Recent Progress in Combinatorics on Words
March 12-16, 2007

Pseudoepisturmian sequences

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Abstract

Theory of Sturmian and episturmian sequences is a subject of great interest in combinatorics on words with applications in several fields. The aim of this talk is to present very recent research results concerning some extensions of episturmian sequences which have been obtained in a joint work with Alessandro De Luca, Michelangelo Bucci, and Luca Q. Zamboni [**?**, **?**, **?**].

As is well known palindromes play an essential role in Sturmian and episturmian sequences. In fact, a Sturmian sequence can be generated by iteration of the operator $^{(+)}$ of right palindromic closure associating to each word $w \in \{a, b\}$ the word $w^{(+)}$ defined as the shortest palindrome having w as a prefix. More precisely, one introduces a map $\psi\colon \{a, b\}^* \to \{a, b\}$ defined as: $\psi(\varepsilon) = \varepsilon$ and for any word v and letter x, $\psi(vx) = (\psi(v)x)^{(+)}$. It has been proved that ψ is injective and that $\psi(\{a, b\}^*)$ is the set of palindromic prefixes of all *standard Sturmian sequences*, called also *central words* [**?**, **?**]. Hence, if one starts with an infinite word $x = x_1x_2 \cdots x_n \cdots$ (*directive word*), where the letters belong to $\{a, b\}$ and there are infinitely many occurrences of both the letters, then one obtains an infinite word $\psi(x) = \lim_{n\to\infty}(x_1 \cdots x_n)$ which is a standard Sturmian sequence. Conversely, any standard Sturmian sequence can be obtained in this way.

The use of iterated palindromic closure in the case of arbitrary directive words over any finite alphabet, gives rise to the class of standard episturmian sequences [**?**, **?**] including both standard Sturmian sequences and the standard Arnoux–Rauzy sequences [**?**]. More precisely for an infinite word $w \in A^\omega$, the following conditions are equivalent (see [**?**, **?**]):

1. There exists an infinite word $\Delta = x_1 x_2 \cdots x_n \cdots \in A^\omega$ such that $w = \lim_{n\to\infty} u_n$, with $u_1 = \varepsilon$ and $u_{i+1} = (u_i x_i)^{(+)}$ for all $i \geq 1$, where $^{(+)}$ is the right palindrome closure operator.

2. w is closed under reversal, and each of its left special factors is a prefix of w.

An infinite word satisfying such conditions is by definition a standard episturmian sequence. Thus the operator$^{(+)}$ of right palindromic closure and the iterated closure operator play an essential role in the combinatorics of of standard Sturmian and episturmian sequences.

If the reversal operator is replaced by an arbitrary involutory antimorphism , then conditions 1 and 2 above are no longer equivalent, but each gives rise to a natural generalization (or extension) of the usual episturmian sequences.

A first extension can be obtained by considering θ-palindromes, i.e., the fix points of an involutory antimorphism of a free monoid, and define accordingly the right θ-palindromic closure $w^{\oplus_\theta}$ of a word w as the shortest θ-palindrome having w as a prefix. Starting with a directive word, one can iterate the application of θ-palindrome closure operator generating an infinite word called *θ-standard sequence.* In [**?**] it has been proved that any θ-standard sequence is a morphic image, by an injective morphism, of the standard episturmian sequence having the same directive word.

A second extension is obtained by generalizing condition 2 above; an infinite word $w \in A^\omega$ is called standard θ-episturmian if it is closed under θ and each of its left special factors is a prefix of w.

Various connections exist amongst these two families of sequences. We shall see that, in general, neither one is a subset of the other.

A further generalization of condition 1 is made by allowing the iterative θ-palindrome closure process to start from an arbitrary word u_0 (called *seed*). In [**?**] we called any infinite word constructed in this way a *θ-standard sequence with seed.* This is a larger class, strictly containing not only θ-standard sequences (as is trivial by the definition), but also standard θ-episturmian sequences. Indeed, one of the main results shows that an infinite sequence s is θ-standard with seed if and only if it is closed under θ and there exists $N \geq 0$ such that any left special factor of s having length $n \geq N$ is a prefix of s.

In general, we shall refer to the words of these families as pseudoepisturmian sequences.

References

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