Defects of fixed points of some substitutions

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Definition of defect

- *Defect* of a finite word w equals

$$|w| + 1 - \text{the number of different palindromes in } w.$$

- ε is considered as a palindrome.
- $n + 1$ is the maximal possible number of palindromes in any word of length n.

Basic facts about substitution

- Substitution is a non-erasing morphism $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ such that there exists $d \in \mathcal{A}$ satisfying $\varphi(d) = dw$ for some non-empty word $w \in \mathcal{A}^*$

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- An infinite word u is *uniformly recurrent*, if for every $n \in \mathbb{N}$, there exists $R(n) > 0$ such that any factor of u of length $\geq R(n)$ contains all the factors of u of length n.

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β-expansion and β-integers

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- Let $\beta > 1$ and $x \geq 0$, any convergent series of the form $x = \sum_{i=-\infty}^{k} x_i \beta^i$, where $x_i \in \mathbb{N}$, is called a *β-representation* of x and denoted $x_k x_{k-1} \ldots x_0 \bullet x_{-1} \ldots$.

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  - models of quasicrystals
  - in non-standard numeration- analogy of $\mathbb{N}$ in $\mathbb{R}$

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  - Let $d_\beta(1)$ be an infinite Rényi expansion of unity. $x_kx_{k-1}\ldots x_0 \bullet x_{-1}\ldots$ is a β-expansion if and only if $x_ix_{i-1}\cdots \prec d_\beta(1)$ for all $i \leq k$.

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  - Let $d_\beta(1)$ be a finite Rényi expansion of unity, i.e., $d_\beta(1) = t_1t_2...t_m$. We define $d_\beta(1)^* := (t_1t_2...t_{m-1}(t_m - 1))^\omega$. Then $x_kx_{k-1}\ldots x_0 \bullet x_{-1}\ldots$ is a β-expansion if and only if $x_ix_{i-1}\cdots \prec d_\beta(1)^*$ for all $i \leq k$.

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- Thurston: Distances in $\mathbb{Z}_\beta^+$ form the set $\{\Delta_k \mid k \in \mathbb{N}\}$, where

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- $\{\Delta_k \mid k \in \mathbb{N}\}$ is finite if and only if $d_\beta(1)$ is eventually periodic.
- If $d_\beta(1)$ is eventually periodic, β is called a *Parry number*. If $d_\beta(1)$ is finite, β is called a *simple Parry number*.

Infinite words associated with β-integers

- Let β be a simple Parry number, $d_\beta(1) = t_1 \dots t_m$. If we associate to each Δ_k the letter k, where $k \in \{0, \dots, m-1\}$, we obtain $u_\beta = \lim_{n\to\infty} \varphi^n(0)$, where the substitution φ is given by

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\begin{aligned}
\varphi(0) &= 0^{t_1}1 \\
\varphi(1) &= 0^{t_2}2 \\
&\dots \\
\varphi(m+p-2) &= 0^{t_{m-1}}(m-1) \\
\varphi(m+p-1) &= 0^{t_m}.
\end{aligned}
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Infinite words associated with β-integers

- Let β be a non-simple Parry number, in particular, let $m \in \mathbb{N}$ and $p \in \mathbb{N}$ be minimal such that $d_\beta(1) = t_1 ... t_m (t_{m+1} ... t_{m+p})^\omega$. If we associate to each Δ_k the letter k, where $k \in \{0, \dots, m+p-1\}$, we obtain $u_\beta = \lim_{n\to\infty} \varphi^n(0)$, where the substitution φ is given by

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- Studied problems for both simple and non-simple Parry numbers:
  - factor and palindromic complexity,
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  - recurrence function and balances (only for quadratic case),
  - the upper bound on the fractional part length resulting from addition of two β-integers.

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Languages closed under reversal

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Language of u_β closed under reversal

- The language $\mathcal{L}(u_\beta)$ for β simple Parry is closed under reversal if and only if $t_1 = \cdots = t_{m-1}$,i.e., $d_\beta(1) = t...ts$, where $t \geq s \geq 1$. The infinite word u_β is the fixed point of the substitution φ given by

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- The language $\mathcal{L}(u_\beta)$ for β non-simple Parry is closed under reversal if and only if $m = p = 1$, i.e., $d_\beta(1) = t_1(t_2)^\omega = ab^\omega$, $a > b \geq 1$. The infinite word u_β is the fixed point of the substitution φ given by

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- The relation between complexity and palindromic complexity:

$$P(n) + P(n+1) = C(n+1) - C(n) + 2 \quad \text{for all } n \in \mathbb{N}.$$

- In general, for uniformly recurrent words with language closed under reversal, we have $P(n) + P(n+1) \leq C(n+1) - C(n) + 2$.

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- Every finite word w contains at most $|w| + 1$ different palindromes (the empty word being considered as a palindrome, too).
- We call a finite word w containing the maximal possible number $|w| + 1$ of palindromes *full*. An infinite word is *full* if all of its prefixes are full.

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- Droubay, Justin, Pirillo have proved that Sturmian and episturmian words are full.
- We will show for fixed points of some well-known substitutions whether they are or are not full.

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- Sturmian case for $b = a - 1$.
- Lemma: A factor p of u_β is a palindrome if and only if $T(p) = 1\varphi(p)$ is a palindrome. Moreover, for every palindrome p containing at least one letter 1, we find a palindrome q shorter than p such that p occurs only as a central factor of the palindrome $1\varphi(q)$.

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- Sturmian case for $b = a - 1$.
- Lemma: A factor p of u_β is a palindrome if and only if $T(p) = 1\varphi(p)$ is a palindrome. Moreover, for every palindrome p containing at least one letter 1, we find a palindrome q shorter than p such that p occurs only as a central factor of the palindrome $1\varphi(q)$.
- Theorem: u_β is full.

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- contradiction with the minimality of v

Defects of u_β for β being a simple Parry number

- u_β is the fixed point of φ defined by
  $\varphi(0) = 0^t 1$
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- $u_{TM} = 011010011|0010110$

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Outline of the talk

1. Definition of defect
2. Basic facts about substitution
3. Words u_β associated with β-integers
4. Palindromes and defects of u_β
5. Defects of fixed points of some other substitutions
6. Importance of relation between palindromes and their images by substitution
7. Conjecture: Fixed points of substitutions contain infinitely many palindromes iff ...
8. Open problem: Is this word full or not?

Conjecture

- Definition: Substitution φ on $\mathcal{A}$ is of class P if there exists a palindrome p, and for every $a \in \mathcal{A}$, a palindrome q_a such that $\varphi(a) = pq_a$ for all $a \in \mathcal{A}$ (or, $\varphi(a) = q_a p$ for all $a \in \mathcal{A}$.)

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- In particular, suppose that Conjecture holds, how long does the palindrome have to be in order to occur only as a central factor of image of a shorter palindrome?

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