Non-Archimedean words and groups

Alexei Miasnikov
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Outline

In this talk I will discuss some new results and methods on free actions of groups. These methods were extensively, though sometimes implicitly, used in our joint with Olga Kharlampovich solution of the Tarski's problems.

It seems, they provide an adequate tool to attack some open problems concerning with the algebraic structure of finitely generated groups acting freely on Lambda-trees.

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The starting point

Theorem (J.-P. Serre, 1980). A group G is free if and only if it acts freely on a tree.

We always assume that a group acts on a metric spaces by isometries.

Free action = no inversion of edges and stabilizers of vertices are trivial.

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$\mathbb{R}$-trees

An $\mathbb{R}$-tree is a metric space (X, p) (where $p : X \times X \to \mathbb{R}$) which satisfies the following properties:

1) (X, p) is geodesic,
2) if two segments of (X, p) intersect in a single point, which is an endpoint of both, then their union is a segment,
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$\mathbb{R}$-trees were introduced by **J.Tits** in 1977.

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Examples

$X = \mathbb{R}$ with usual metric.

A geometric realization of a simplicial tree.

$X = \mathbb{R}^2$ with metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$

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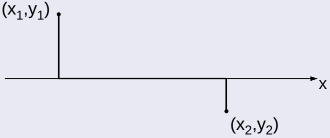

Lyndon's conjecture

A group G acts on an $\mathbb{R}$-tree by isometries.

An action is free if there are no inversions of arcs and the stabilizer of each point is trivial.

$\mathbb{R}$-free groups = groups acting freely on $\mathbb{R}$-trees.

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Every $\mathbb{R}$-free group is a subgroup of $\mathbb{R} * \cdots * \mathbb{R}$.

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Alperin and **Moss (1985)**, and **Promislow (1985)**:

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Finitely generated $\mathbb{R}$-free groups

Rips' Theorem [Rips, 1991 - not published]

A f.g. group is $\mathbb{R}$-free if and only if it is a free product of surface groups (except for the non-orientable surfaces of genus 1,2, 3) and free abelian groups of finite rank.

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Ordered abelian groups

Λ = an ordered abelian group.

Examples:

Archimedean case:

$\Lambda = \mathbb{R}$, $\Lambda = \mathbb{Z}$ with the usual order.

Non-Archimedean case:

$\Lambda = \mathbb{Z}^2$ with the right lexicographic order:

$$(a, b) < (c, d) \iff b < d \text{ or } b = d \text{ and } a < c.$$

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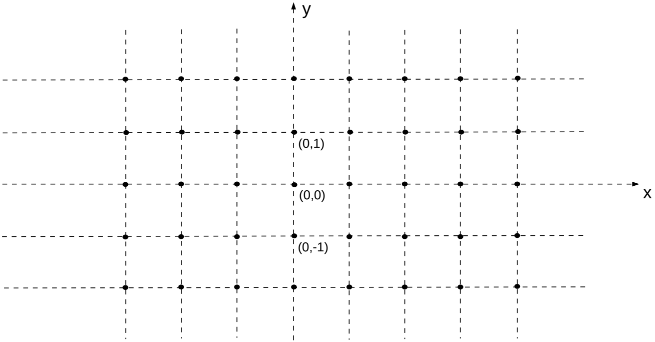
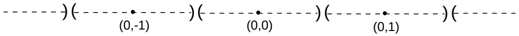


One-dimensional picture



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Morgan and Shalen (1985) defined Λ-trees:

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Theory of Λ-trees

Alperin and Bass (1987) developed the theory of Λ-trees and stated the fundamental research goals:

Find the group theoretic information carried by an action on a Λ-tree.

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The Fundamental Problem

The following is a principal step in the Alperin-Bass' program:

Open Problem [Rips, Bass]

Describe finitely generated groups acting freely on Λ-trees.

Here "describe" means "describe in the standard group-theoretic terms".

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Theorem (Bass, 1991)

A finitely generated $(\Lambda \oplus \mathbb{Z})$-free group is the fundamental group of a finite graph of groups with properties:

- vertex groups are Λ-free,
- edge groups are maximal abelian (in the vertex groups),
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Since $\mathbb{Z}^n \simeq \mathbb{Z}^{n-1} \oplus \mathbb{Z}$ this gives the algebraic structure of $\mathbb{Z}^n$-free groups.

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Actions on $\mathbb{R}^n$-trees

Theorem [Guirardel, 2003]

A f.g. freely indecomposable $\mathbb{R}^n$-free group is isomorphic to the fundamental group of a finite graph of groups, where each vertex group is f.g. $\mathbb{R}^{n-1}$-free, and each edge group is cyclic.

However, the converse is not true.

Corollary A f.g. $\mathbb{R}^n$-free group is hyperbolic relative to abelian subgroups.

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Theorem [Kharlampovich, Miasnikov]

Finitely generated fully residually free groups are $\mathbb{Z}^n$-free.

Theorem [Martino and Rourke, 2005]

Let G_1 and G_2 be $\mathbb{Z}^n$-free groups. Then the amalgamated product $G_1 *_C G_2$ is $\mathbb{Z}^m$-free for some $m \in \mathbb{N}$, provided C is cyclic and maximal abelian in both factors.

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Examples of $\mathbb{Z}^n$-free groups:

$\mathbb{R}$-free groups,

$\langle x_1, x_2, x_3 \mid x_1^2 \; x_2^2 \; x_3^2 = 1 \rangle$ is $\mathbb{Z}^2$-free (but is neither $\mathbb{R}$-free, nor fully residually free).

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Length functions

Length functions were introduced by **Lyndon (1963)**.

Let G be a group. A function $l : G \to \Lambda$ is called a length function on G if

(L1) $\forall\, g \in G: \; l(g) \geqslant 0$ and $l(1) = 0$,

(L2) $\forall\, g \in G: \; l(g) = l(g^{-1})$,

(L3) the triple $\{c(g,f), c(g,h), c(f,h)\}$ is isosceles for all $g, f, h \in G$, where $c(f,g)$ is the Gromov's product:

$$c(g,f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f)).$$

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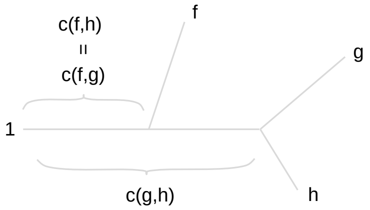
Examples of length functions

In a free group F, the function $f \to |f|$ is a $\mathbb{Z}$-valued length function.

For $f, g, h \in F$ we have



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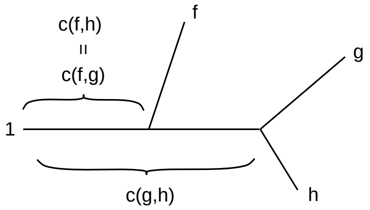

Free length functions

A length function $l : G \to \Lambda$ is free if $l(g^2) > l(g)$ for every non-trivial $g \in G$.

Theorem [Lyndon, 1963]

A group G has a free length function in $\mathbb{Z}$ if and only if G is free.

Chiswell (1976) established a connection between real-valued length functions and actions on metric spaces, which happen to be $\mathbb{R}$-trees **(Imrich, 1977)**.

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Chiswell's Theorem

Let (X, d) be a Λ-tree and $x \in X$. If G acts on (X, d) then $L(g) = d(x, gx)$ is a Lyndon length function on G.

Theorem [Chiswell]

Let $L : G \to \Lambda$ be a Lyndon length function on a group G. Then there exists a Λ-tree (X, d), $x \in X$, and an isometric action of G on X such that $L(g) = d(x, gx)$ for all $g \in G$.

Notice that $L(g) = d(x, gx)$ is free iff the action of G on X is free.

This gives another approach to free actions on Λ-trees.

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Infinite words

Let A be a discretely ordered abelian group with a minimal positive element 1_A and $X = \{x_i \mid i \in I\}$ be a set.

An ***A*-word** is a function

$$w : [1_A, \alpha] \to X^{\pm}, \quad \alpha \in A.$$

$|w| = \alpha$ is called the length of w.

w is **reduced** $\iff$ no subwords xx^{-1}, $x^{-1}x$ $(x \in X)$.

$R(A, X) =$ the set of all reduced A-words.

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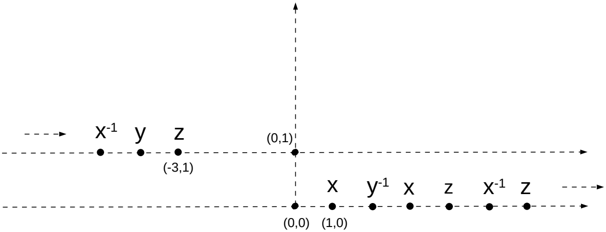
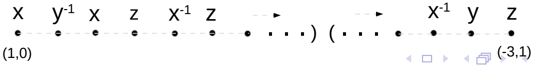
Example.

Let $X = \{x, y, z\}$, $A = \mathbb{Z}^2$

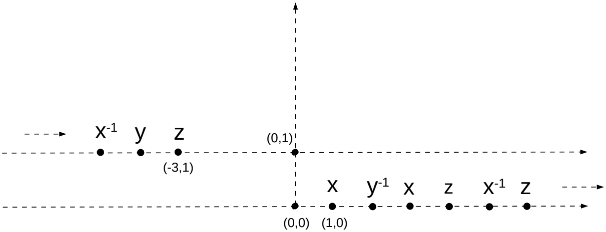


In "linear" notation



Example.

Let $X = \{x, y, z\}$, $A = \mathbb{Z}^2$



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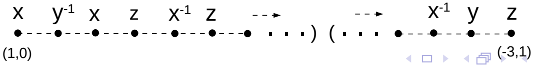
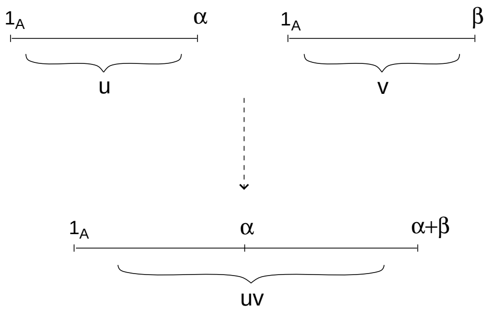
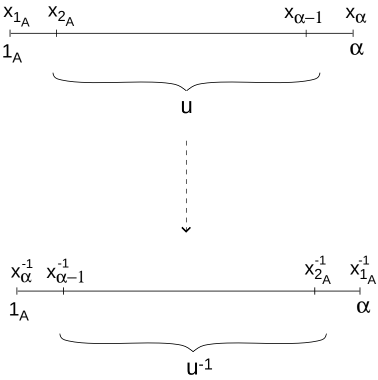


Concatenation of A-words:

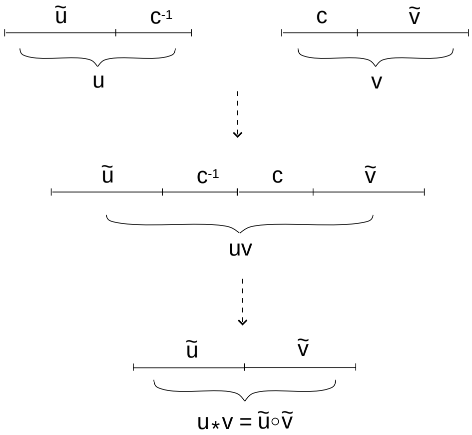


We write $u \circ v$ instead of uv in the case when uv is reduced.

Inversion of A-words:



Multiplication of A-words:



Multiplication of A-words is partial

The product of u and v is defined iff the common initial subword

$$c = com(u^{-1}, v)$$

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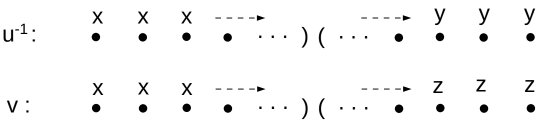
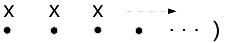
Example

Let $u, v \in R(\mathbb{Z}^2, X)$



Hence, the common initial part of u^{-1} and v is



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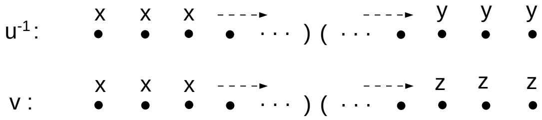

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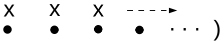

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Cyclic decompositions

$v \in R(A, X)$ is **cyclically reduced** if $v(1_A)^{-1} \neq v(|v|)$.

$v \in R(A, X)$ admits a **cyclic decomposition** if

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From Non-Archimedean words - to length functions

Theorem [Myasnikov-Remeslennikov-Serbin]

Let Λ be a discretely ordered abelian group and X a set. If G is a subgroup of $CDR(\Lambda, X)$ then the function $L_G : G \to \Lambda$, defined by $L_G(g) = |g|$, is a free Lyndon length function.

Corollary.

To show that a group G acts on a Λ-tree - embed G into $CDR(\Lambda, X)$.

Which Λ-free groups embed into $CDR(\Lambda, X)$?

All of them!

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From Length functions - to Non-Archimedean words

Theorem [Chiswell]

Let Λ be a discretely ordered abelian group. If $L : G \to \Lambda$ is a free Lyndon length function on a group G then there exists an embedding $\phi : G \to CDR(\Lambda, X)$ such that $|\phi(g)| = L(g)$ for every $g \in G$.

Corollary. Let Λ be an arbitrary ordered abelian group. If $L : G \to \Lambda$ is a free Lyndon length function on a group G then there exists a length preserving embedding $\phi : G \to CDR(\Lambda', X)$, where $\Lambda' = \Lambda \oplus \mathbb{Z}$ with the lex order.

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The standard way:

Infinite words $\Longrightarrow$ Lyndon length functions $\Longrightarrow$ Free actions

Direct construction:

If $G \hookrightarrow CDR(\Lambda, X)$ then G acts by isometries on a (canonical) Λ-tree Γ_G labeled by letters from $X^{\pm}$.

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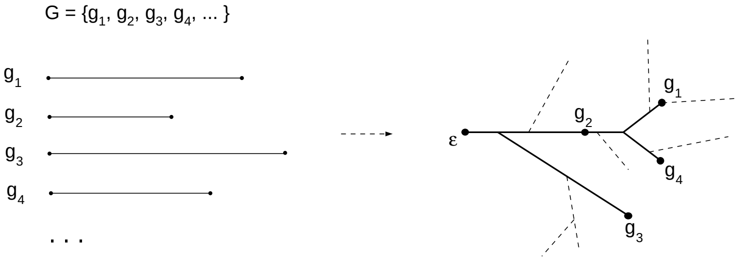
The tree Γ_G

Construction of Γ_G via foldings:



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Theorem [KMS]

Let $G \hookrightarrow CDR(\Lambda, X)$. Then:

- Γ_G is a Λ-tree,
- G acts freely on Γ_G,
- $L_\epsilon(g) = |g|$,
- Γ_G is minimal $\iff$ G contains a cyclically reduced element,
- If (Z, d') is a Λ-tree on which G acts freely by isometries, and $w \in Z$ is such that $L_w(g) = |g|$, then there is a unique G-equivariant isometry $\mu : \Gamma_G \to Z$ such that $\mu(\epsilon) = w$, and $\mu(\Gamma_G)$ is the subtree of Z spanned by the orbit $G \cdot w$ of w.

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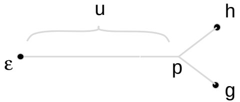
Complete free actions

Complete subgroups

A subgroup $G \leq CDR(\Lambda, X)$ is complete if G contains the common initial segment $com(g, h)$ for every pair of elements $g, h \in G$.



Example. $G = \langle a^{-1}ba, a^{-2}ba^2 \rangle \leq F(a, b)$.

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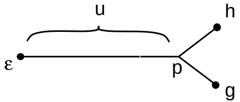

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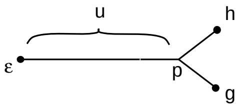

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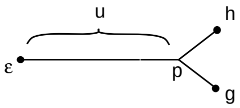

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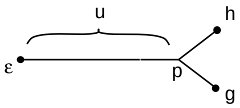
Branch points and completeness

A vertex $p \in \Gamma_G$ is a branch point if it is the terminal endpoint of the common initial segment $u = com(g, h)$ of $g \neq h \in G$.



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Let G act freely on a Λ-tree Γ. Then

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Motivation: Groups with complete length functions (actions) have nice properties:

- Nielsen cancellation techniques.
- Analog of Stallings' foldings.
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Structure of complete $\mathbb{Z}^n$-free groups

Theorem [Kharlampovich, Myasnikov, Serbin]

Let G be a finitely generated complete $\mathbb{Z}^n$-free group. Then

$$G \simeq \langle H, T \mid t_i^{-1} C_i t_i = D_i,\ t_i \in T \rangle,$$

where

- H is a complete $\mathbb{Z}^{n-1}$-free group,
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Proof is based on Nielsen cancellation techniques and Stallings pregroups method.

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Complete HNN-extensions

Theorem [Kharlampovich, Myasnikov, Serbin]

Let H be a complete $\mathbb{Z}^n$-free group. If C, D are maximal abelian subgroups of H and $\phi : C \rightarrow D$ is an isomorphism such that

- ϕ is length preserving,
- c is not conjugate to $\phi(c)^{-1}$ in H for any $c \in C$.

then

$$G = \langle H, s \mid s^{-1} C s \stackrel{\phi}{=} D \rangle,$$

is a complete $\mathbb{Z}^{n+1}$-free group and the standard embedding $H \rightarrow G$ is length preserving.

Idea of the proof

Let $F = F(x, y)$, $u, v \in F$ cyclically reduced elements, $|u| = |v|$ and v is not conjugate to u^{-1}.

Embed $G = \langle F, s \mid s^{-1}us = v \rangle$ into $CDR(\mathbb{Z}^2, X)$:

- $F(X) \subset CDR(\mathbb{Z}^2, X)$ as finite words.
- Need to send s to an infinite word w such that $uw = wv$

  Non-standard argument: put $w = u^\infty v^\infty$ then

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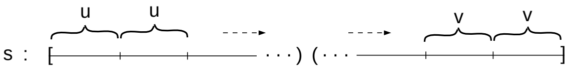
Non-standard argument revised:

Map s to the following infinite word



Clearly, $u \circ s = s \circ v$.

Description of complete free $\mathbb{Z}^n$-actions

Theorem [KMS]

A finitely generated group G is a complete $\mathbb{Z}^n$-free group if and only if G can be represented as a union of a finite series of groups

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- G_i is complete $\mathbb{Z}^i$-free (so G_1 is free),
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Theorem [KMS]

Every f.g. $\mathbb{Z}^n$-free group G embeds into a complete $\mathbb{Z}^n$-free group G^*. Moreover, this embedding preserves the length and can be constructed effectively.

A (minimal) such group G^* is called a **completion** of G.

Apply the standard Bass-Serre theory to study G as a subgroup of G^*.

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Problem. Develop a theory of completions: uniqueness under minimality conditions, effective constructions, etc.

Algorithmic properties of $\mathbb{Z}^n$-free groups.

Motto: Algorithmic properties of f.g. $\mathbb{Z}^n$-free groups are similar to the ones of free groups.

Main ideas:

- Go to the completion.
- Replace finite words with the infinite ones.
- Repeat the known proofs for the standard free groups.

Hint: Take this idea literally.

The main illustration: fully residually free groups.

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The Main Conjecture.

Why $\mathbb{Z}^n$-free groups?

Conjecture

Every finitely generated Λ-free group is $\mathbb{Z}^n$-free.

How probable it is?

All known finitely generated Λ-free groups are $\mathbb{Z}^m$-free.

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Theorem [Kharlampovich, M., Serbin]

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Reflection on Non-Archimedian words

Pro: The set of reduced words $R(A, X)$ provides many useful techniques to deal with group actions, equations in groups, etc.

Con: $R(A, X)$ is not a group - it has only partial multiplication.

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Non-Archimedian "free" words

Let $A = \mathbb{Z}^\omega$ be a free abelian group of infinite rank with lex order.

$W(A, X)$ - the set of all Non-Archimedean words of the type $[\alpha, \beta] \to X^{\pm 1}$ over A.

Two words

$u : [\alpha, \beta] \to X^{\pm 1}, \quad v : [\gamma, \delta] \to X^{\pm 1}$

are **equivalent** if v is a "shift" of u by some $a \in A$:

Concatenation is a well-defined operation on

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Monoid $M(A, X)$

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$R(A, X) \hookrightarrow M(A, X)$

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Let's try again to construct the analog of a free group from $M(A, X)$.

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$$M_{inv}(A, X) = M(A, X)/\{uu^{-1} = 1 \mid u \in R(A, X)\}$$

Let $\phi : M(A, X) \to M_{inv}(A, X)$ be the standard epimorphism.

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Claim: The image $\phi(R(A,X)^*)$ is a unique maximal subgroup in $M_{inv}(A,X)$.

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The group $F(A,X) = \phi(R(A,X)^*)$ is called the Non-Archimedean free group with basis X over A.

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Non-Archimedean boundaries of $F(A, X)$

The set of reduced Non-Archimedean words $R(A, X)$ forms the Non-Archimedean boundary of the free Non-Archimedean group $F(A, X)$.

Non-Archimedean free groups and free actions

Theorem, [Diekert, M.]

The canonical projection $\phi : R(A, X)^* \to F(A, X)$ is injective on $R(A, X)$.

Corollary.

- $F(A, X)$ is the canonical extension of $F(X)$
- $F(A, X)$ contains all finitely generated groups acting freely on A-trees.

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Algebraic structure of $M(A, X)$ and $F(A, X)$

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What is the algebraic structure of the free Non-Archimedean monoid $M(A, X)$?

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