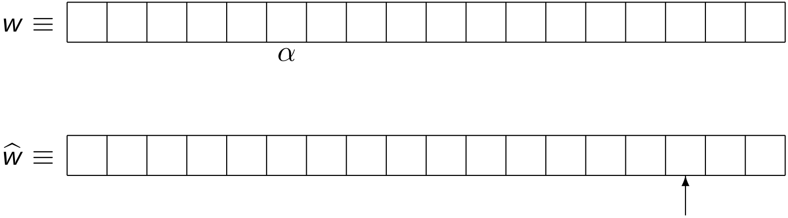

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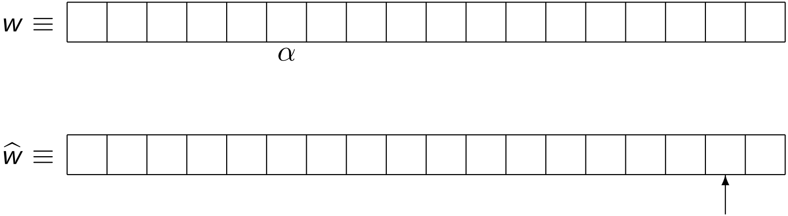

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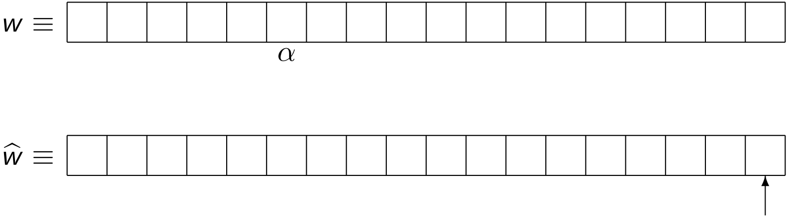

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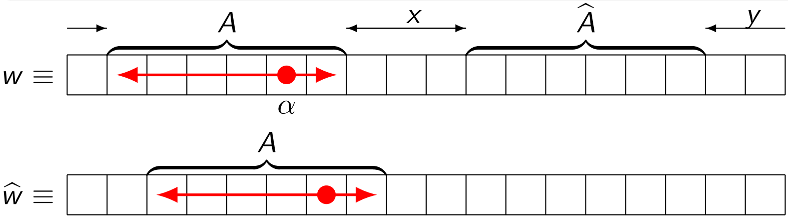

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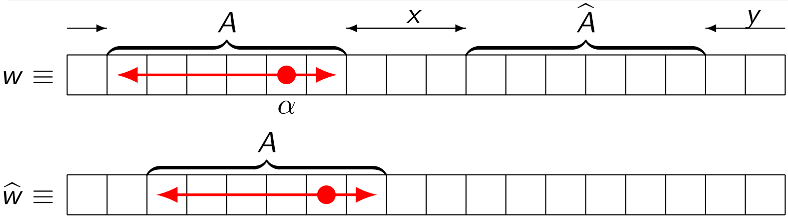


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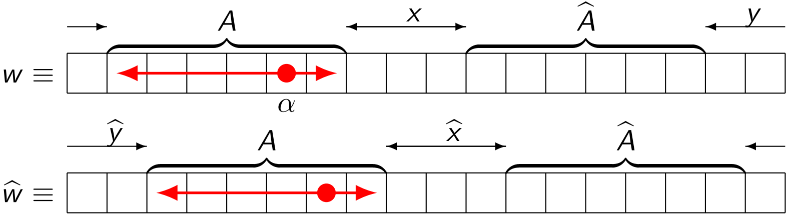


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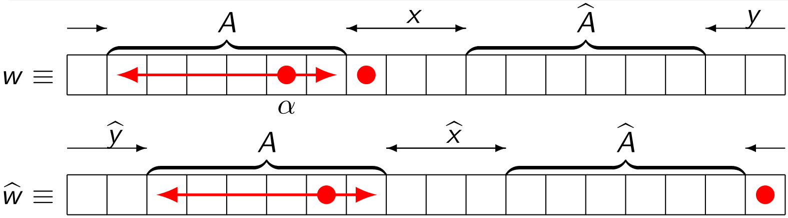


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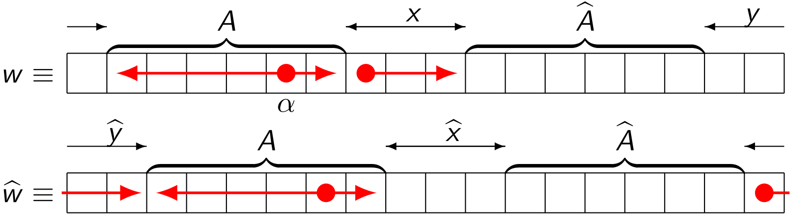


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**Lemma**

*Let $w$ be a $k$-square-free word coding a polyomino, and let $\alpha$ be a position in $w$. The number of admissible factors overlapping $\alpha$ in $w$ is bounded by $4k + 2\log(n)$.*

# Detecting pseudo-hexagons

**Theorem**

*Let $w$ be a $k$-square-free word coding a polyomino, with $k \in \mathcal{O}(\sqrt{n})$. Determining if $w$ codes a pseudo-hexagon is decidable in linear time.*

# Detecting pseudo-hexagons

*Input* : $w \in \Sigma^*$ coding a polyomino $p$.

**Build** $L_1$ the list of all admissible factors that overlap the position $\alpha$.

$\beta :=$ (the position of the rightmost letter of $w$ include in a factor of $L_1$) $+ 1$.

**Build** $L_2$ the list of all admissible factors that overlap the position $\beta$.

**For** all $X \in L_1$ **do**

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    **If** $w \equiv XYx\widehat{X}\widehat{Y}y$ **then**

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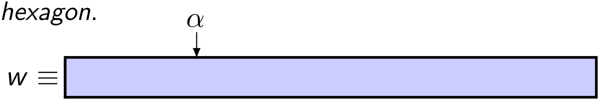

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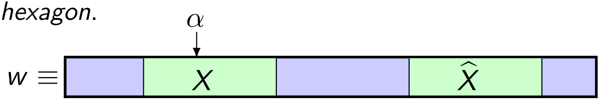

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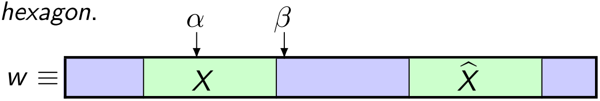

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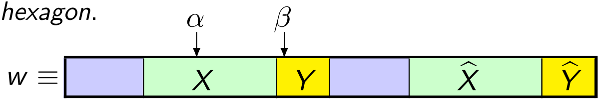

# Detecting pseudo-hexagons

*Input* : $w \in \Sigma^*$ coding a polyomino $p$.

**Build** $L_1$ the list of all admissible factors that overlap the position $\alpha$.<br>
$\beta :=$ (the position of the rightmost letter of $w$ include in a factor of $L_1$) $+ 1$.<br>
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      **If** longest common extention$(w, \widehat{w}, i, j) = |x|$ **then**<br>
        $p$ is a *speudo-hexagon*.<br>
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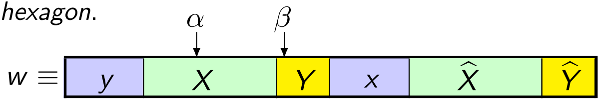

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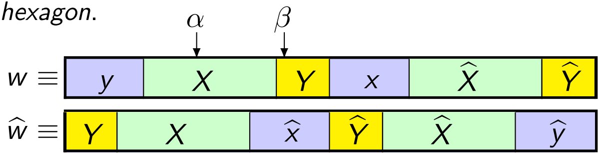

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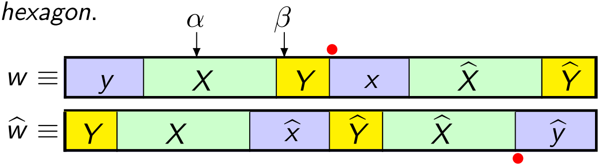

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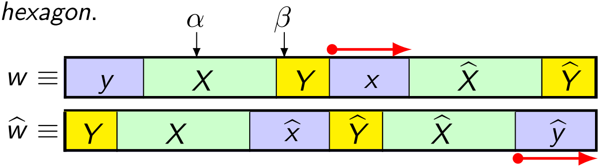

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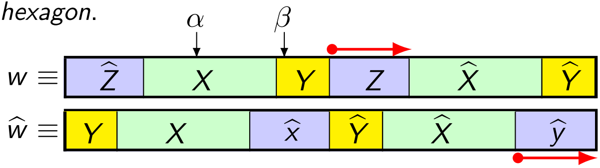

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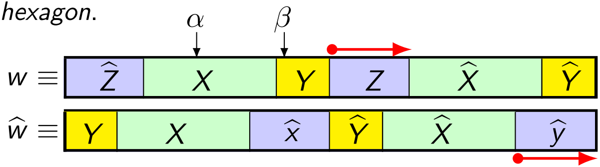
$$\mathcal{O}\left(n + (k + \log n)^2\right) = \mathcal{O}(n)$$



# *THANK YOU*!