Sturmian fixed points of morphisms

V. Berthé

LIRMM-CNRS-Montpellier-France
berthe@lirmm.fr
http://www.lirmm.fr/~berthe

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A notion of duality for Sturmian morphisms

- On substitution invariant Sturmian words: an application of Rauzy fractals, B., Ei, Ito and Rao
- On an involution of Christoffel words and Sturmian morphisms, B., de Luca, Reutenauer

Sturmian fixed points

Question

For which α and ρ is the Sturmian word $s_{\alpha,\rho}$ a fixed point of some non-trivial substitution?

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For which α and ρ is the Sturmian word $s_{\alpha,\rho}$ a fixed point of some non-trivial substitution?

- By non-trivial substitution, we mean a substitution that is distinct from the identity.
- By $s_{\alpha,\rho}$ is substitution invariant, we mean that $s_{\alpha,\rho}$ a fixed point of some non-trivial substitution.

Sturmian words

Notation

Let $0 < \alpha < 1$. Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ denote the one-dimensional torus. The rotation of angle α of $\mathbb{T}^1$ is defined by $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1,\ x \mapsto x + \alpha$.
Let

$$\underline{I}_1 = [0, 1-\alpha),\quad \underline{I}_2 = [1-\alpha, 1);\quad \overline{I}_1 = (0, 1-\alpha],\quad \overline{I}_2 = (1-\alpha, 1].$$

We define the two Sturmian words:

$$\underline{s}_{\alpha,\rho}(n) = \begin{cases} 1 & \text{if } R_\alpha^n(\rho) \in \underline{I}_1, \\ 2 & \text{if } R_\alpha^n(\rho) \in \underline{I}_2, \end{cases}$$

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Characteristic word

The characteristic word is obtained for $\rho = \alpha$. One has

$$s_{\alpha,\alpha} := \underline{s}_{\alpha,\alpha} = \overline{s}_{\alpha,\alpha}.$$

Sturm number and characteristic case

For a number x in a quadratic field, we denote by x' the conjugate of x in this field.

Theorem [Crisp *et al.*]

Let $0 < \alpha < 1$ be an irrational number. Then the following two conditions are equivalent:

- the characteristic word $s_{\alpha,\alpha}$ is substitution invariant;
- α is a quadratic irrational with $\alpha' \notin [0,1]$.

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Sturm number [Allauzen]

A quadratic number α with $0 < \alpha < 1$ and $\alpha' \notin [0, 1]$ is called a Sturm number.

Homogeneous vs nonhomegeneous case

- The homogeneous case corresponds to $\rho \in \mathbb{Z}\alpha \bmod 1$
  [Berstel, Séébold, Brown, Fagnot,Lothaire...]

- The nonhomogeneous case corresponds to $\rho \notin \mathbb{Z}\alpha \bmod 1$
  [Komatsu,Parvaix...]

Yasutomi's characterization

Theorem [Yasutomi'97]

Let $0 < \alpha < 1$ and $0 \leq \rho \leq 1$. Then $s_{\alpha,\rho}$ is substitution invariant if and only if the following two conditions are satisfied:

- α is an irrational quadratic number and $\rho \in \mathbb{Q}(\alpha)$;
- $\alpha' > 1,\ 1 - \alpha' \leq \rho' \leq \alpha'$ or $\alpha' < 0,\ \alpha' \leq \rho' \leq 1 - \alpha'$.

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Corollary

Let α be a Sturm number. Then

- for any $\rho \in \mathbb{Q} \cap (0,1)$, $\underline{s}_{\alpha,\rho} = \overline{s}_{\alpha,\rho}$ is substitution invariant.
- [Yasutomi, Fagnot] Let $\rho \in [0,1)$. The Sturmian word $\underline{s}_{\alpha,\{n\alpha\}}$ (resp. $\overline{s}_{\alpha,\{n\alpha\}}$) is substitution invariant if and only if $n = -1, 0, 1$. In total we obtain exactly five substitution invariant Sturmian words

$$\{21s_{\alpha,\alpha}, 12s_{\alpha,\alpha}, 2s_{\alpha,\alpha}, 1s_{\alpha,\alpha}, s_{\alpha,\alpha}\}$$

in the homogeneous case.

Proofs of Yasutomi's characterization

- [Yasutomi]:
  Yasutomi defines three transformations from $[0,1]^2$ to $[0,1]^2$:

$$T_1(\alpha,\rho) = (\frac{\alpha}{1+\rho}, \frac{\rho}{1+\alpha}), \quad T_2(\alpha,\rho) = (\frac{1}{2-\alpha}, \frac{\rho}{2-\alpha}),$$

$$T_3(\alpha,\rho) = (1-\alpha, 1-\rho).$$

  A Sturmian word $s_{\alpha,\rho}$ is substitution invariant if and only if there exists a sequence $S_1,\ldots,S_n$ with $S_i \in \{T_1, T_2, T_3\}$ such that $(\alpha,\rho) = S_1 \circ \cdots \circ S_n(\alpha,\rho)$
- [Baláži, Masáková, Pelantová]: cut and project schemes, quasicrystals
- [B., Ei, Ito, Rao]: dual substitutions

Sturmian morphisms

Let σ be a substitution over $\{1,2\}$ and let $M_\sigma = (m_{ij})$ be its incidence matrix, where m_{ij} counts the number of occurrences of the letter i in $\sigma(j)$.
We shall call determinant of a Sturmian morphism the determinant of its incidence matrix.

Sturmian morphisms

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We shall call determinant of a Sturmian morphism the determinant of its incidence matrix.
The monoid of Sturmian morphisms is called the Sturmian monoid and is denoted by St.
Let St_0 denote the special Sturmian monoid, which is the submonoid of St of endomorphisms whose determinant is 1.
The monoid St_0 is generated by the endomorphisms $G, D, \tilde{G}, \tilde{D}$ which are respectively:

$$G = (x, xy), \ D = (yx, y), \ \tilde{G} = (x, yx), \ \tilde{D} = (xy, y).$$

Invertible substitution

Definition

A substitution is said to be invertible if it is an automorphism of the free group F_2.

Theorem [Mignosi-Séébold, Wen-Wen]

A word is a Sturmian substitution invariant word if and only if it is a fixed point of some primitive and invertible substitution.

Theorem [Séébold]

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a primitive unimodular matrix with non-negative entries. The number of invertible substitutions with incidence matrix M is equal to $a + b + c + d - 1$.

Sturm number

Property

A number $\alpha \in (0,1)$ is a Sturm number if and only if there exists a 2×2 primitive unimodular matrix M with non-negative integral entries such that $(1-\alpha, \alpha)$ is an expanding eigenvector of M.

Consequently, if the Sturmian word $s_{\alpha,\rho}$ is substitution invariant, then this implies that α is a Sturm number.

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- Let σ be a primitive unimodular substitution over $\{1,2\}$. Let β be the maximal eigenvalue of its incidence matrix M_σ. Its algebraic conjugate β' is also an eigenvalue of M_σ.
- Furthermore, the vector of densities of the letters 1 and 2 denoted by $(1-\alpha, \alpha)$, with $0 \leq \alpha \leq 1$, is easily proven to be an expanding eigenvector, i.e., an eigenvector associated with the expanding eigenvalue β.
- Since α is (irrational) quadratic, the vector $(1-\alpha', \alpha')$ is an eigenvector associated with the eigenvalue β'.
- By Perron-Frobenius' theorem, coordinates $1-\alpha'$, α' cannot both be positive, hence $\alpha'(1-\alpha') \leq 0$, which implies that $\alpha' \notin]0,1[$. Hence α is a Sturm number.

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Example

Let σ be the substitution $1 \mapsto 121$, $2 \mapsto 12$, i.e., the square of the Fibonacci substitution. This substitution admits as a unique fixed point the Fibonacci word $s_{\alpha,\alpha}$, with $\alpha = \frac{3-\sqrt{5}}{2}$, whose first terms are

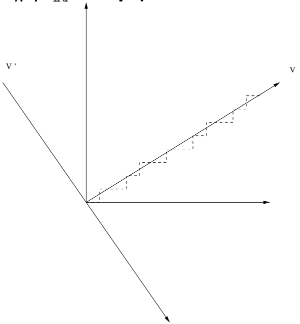
$$121121211211212112121$$

One has $M_\sigma = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\beta = \frac{3+\sqrt{5}}{2}$, and $\beta' = \frac{3-\sqrt{5}}{2} = \alpha = \frac{1}{\beta} > 0$.

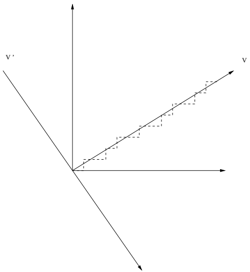
Rauzy fractals

Let σ be a primitive substitution with determinant 1 over $\{1, 2\}$.
Let $\vec{e}_1, \vec{e}_2$ be the canonical basis of $\mathbb{R}^2$. Let $f : \{1, 2\}^* \to \mathbb{Z}^2$ be the Parikh map, defined by $f(w) = |w|_1\vec{e}_1 + |w|_2\vec{e}_2$.
Let V be the expanding eigenspace of the matrix M_σ corresponding to the eigenvalue β, and V' the contracting eigenspace corresponding to β'.
Then $V \oplus V' = \mathbb{R}^2$ is a direct sum decomposition of $\mathbb{R}^2$. According to this direct sum, we define the projection

$$\pi : \mathbb{R}^2 \to V'.$$



Rauzy fractals



Let $s = (s_k)_{k \geq 0}$ be a fixed point of σ^2. We first define

$$Y = \{f(s_0 \ldots s_{k-1});\ k \geq 0\}.$$

We then divide Y into two parts:

$$Y_1 = \{f(s_0 \ldots s_{k-1});\ s_k = 1\},\ Y_2 = \{f(s_0 \ldots s_{k-1});\ s_k = 2\}.$$

Projecting Y_1, Y_2 onto the contracting eigenspace V' and taking the closures, we get

$$X_1 = \overline{\pi(Y_1)},\ X_2 = \overline{\pi(Y_2)}.$$

We call X_1 and X_2 the Rauzy fractals of the substitution σ.

Rauzy fractals of invertible substitutions

Connectedness

Let σ be a primitive invertible substitution. Then there exists $h \in \mathbb{Z}$ such that the Rauzy fractals satisfy

$$X_1 = [-1 + \alpha + h, h], \quad X_2 = [h, \alpha + h],$$

where $(1 - \alpha, \alpha)$ is the Perron-Frobenius eigenvector with positive entries of the incidence matrix M_σ.
Furthermore, if $\overline{s}_{\alpha,\rho}$ or $\underline{s}_{\alpha,\rho}$ is a fixed point point of σ^2, then

$$\rho = 1 - \alpha - h.$$

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Furthermore, if $\overline{s}_{\alpha,\rho}$ or $\underline{s}_{\alpha,\rho}$ is a fixed point point of σ^2, then

$$\rho = 1 - \alpha - h.$$

Remark

Let $0 < \alpha < 1$ be an irrational number and $0 \leq \rho \leq 1$. Then $\underline{s}_{\alpha,\rho}$ is substitution invariant if and only $\overline{s}_{\alpha,\rho}$ is also substitution invariant.

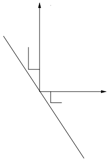
Stepped surface

Recall that V' is the contracting eigenline of M_σ. We denote the upper closed half-plane delimited by V' as $(V')^+$, and the lower open half-plane delimited by V' as $(V')^-$. We define

$$S = \{[z, i^*]; \quad z \in \mathbb{Z}^2, z \in (V')^+ \text{ and } z - \vec{e}_i \in (V')^-\},$$

where the notation $[z, i^*]$, for $z \in \mathbb{Z}^2$ and $i^* \in \{1^*, 2^*\}$, $\overline{[z, 1^*]}$ (resp. $\overline{[z, 2^*]}$) is defined as the closed line segment from z to $z + \vec{e}_2$ (resp. to $z + \vec{e}_1$).
Then the stepped surface $\overline{S}$ of V' is defined as the broken line consisting of the following segments

$$\overline{S} = \bigcup_{[z,i^*]\in S} \overline{[z, i^*]}.$$



Stepped surface

Projecting the stepped surface $\overline{S}$ onto V', we first obtain a tiling $\mathcal{J}'$ of V' by two intervals of length $1-\alpha$ and α.
We label the tiles of $\mathcal{J}$ by the two-sided sequence $(T_k)_{k\in\mathbb{Z}}$.
We define the two-sided word $(t_n)_{n\in\mathbb{Z}}$ as:

$$\forall n \in \mathbb{Z},\ t_n = \begin{cases} 1, & \text{if } |T_n| = 1-\alpha \\ 2, & \text{if } |T_n| = \alpha. \end{cases}$$

One checks that $(t_n)_{n\in\mathbb{Z}}$ is the upper two-sided cutting sequence of the line $V' : y = \frac{1-\alpha'}{\alpha'}x$. Hence

$$t_{-1}t_{-2}t_{-3}\cdots = 1s_{\gamma,\gamma}, \quad t_0t_1t_2\cdots = 2s_{\gamma,\gamma},$$

with

$$\gamma = \frac{\alpha'-1}{2\alpha'-1},$$

called dual angle.

Self-similar structure of Rauzy fractals

Let σ be a primitive substitution over $\{1, 2\}$ and let β be the Perron-Frobenius eigenvalue of M_σ. We recall that $|\beta'| < 1$.
The Rauzy fractals X_1 and X_2 have a self-similar structure:

both $\frac{1}{\beta'} X_1$ and $\frac{1}{\beta'} X_2$ are unions of translated copies of X_1 and X_2.
[Arnoux-Ito,Sirvent-Wang]

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[Arnoux-Ito, Sirvent-Wang]

Example

Let σ be the substitution $1 \mapsto 121$, $2 \mapsto 12$.
One checks that

$$\frac{X_1}{\beta'} = [-1, 1/\alpha - 2] = [-1, 1 - \alpha] = (X_1 + \alpha - 1) + X_1 + X_2,$$

$$\frac{X_2}{\beta'} = [1/\alpha - 2, 1/\alpha - 1] = [1 - \alpha, 2 - \alpha] = (X_1 + 1) + (X_2 + 1).$$

Set equation

Let $X_1 = [-1+\alpha+h, h], X_2 = [h, h+\alpha]$ be the Rauzy fractals of the primitive invertible substitution σ. Then

$$\frac{X_1}{\beta'} = \left(\bigcup_{T\in\mathcal{D}_1} T\right) + h, \quad \frac{X_2}{\beta'} = \left(\bigcup_{T\in\mathcal{D}_2} T\right) + h,$$

where $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_1 \cup \mathcal{D}_2$ are connected patches of the tiling $\mathcal{J}$.

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One has $\mathcal{D}_1 = \{T_{-2}, T_{-1}, T_0\}$, $\mathcal{D}_2 = \{T_1, T_2\}$, $\frac{X_1}{\beta'} = h + T_{-2} + T_{-1} + T_0$, $\frac{X_2}{\beta'} = h + T_1 + T_2$.

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where $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_1 \cup \mathcal{D}_2$ are connected patches of the tiling $\mathcal{J}$.

We assume that the determinant of M is equal to 1. One has $1/\beta' = \beta > 0$ so that $\frac{X_1}{\beta'}$ is on the left side of $\frac{X_2}{\beta'}$. There exists k with $1 \leq k \leq a+b+c+d-1$ such that

$$\begin{aligned} \mathcal{D}_1 &= \{T_{-k}, T_{-k+1}, \ldots, T_{-k+a+b-1}\}, \\ \mathcal{D}_2 &= \{T_{-k+a+b}, T_{-k+a+b+1}, \ldots, T_{-k+a+c+b+d-1}\}. \end{aligned} \tag{1}$$

There are $a+b+c+d-1$ invertible substitutions with incidence matrix M in a one-to-one correspondence with the set equations determined by (1).
We denote these substitutions by σ_k, $1 \leq k \leq a+b+c+d-1$.

Intersection point

Projecting the stepped surface $\overline{S}$ onto V', we first obtain a tiling $\mathcal{J}'$ of V' by two intervals of length $1-\alpha$ and α.
We label the tiles of $\mathcal{J}$ by the two-sided sequence $(T_k)_{k\in\mathbb{Z}}$.
We furthermore define the two-sided sequence $(g_k)_{k\in\mathbb{Z}}$ as the sequence of left endpoints of tiles T_k (one has $g_0=0$).

Let M be a 2×2 primitive matrix with non-negative entries such that $\det M = 1$. Let σ_k, $1\le k\le a+b+c+d-1$, be the invertible substitutions with incidence matrix M, and let $X_1=[-1+\alpha+h_k, h_k]$, $X_2=[h_k, \alpha+h_k]$ be the Rauzy fractals of σ_k^2. Let β be the Perron-Frobenius eigenvalue of M. Then

$$h_k=\frac{g_{-k+a+b}}{\beta-1}.$$

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$$h_k = \frac{g_{-k+a+b}}{\beta-1}.$$

- One has $\frac{X_1}{\beta'} \cap \frac{X_2}{\beta'} = \{(\beta')^{-1}h_k\} = \{\beta h_k\}$.
- This intersection point is the left endpoint of the interval $\cup\{T+h_k;\ T\in\mathcal{D}_2\}$, i.e., the left endpoint of $T_{-k+a+b}+h_k$. So we get $g_{-k+a+b}+h_k = \beta h_k$, and $h_k = \frac{g_{-k+a+b}}{\beta-1}$.

Sketch of the proof of Yasutomi's characterization

- Let σ be an invertible substitution with Perron-Frobenius postive eigenvector $(1-\alpha, \alpha)$. Then α is a Sturm number, and the Rauzy fractals X_1, X_2 are intervals with length $1-\alpha$ and α, respectively. Suppose $s = \overline{s}_{\alpha,\rho}$ or $s = \underline{s}_{\alpha,\rho}$ is a fixed point of σ^2. One checks that $\rho = 1 - \alpha - h$, where $\{h\} = X_1 \cap X_2$.
- Let V' be the contracting line $y = \frac{1-\alpha'}{\alpha'} x$, where α' is the algebraic conjugate of α. A broken line in $\mathbb{R}^2$, the so-called stepped surface, is associated with line V', defined as a discretization of V'.
- The sets X_1, X_2 have a self-similar structure: indeed they satisfy a set equation which is controlled by the stepped surface of V'. Hence, by connectedness and self-similarity of Rauzy fractals, we express the intersection $X_1 \cap X_2$ in terms of the stepped surface.
- Then we show that the stepped surface is associated with the so-called dual rotation R_γ with $\gamma = \frac{\alpha'-1}{2\alpha'-1}$. An arithmetic characterization of the stepped surface is obtained. This allows us to get an algebraic description of the intersection set $X_1 \cap X_2$ for an invertible substitution σ, which yields a proof of Yasutomi's Theorem.

Rigid words

Definition

An infinite word generated by a substitution is rigid if all the subtitutions which generate this word are powers of the same unique substitution.

Theorem [Séébold]

Let $s_{\alpha,\rho}$ be a substitution invariant Sturmian word. There exists a substitution such that all the substitutions that fix this word are powers of σ.

- Combinatorial proof based on Rauzy rules
- Arithmetic proof based on Pell's equations and Dirichlet's theorem on the group of units of $\mathbb{Q}(\sqrt{D})$ which is isomorphic to $\mathbb{Z}$.

Christoffel words

[On an involoution of Christoffel words and Sturmian morphisms, B., de Luca, Reutenauer]

Definition

Let p and q be positive relatively prime integers and $n = p + q$.
Given an ordered two-letter alphabet $\{x < y\}$, the Christoffel word w of slope $\frac{p}{q}$ on this alphabet is defined as $w = x_1 \cdots x_n$, with

$$x_i = \begin{cases} x & \text{if} \quad (i-1)p \in \{0, 1, \ldots, q-1\} \mod n \\ y & \text{if} \quad (i-1)p \in \{q, q+1, \ldots, n-1\} \mod n \end{cases}$$

for $i = 1, \ldots, n$.

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for $i = 1, \ldots, n$.

Remark

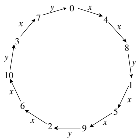
One has

$$(i-1)p \in \{0, 1, \ldots, q-1\} \mod n \text{ iff } ip \mod n > (i-1)p \mod n.$$

Christoffel words

In other words, label the edges of the Cayley graph of $\mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\}$ with generator p as follows:

- the label of the edge $h \to k$ (where $h + p \equiv k \mod n$) is x if $h < k$ and y if $h > k$;
- then read the word w, of length n, starting with the label of the edge $0 \to p$.



Remark

Note that

$$|w|_x = q \text{ and } |w|_y = p.$$

Thus, if we know the number of occurrences of each letter in w, we know its slope, hence w.

Dual Christoffel word

Definition

Given the proper Christoffel word of slope $\frac{p}{q}$, we define the dual Christoffel word w^* of slope $\frac{p^*}{q^*}$, where p^* and q^* are the respective multiplicative inverses in $\{0, 1, \ldots, n-1\}$ of p and q.

- These inverses exist since p and q are relatively prime.
- A Christoffel word and its dual have the same length.

Example

Let

$$w = xxyxxyxxyxy$$

be the Christoffel word of slope $\frac{4}{7}$, on the alphabet $\{x < y\}$, with $p = 4,\ q = 7,\ n = 11$ and its dual Christoffel word of slope $\frac{3}{8}$ on the alphabet $\{x < y\},\ p^* = 3,\ q^* = 8,\ n = 11$:

$$0 \xrightarrow{x} 3 \xrightarrow{x} 6 \xrightarrow{x} 9 \xrightarrow{y} 1 \xrightarrow{x} 4 \xrightarrow{x} 7 \xrightarrow{x} 10 \xrightarrow{y} 2 \xrightarrow{x} 5 \xrightarrow{x} 8 \xrightarrow{y} 0$$

Lyndon factorization

Lyndon Factorization

Each Christoffel word w is a Lyndon word. Thus, a proper Christoffel word has a standard factorization

$$w = w_1 w_2,$$

where w_1, w_2 are also Christoffel words and $w_1 < w_2$ in lexicographic order.

Factorization and Cayley graph

Let w be a proper Christoffel word of slope $\frac{p}{q}$ and

$$w = w_1 w_2$$

its factorization in an increasing product of two Christoffel words. Then

$$|w_1| = p^* \text{ and } |w_2| = q^*.$$

Moreover, w_1 (resp. w_2) is the label of the path from 0 to 1 (resp. 1 to 0) in the previous Cayley graph.

Central words

Palindrome

If w is a proper Christoffel word on the alphabet $\{x < y\}$, then $w = xuy$, where u is a palindrome.

This is easily seen on the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ by reversing edges.

Central words

The words u such that xuy are Christoffel words (necessarily proper) on the alphabet $\{x < y\}$ are called central words.

Standard word

It is a word on the alphabet $\{x, y\}$ which is either a letter or of the form

$$uxy \text{ or } uyx,$$

where u is a central word.

Central words

The central words on the alphabet $\{x, y\}$ have been completely characterized:

they are the words which for some relatively prime positive integers p and q are of length $p + q - 2$ and have periods p and q [de Luca-Mignosi].

The set of central words is equal to the set of palindromic prefixes of standard Sturmian sequences [Berstel-Séébold].

Periodicity and central words [de Luca-Mignosi]

Let $w = xuy$ be the Christoffel word of slope $\frac{p}{q}$ on the alphabet $\{x < y\}$. Then the central word u has the periods p^* and q^* where $pp^*, qq^* \equiv 1 \mod p + q$.

Dual word and Fine and Wilf Theorem

Periodicity and central words [de Luca-Mignosi]

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The dual of a Christoffel word may be read on the graph which defines this latter word. For example, take the graph below, and remove the vertices 0 and $n-1=10$, the labels and the orientation (cf. the proof of the theorem of Fine and Wilf by [Choffrut-Karhumäki]).

3 —— 6 —— 9 —— 1 —— 4 —— 7   2 —— 5 —— 8

This graph expresses the equality of letters according to their positions in a word of length 9 with periods 3 and 8. The central word of the dual of this Christoffel word

$$xyxxyxxyx$$

(since the x's are in positions $3, 6, 9, 1, 4, 7$ and the y's in positions $2, 5, 8$). We thus recover the Christoffel word

$$xxyxxyxxyxy.$$

Palindromic closure

The right palindromic closure w^+ of a word w is defined as the unique shortest palindrome having w as a prefix.

This word exists and is equal to $uv\tilde{u}$, where $w = uv$, $\tilde{u}$ is the mirror image of u, and v is the longest palindromic suffix of w.

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Example

For example, $(xyxxyxx)^+ = xyxxyxx.yx$.

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The right iterated palindromic closure of w is denoted by $\mathrm{Pal}(w)$ and is defined recursively by $\mathrm{Pal}(w) = (\mathrm{Pal}(u)z)^+$, where $w = uz, z$ the last letter of w, together with the initial condition $\mathrm{Pal}(\varepsilon) = \varepsilon$.

Theorem [Carpi-de Luca]

- The set $\{\mathrm{Pal}(v), v \in \{x, y\}^*\}$ coincides with the set of central words.
- If $w = \mathrm{Pal}(v)$, then v is uniquely defined by w; v is called the directive word of w.
- $\mathrm{Pal}(\tilde{v})$ is the dual central word of the central word $\mathrm{Pal}(v)$.

Arithmetic expression

Let matrix M be the image of $v \in \{x, y\}^*$ under the multiplicative morphism $\mu : x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Let $u = \mathrm{Pal}(v)$ and let w be the Christoffel word xuy. Let $w = w_1 w_2$ be its decomposition into two Christoffel words with $w_1 < w_2$ in lexicographic order. Then

$$M = M_v = \begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{pmatrix}.$$

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If $u = \mathrm{Pal}(v)$ and $M_v = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, then $|u|_x = a + b - 1$, $|u|_y = c + d - 1$ and u has the relatively prime periods $a + c$ and $b + d$. Moreover the Christoffel word $w = xuy$ has slope $\frac{c+d}{a+b}$ and its dual has slope $\frac{c+a}{d+b}$.

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Let p/q and p'/q' be two positive rational numbers, in irreducible form. Then the paths in the Stern-Brocot tree defining the corresponding nodes are mirror each of another if and only if $p + q = p' + q'$ and $pp', qq' \equiv 1 \mod p + q$.

Back to Sturmian morphisms

A Sturmian morphism is an endomorphism of the free monoid $\{x, y\}^*$ that sends each Sturmian sequence onto a Sturmian sequence.

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Definition

Given two Sturmian morphisms f and f', f' is a right conjugate of f if for some word $w \in \{x, y\}^*$, one has

$$uw = wu', \; vw = wv'$$

where $f = (u, v)$ and $f' = (u', v')$.

Theorem [Nielsen]

Two automorphisms f and f' of the free group with two elements have the same commutative image iff

$$f = \varphi f'$$

for some inner automorphism φ of F_2.
Two Sturmian morphisms f and f' are conjugate if and only if they have the same commutative image.

Back to Sturmian morphisms

A Sturmian morphism is an endomorphism of the free monoid $\{x, y\}^*$ that sends each Sturmian sequence onto a Sturmian sequence.

Characterization

A morphism f is Sturmian if and only if it sends each Christoffel word onto the conjugate of a Christoffel word.

Remark

An endomorphism of the free monoid on x and y is Sturmian if and only if it sends the three Christoffel words

$$xy,\; xxy,\; xyy$$

onto conjugates of Christoffel words.

Conjugacy classes

- Take a Sturmian morphism with determinant 1.
- Since xy is a Christoffel word, $f(xy)$ is conjugate to a proper Christoffel word w. We say that w is the Christoffel word associated with f.
- If f and f' are conjugate Sturmian morphisms, then $f(xy)$ and $f'(xy)$ are conjugate words, so that f and f' have the same associated Christoffel word.
- Conversely, suppose that f and f' with determinant 1 have the same Christoffel word. Then, $f(xy)$ and $f'(xy)$ are conjugate. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ be the matrices associated to f, f'. Then $a + b = |f(xy)|_x = |f'(xy)|_x = a' + b'$, and similarly, $c + d = c' + d'$. Thus, these matrices are equal and, by Nielsen's theorem, f and f' are conjugate.

Property

The map $f \mapsto f(xy)$ induces a bijection between conjugacy classes of Sturmian morphisms of determinant 1, and conjugacy classes of Christoffel words.

Dual Sturmian morphisms

Denote by (x^{-1}, y) the automorphism of the free group F_2 sending x onto x^{-1} and y onto y.
Recall that $G, D, \tilde{G}, \tilde{D}$ are defined as

$$G = (x, xy),\ D = (yx, y),\ \tilde{G} = (x, yx),\ \tilde{D} = (xy, y).$$

Theorem [B., de Luca, Reutenauer]

The mapping

$$f \mapsto f^* = (x^{-1}, y) f^{-1} (x^{-1}, y)$$

is an involutive anti-automorphism of the Sturmian monoid of morphisms with determinant 1, that exchanges D and $\tilde{D}$ and fixes G and $\tilde{G}$.
It sends conjugacy classes of morphisms onto conjugacy classes.

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It sends conjugacy classes of morphisms onto conjugacy classes.

Theorem

Let $w = xuy$ be a Christoffel word of length n and slope $\frac{p}{q}$, and w^* the dual word.
Let $\{f_1, \ldots, f_{n-1}\}$ (resp. $\{f'_1, \ldots, f'_{n-1}\}$) be the conjugacy class of Sturmian morphisms associated with w (resp. w^*), in the previous description. Then

$$f_i^* = f'_{ip},$$

where the subscript is taken modulo n.

Self-dual words

- The Christoffel word $x\mathrm{Pal}(v)y$ is self-dual if and only if the directive word v of $\mathrm{Pal}(v)$ is a palindrome.
- This implies that $\mathrm{Pal}(v)$ is harmonic [Carpi, de Luca].
- A Christoffel word of slope p/q is self-dual if and only if $p^2 \equiv 1 \mod (p+q)$.

Dual christoffel words

This duality equivalently consists in

- using the slope of the word, and changing the numerator and the denominator respectively in their inverses modulo the length;
- using the cyclic graph allowing the construction of the word, by interpreting it in two ways
  - one as a permutation and its ascents and descents, coded by the two letters of the word,
  - the other in the setting of the Fine and Wilf periodicity theorem;
- by using central words and generation through iterated palindromic closure, by reversing the directive word.

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- by using central words and generation through iterated palindromic closure, by reversing the directive word.

- This involution extends to Sturmian morphisms: it preserves conjugacy classes of these morphisms, which are in bijection with Christoffel words.
- This involution on morphisms is the restriction of some conjugation of the automorphisms of the free group.

Higher-dimensional case

- Words
  - Fine and Wilf theorem for three letter words and generalized Euclid's algorithms

    [Castelli-Mignosi-Restivo, Zamboni-Tijdeman]
- Multidimensional words
  - Multididimensional Sturmian words [B.-Vuillon]
  - Generalized substitutions [Arnoux-Ito]
  - Discrete geometry [B.-Fernique]