An Application of Word Combinatorics to Decision Problems in Group Theory

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In this talk we develop a graph theoretical test on graphs corresponding to subgroups of one-relator groups with small cancellation condition which, if successful, implies that the subgroup under consideration has solvable membership problem with a simple solution. The proof of the solvability of the membership problem relies on word combinatorics in an essential way.

One way to describe a group G is by giving a set S of elements of G and a set of defining relations among them such that S generates G, in the sense that every element of G is the product of elements of S or their inverses and every relation among elements of S is a consequence of the given relations. More formally, let X be a set in one to one correspondence, say θ_0, with S and let $F = F(X)$ be the free group, freely generated by X. Then θ_0 extends uniquely to a homomorphism θ from $F(X)$ onto G. Therefore, if $N = Ker\ \theta$, then $G \cong F\big/N$.

$$\theta : F(X) \to G, \quad F\big/Ker\ \theta \cong G \tag{1}$$

Now, the relations which hold among the generators of G in S are precisely the images of the elements of N by θ and if $\mathcal{R}$ is a subset of N which normally generates N, in the sense that every element of N is a finite product of conjugates of elements of $\mathcal{R}$ and their inverses, then the images of $\mathcal{R}$ form a set of defining relations for G. In this case N is *the normal closure of* $\mathcal{R}$, denoted by $\ll \mathcal{R} \gg^F$. Elements of N also called *consequences of* $\mathcal{R}$. The triple $(X, \mathcal{R}, \theta)$ defines G completely. Such a description of G is a *presentation of* G. (See [Jo].) Usually, θ is clear from the context and then we denote the presentation by $\mathcal{P} = \langle X|\mathcal{R}\rangle$. In these terms $F(X)$ has presentation $\langle X|-\rangle$, where " $-$ " denotes the empty set.

Sometimes it is possible to deduce properties of groups from their presentations, if we can solve the following fundamental decision problem, the Word Problem.

(WP): *Given an element g in the group G. Decide whether $g = 1$ or $g \neq 1$ in G.*

However, in general this problem is not solvable. (See [L-S].) Let us see what kind of difficulties may occur: let g be an element of a group G which has a presentation $\langle X|\,\mathcal{R}\rangle$, with X and $\mathcal{R}$ finite and let W be a word in F with $\theta(W) = g$. We may assume that W is reduced, in the sense that no xx^{-1} or $x^{-1}x$ occurs in W, for every $x \in X$. We would like to check whether $g = 1$ or $g \neq 1$ in G. Assume $g = 1$. Then $W \in N$ and following the above notation and writing R^f in place of $f^{-1}Rf$, we have

$$W = R_1^{\varepsilon_1 f_1} \cdots R_k^{\varepsilon_k f_k}, \quad R_i \in \mathcal{R}, \quad \varepsilon_i \in \{1, -1\} \quad \textit{and} \quad f_i \in F, \quad i = 1, \ldots, k \qquad (2)$$

Now the left hand side of (2) is a reduced word while the right hand side in general is not. (See Example 1.) Hence the reduced word W is obtained from the right hand side by free cancellations which replace xx^{-1} and $x^{-1}x$, $x \in X$ by the empty word. If we could know that after carrying out the cancellations at least one letter from each $R_i^{\varepsilon_i f_i}$ survives, we could know that W is the product of at most $|W|$ conjugates of relations, where $|W|$ is the length of W, i.e. the number of letters in W with multiplicity, as usual. It is not difficult to show that $|f_i|$ can be bounded from above by $2k|W||R| + |W|$, where $|R|$ is the length of the longest relator in $\mathcal{R}$. (See [L-S, p. 239].) Hence, we could produce *all* the possible products of conjugates of R_i which may be candidates for the right hand side of (2) and check if one of them coincides with W in the free group $F(X)$, after cancellations. If the answer is "NO" then $g \neq 1$ and if the answer is "YES" then we get a proof for $g = 1$. But in general, it is not the case that a letter from each conjugate of $R_i^{\pm 1}$ survives the cancellations.

It turns out however, that we still can use the naive idea of requiring the survival of one letter from each conjugate of R_i in (2), in a weaker form: instead, we require that whenever a word is cancelled out in the right hand side of (2) it is short, relative to the length of the relators involved. Then it follows from theorems in Combinatorial Group Theory that the number k of relators in the right hand side of (2) is bounded from above by $c|W|$, where c is a known constant, depending on the presentation. This clearly solves the word problem for G and definitely allows complete cancellation of relators in (2).

Example 1. ([De]) Let $\mathcal{P} = \langle a_1, b_1, a_2, b_2 | R \rangle$, $R = a_1^{-1}b_1^{-1}a_1b_1a_2^{-1}b_2^{-1}a_2b_2$. In order to check the amount of cancellation between two relators we have to consider all the configurations in which cancellation between letters of the relators can occur. For example, in order to see how a_1^{-1} (which is the first letter of R) cancels with a_1 (which is the third letter of R) we have to consider a conjugate R' of R which ends with a_1 and then the last letter (a_1) of R' cancels the first letter (a_1^{-1}) of R:

$$\begin{aligned} R' &= b_1a_2^{-1}b_2^{-1}a_2b_2a_1^{-1}b_1^{-1}a_1 = \left(b_1^{-1}a_2^{-1}b_2^{-1}a_2b_2\right) R \left(b_1^{-1}a_2^{-1}b_2^{-1}a_2b_2\right)^{-1} \quad \text{and} \\ R' \cdot R &= \left(b_1a_2^{-1}b_2^{-1}a_2b_2a_1^{-1}b_1^{-1}\right)\left(a_1a_1^{-1}\right)\left(b_1^{-1}a_1b_1a_2^{-1}b_2^{-1}a_2b_2\right) \end{aligned}$$

Such conjugates (i.e. by initial or terminal subwords of R) are *the cyclic conjugates of* R. The word R has $|R|$ cyclic conjugates ($|R| = 8$), and similarly R^{-1} has eight cyclic conjugates, which can be obtained from $R^{\pm 1}$ by writing $R^{\pm 1}$ on a circle, rather than on a straight line, and start reading each time in a different place. The word R written on a circle is called *the cyclic word corresponding to* R and we shall denote it by $\widehat{R}$. The word R is cyclically reduced if $|R| = |\widehat{R}|$. It is easy to check that if R_1 and R_2 are two cyclic conjugates of $R^{\pm 1}$ with $R_1 \cdot R_2 \neq 1$ in F then in forming the product R_1R_2 at most one letter may cancel out in R_1 and one in R_2. Therefore, the ratio of the length of the cancelled word and $|R|$ is $\frac{1}{8}$ and this amounts as "small". For this example the theory shows that the number k of relators in (2) is bounded from above by $\frac{1}{6}|W|$. The theory which deals with result of this type is *Small Cancellation Theory*.

There are three classical small cancellation conditions under which the theory works: the metric condition $C'(\lambda)$, where $0 < \lambda < 1$, and the combinatorial conditions $C(p)$ and $T(q)$, where $p, q \geq 3$ are natural numbers. The condition $C'(\lambda)$ requires that if R_1 and R_2 are two relators or their inverses and U is a word cancelled out in forming the product $R_1 \cdot R_2$, then $\frac{|U|}{|R_1|} < \lambda$ and $\frac{|U|}{|R_2|} < \lambda$. For example, the group presentation in

Example 1 satisfies this condition with $\lambda = \frac{1}{7}$, since $\frac{1}{8} < \frac{1}{7} = \lambda$. The condition $C(p)$ requires that if $\mathcal{P} = \langle X | \mathcal{R} \rangle$ is a presentation and $R^{\pm 1} \in \mathcal{R}$ is any relator or one of its cyclic conjugates and R can be decomposed to $R = P_1 \cdots P_s$, reduced as written, such that each P_i is a subword of a relator or a cyclic conjugate of a relator or its inverse which occurs in a different relation or in R in a position different from that in R, then $s \geq p$. The subword P_i are called *pieces*. (See [L-S, p. 240-241].) Clearly, $C'(1/n)$ implies $C(n+1)$ and if $p_1 > p_2$ then $C(p_2)$ implies $C(p_1)$. The condition $T(q)$ requires the following. Let $3 \leq h < q$. Suppose $R_1, \ldots, R_n$ are cyclic conjugates of relators in R or their inverses, such that no successive elements R_i, R_{i+1} form on inverse pair. Then at least one of the two products $R_1 \cdots R_{h-1} R_h$ and $R_h R_1$ is reduced as written. A basic theorem of small cancellation theory states that if $\mathcal{P}$ is a presentation which satisfies the condition $C(p)$ with $p \geq 6$ or $C(4) \& T(4)$, then the word problem for $\mathcal{P}$ is solvable. In this work we shall assume that both conditions $C'(1/5)$ and $T(4)$ are satisfied by the relevant presentations.

A natural extension of the word problem is the Membership Problem:

(MP): *Given a subgroup H of the group G (for example, by generators) and an element $g \in G$. Decide whether $g \in H$.*

The (WP) is a special case of the (MP), where $H = \{1\}$. The Membership Problem is not solvable in general, even in small cancellation groups. The present work shows how word combinatorics together with tools from small cancellation theory enables one to solve the Membership Problem for certain kind of subgroup in the class of groups which can be defined by a single relator (one-relator groups). This class of groups contains important subclasses like surface groups, fundamental groups of three-manifolds. It was the first class of groups in combinatorial group theory for which definitive positive results were obtained, mostly due to the pioneering works of Max Dehn and Wilhelm Magnus. (See [L-S].) Among many other things Magnus proved that if $\mathcal{P} = \langle x_1, \ldots, x_n | \, R \rangle$ presents the one-relator group G and $H \subseteq \langle x_{i_1}, \ldots, x_{i_k} \rangle$, $k \leq n-1$ then the membership problem for H in G is solvable. In spite of the much research made on one-relator groups, essentially for no other classes of subgroups of G is the membership problem known to be solvable. The main result of the present work produces new infinite families of such subgroups.

Before we formulate the Main Theorem, we recall the notion of Whitehead graphs from [L-S]. Thus, the Whitehead graph of a word W in $F(X)$ is, by definition, the graph on $2|X|$ vertices which are labeled by elements of $X \cup X^{-1}$ in which two vertices, say v_1, with label a_1 and v_2 with label a_2 ($a_1, a_2 \in X \cup X^{-1}$), are connected by an edge if $a_1^{-1}a_2$ is a subword of W or of W^{-1}. We shall denote by $Wh(W)$ the graph obtained from the Whitehead graph by identifying edges with the same endpoints. If T is a set of words, we denote by $Wh(T)$ the union of $Wh(t)$, $t \in T$. Our Main Theorem loosely says that if $Wh(\widetilde{H})$ does not contain a large portion of $Wh(R)$ then H has solvable membership problem. Denote by $\mathcal{E}(W)$ the set of edges of $Wh(W)$ and similarly let $\mathcal{E}(T)$ the set of edges of $Wh(T)$.

Main Theorem. *Let $\mathcal{P} = \langle X | R \rangle$ be a one-relator presentation of a one-relator group G with $|R| \geq 5$ which satisfies the small cancellation conditions $C'(1/5)$ and $T(4)$. Let $F = F(X)$ and let $\theta : F \to G$ be the canonical homomorphism introduced in (1). Let $Y \subseteq F$ be a finite subset of F, let $\widetilde{H}$ be the subgroup of F generated by Y and let $H = \theta(\widetilde{H})$. Let $Z = \mathcal{E}(\widehat{R})$ and let $K = \mathcal{E}(H)$. If*

$$|Z| - |Z \cap K| > 3 \qquad (**)$$

then H has solvable Membership Problem in G.

Three remarks are in order here. First, observe that Magnus' result mentioned above, and this result are of similar type in that Magnus' result is on subwords of length one, namely the generators, and in our Main Theorem we consider subwords of length two, instead. Next, with the cost of more work on the side of small cancellation theory, we can replace the metric condition $C'(1/5)$ by the combinatorial condition $C(6)$. Finally, when checking condition $(**)$ we do not have to check each element of H when forming K, because there is a standard way to construct a graph from which K can be easily read off.

Example 2. Let $F = \langle a, b, c, d | - \rangle$, $R = P^2U_1U_2$, where $P = ab^{-2}a^2c^3acd^{-1}a$, $U_1 = da^{-3}db$ and $U_2 = c^{-1}dbc^{-1}b$. Let $H = \langle U_1, U_2 \rangle$. Then $\theta(H)$ has solvable membership problem, by the Main Theorem. To see this, consider first the Whitehead graph $Wh(R)$

of the cyclic word $\widehat{R}$. It is depicted in Fig. 1, where $\bar{a}$ denotes a^{-1} and similarly for $\bar{b}$, $\bar{c}$ and $\bar{d}$.

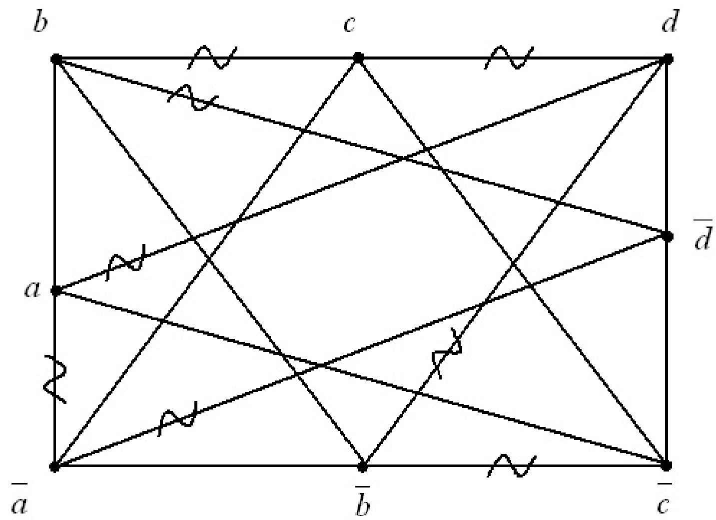

Figure 1.

Thus, $Wh(R)$ is a 4-regular graph with $\dfrac{8 \cdot 4}{2} = 16$ edges. Next, we consider the Whitehead graphs of the elements of $\widetilde{H}$. Since U_1 and U_2 have length greater than two and since the only cancellation in the products $U_i^{\varepsilon_i} U_j^{\varepsilon_j}$, where $\varepsilon_i, \varepsilon_j \in \{1, -1\}$, $i, j \in \{1, 2\}$ and $i = j \Rightarrow \varepsilon_i = \varepsilon_j$ are in $U_1 U_2^{-1}$, in which case only one letter is cancelled out (b), hence the Whitehead graph of every element in H is contained in the union of the Whitehead graphs of U_i and $U_i^{\varepsilon_i} U_j^{\varepsilon_j}$, $\varepsilon_i, \varepsilon_j \in \{1, -1\}$, $i, j \in \{1, 2\}$. These are marked in Fig. 1. by a tilda. Thus, we see that $|Wh(L)| \leq 8$ for every element L in H. But $|Wh(\widehat{R})| = 16$ and $8 < 16 - 3 = 13$. Therefore, we can not have $\Gamma \subseteq Wh(L)$ for any subgraph Γ of $Wh(R)$ which misses three edges or less. Consequently, if $\mathcal{R}$ satisfies $C(6)\&T(4)$ then $\theta(H)$ has solvable mempership problem in G, by the Main Theorem (see remark above). But since $Wh(R)$ contains no closed curves of length less than or equal three, it follows that $\mathcal{R}$ satisfies $T(4)$ and it is easy to check that it satisfies the condition $C(6)$ as well. Thus, $\theta(H)$ has solvable membership problem in G, by the Main Theorem and the second remark following the Theorem.

Finally, a few remarks on the method of proof. We use small cancellation theory as developed in [J1]. The main tool of the theory is van Kampen diagrams. These are planar cell complexes, labeled by elements of F, which describe precisely and efficiently the cancellations that occur in the right hand side of equation (2), when forming W. For every reduced word W in N there is such a diagram which has W as one of its boundary labels. (See [L-S, Ch.V]). We apply this theory in order to show that if $u \in G$ is an element represented by a reduced word U in F and $u \in H$ then there is a word V in $\widetilde{H}$ such that $|V| \leq 9|U||R|$ and $v = u$ in G, where v is the image of V in G. Since G has solvable word problem this solves the membership problem for H in G.

A central ingredient of the theory is Greendlinger's Lemma, which guaranties the existence of at least two Greendlinger regions (*two-cells*) in every van Kampen diagram M which has at least two regions. These are regions with the property that their boundary has a large common portion with the boundary of M. Since the label of the boundary ∂M of M is a consequence of $\mathcal{R}$ and for every consequence there is such a diagram, this means in algebraic terms that every consequence of $\mathcal{R}$ contains a large portion of one of the defing relations. (See [L-S, Ch. V].) If the small cancellation condition is strong enough, this result alone is enough to solve the word problem.

However there are problems, the solution of which requires a more precise information than just a large portion of the defining relator. Suppose for example that the set X of generators is partitioned into subsets: $X = \bigcup_{i \in I} X_i$, I an index set, (this happens, for example, when F_1 is a free product) and we would like to show that every consequence of R contains at least one letter from each X_i, $i \in I$. (This is what needed for the proof of Magnus' Freiheitssatz for one-relator products.) Clearly, Greendlinger's lemma is not helpful because even if the label Q of the common boundary of a Greendlinger region with ∂M is long, this does not guarantee that every X_i is represented in Q.

Recently we developed a method to resolve difficulties of this type in one-relator quotients, where F_1 is as above and $\mathcal{R}$ satisfies a certain small cancellation condition. Using versions of it we proved several results of different nature: in [J2] Magnus' Freiheitssatz for one-relator free products with more than two components is proved, under a certain small cancellation condition, without restriction on the components. (Notice that all the results in the literature make assumption on the components.) In [J8] we solved the membership problem for Magnus subgroups of one-relator free products with small cancellation. In [J9] we proved the appropriate version of Magnus's Freiheitssatz for Magnus subsemigroups of one-relator groups with small cancellation. In [J5] we completely classified all exceptional Magnus intersections (i.e. pairs of Magnus subgroups whose intersection is not a Magnus subgroup) in one-relator free products with small cancellation. (This result has consequences on solvability of equations (see [J10]).) In [J11] we classify non-malnormal Magnus subgroups in one-relator groups and free products with small cancellation.

In the present work we extend the method in order to describe the relationship between Whitehead graphs of defining relations and their consequences.

In all these results we rely heavily on the assumption that the given group is defined by a single defining relator. No doubt, this is strong combinatorial assumption, but how to connect it to small cancellation theory in a useful way? We resolved this problem by massive application of word combinatorics. The appearance of word combinatorics in the context of small cancellation theory is quite natural: for example, there are standard results, like Lyndon-Schützenberger's theorem for periodic words, which guaranteers that long subwords which occur more than once in the word occur in a very special configurations, which can be avoided by an appropriate small cancellation theory. (See also [C-J1] and [C-J2] for other types of words.)

Yet, the application of word combinatorics in the present work and the previously mentioned works is of different nature: we apply word combinatorics in order to improve Greendlinger's Lemma for the one-relator case in the following way: suppose for example, we would like to prove Magnus' Freiheitssatz for one-relator free products with several components; $G = \langle G_1 * \cdots * G_m \,|\, R\rangle$, $m \geq 3$. (This problem is similar to ours, but simpler.) We have to show that every consequence of R contains a letter from each G_i, $i = 1, \ldots, m$. Let C be a consequence of R and let M be a van Kampen diagram with C as a boundary label. It follows by standard small cancellation theory that under the condition $C'(1/5)\&T(4)$ a diagram M has a Greendlinger region D which either has only one neighbouring region E in M or it has two neighbours E_r and E_ℓ which have common boundary paths with D, having labels P_1 and P_2, respectively. (See Fig. 2 and Fig. 3, respectively.)

Suppose for simplicity the first and let P be the label of the common boundary path

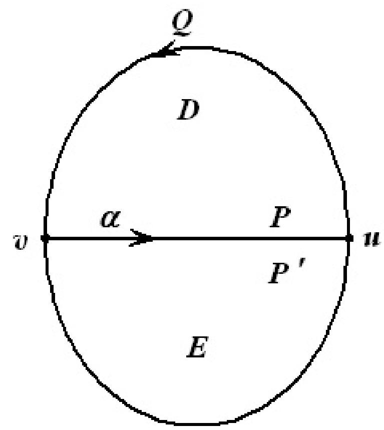

Figure 2.

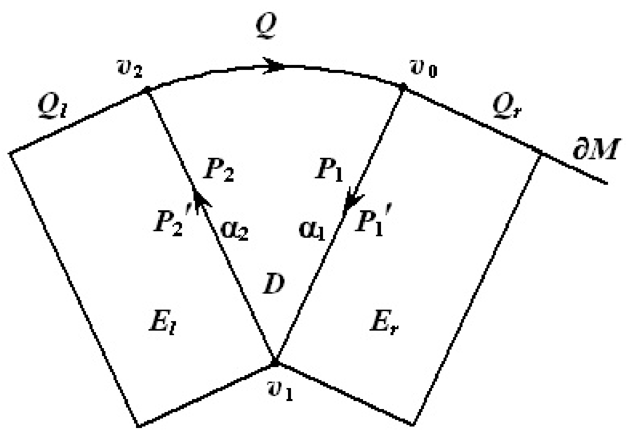

Figure 3.

$\partial E \cap \partial D$ of E and D and let Q be the label of the common boundary path of D and M. Thus $R_1 := QP$ is a boundary label of D, hence a cyclic conjugate of R^{ε}, $\varepsilon \in \{1, -1\}$. We claim that every letter in P necessarily occurs in Q. Now P is a common label of ∂D and ∂E. Therefore, P occurs as a subword of R_1 and also as a subword of a boundary label R_2 of E. But since

R is the only defining relator of G, R_1 and R_2 are cyclic conjugates of $R^{\pm 1}$.

Therefore, in addition to the above mentioned occurrence of P in R_1, P^{ε}, $\varepsilon \in \{1, -1\}$, has another occurrence in R_1, which we denote by P', which comes from the occurrence of P as a subword of R_2. (These occurrences are different because we assume that our diagrams are reduced.) Now, since $R_1 = QP$ and P' is a subword of the cyclic word $\widehat{R}_1$, either P_1' is a subword of Q in which case we are done, or else P' overlaps non-trivially with P. In this case $\varepsilon = 1$ and we have the following word equations: $P' = AB$, $Q = Q_0A$ and $P = BC$. In particular, $AB = BC(= P)$ hence by the well known (and easy) result from word combinatorics we get that $P = (KL)^{\alpha}K$, $\alpha \geq 1$ and $A = KL$, for certain subwords K and L of P. But then $Q = Q_0A = Q_0KL$ and since all the different letters occurring in P already occur in KL, we get that all the letters in P occur in Q, as required. Observe that we used here word combinatorics in order *to shift the letters of P into Q.*

In the case when D has two neighbours E_r and E_ℓ it is not always true that Q contains all the letters of P_1 and P_2. In fact we prove that

Q_rQQ_ℓ contains every letter of R, where Q_r and Q_ℓ are the labels of $\partial E_r \cap \partial M$ and $\partial E_\ell \cap \partial M$, respectively. $(*)$

This requires the development of a rather complicated combinatorial machinery on words. We describe it very briefly. First, observe that in the above word equations we were interested not so much in their solution but rather the values of the function $\mathrm{Supp} : F_1 \to \mathcal{L}$, where $\mathcal{L}$ is the set of all the subsets of $\{1, \ldots, m\}$ and for a word W in F_1, $\mathrm{Supp}(W)$ is the set of the indices i for which G_i contributes a letter in W. We rely very heavily on this observation when proving $(*)$. In the present work we deal with Whitehead graphs hence we have to introduce a new function $\sigma : F_1 \to \mathcal{K}$, where $\mathcal{K}$ is the set of all the subsets of the set of words in F_1 with length two. (This corresponds to the edges of the corresponding Whitehead graph.)

Finally, we point out that the above mentioned improved version of Greendlinger's Lemma holds true under the small cancellation condition $C(6)$, with a few known exceptions ([J6]), however the corresponding word combinatorics is incomparably much more complicated. Also, we would like to point out that the method of the present work is applicable to one-relator free products ([J2] and [J5]), one-relator amalgamated free products ([J4]), and one-relator HNN-extensions ([J3]). In general, to deal with relative presentations, like in one-relator free products, one has to deal with *partial words* rather than *ordinary words*. For these even the Fine-Wilf theorem has no complete analogue (see [B-H]). However, in [J3], [J4] and [J5] we are able to avoid using partial words, although in general this is not possible. In fact, in [J3], [J4] and [J5] we are interested primarily in the *combinatorics* of certain set theoretical *labels* attached to the words, and we use words as "carriers of the labels", the combinatorics of which faithfully reflects the combinatorics of the labels. The point is that these labels can be defined in the "holes" occuring in the partial words, hence combinatorics of ordinary words suffices (see [J6]). Also, with some care, the method works for certain subsemigroups of one-relator groups and one-relator semigroups (see [J7]).

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