Matrices of 3iet preserving morphisms

P. Ambrož

joint work with Z. Masáková and E. Pelantová

Doppler Institute and Department of Mathematics,
FNSPE, Czech Technical University

Atelier sur les progrès récents en combinatoire des mots

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Outline

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Finite and infinite words

Notation and properties

$\mathcal{A}$ finite alphabet, $\mathcal{A}^*$ monoid of finite words, $\mathcal{A}^{\mathbb{Z}}$ set of biinfinite words

Let $u = (u_n), v = (v_n) \in \mathcal{A}^{\mathbb{Z}}$. The distance between u and v is

$$d(u, v) := \frac{1}{1 + \min\{j \in \mathbb{N} \mid u_j \neq v_j \text{ or } u_{-j} \neq v_{-j}\}}.$$

$d(u, v)$ is metric, $(\mathcal{A}^{\mathbb{Z}}, d)$ is compact metric space.

The density of a letter $a \in \mathcal{A}$ in $u \in \mathcal{A}^{\mathbb{Z}}$ is

$$\rho(a) := \lim_{n \to \infty} \frac{\#\{i \mid -n \leq i \leq n, u_i = a\}}{2n + 1},$$

if the limit exists.

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Finite and infinite words

Notation and properties

A biinfinite word u is called sturmian if $C_u(n) = n + 1$ for all $n \in \mathbb{N}$ and the densities of letters are irrational.

Example. The word $\cdots 111|000 \cdots$ has the complexity $\mathcal{C}(n) = n + 1$ for all $n \in \mathbb{N}$.

Lemma

Let $u \in \{0,1\}^{\mathbb{Z}}$ be a limit of a sequence of sturmian words $(u^{(m)})$. Then u is either sturmian or the densities of letters are rational.

Proof.
Let $w, \widehat{w}$ be factors of the same length in u. Since $u = \lim_{m\to\infty} u^{(m)}$ $\exists m_0 \in \mathbb{N}$ such that $w, \widehat{w}$ are factors of $u^{(m_0)}$, which is sturmian.
Therefore $\big||w|_0 - |\widehat{w}|_0\big| \leq 1$ and u is balanced.
If, moreover, the densities are irrational, then u is sturmian.

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Outline

Interval exchange transformation
Definition

r interval exchange transformation T **[Katok, Stepin, Keane]**

Let $I_1, \dots, I_r$ be intervals, $I := I_1 \cup \dots \cup I_r$

$\pi : \{1, \dots, r\} \to \{1, \dots, r\}$ a permutation of $\{1, \dots, r\}$

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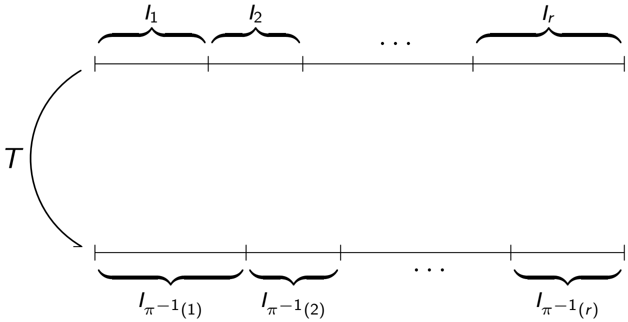

Interval exchange transformation

Orbits

Let T be a r interval exchange transformation, I its domain, $x_0 \in I$.
When is the orbit $\{T^n(x_0) \mid n \in \mathbb{Z}\}$ dense in I?

For general r, partial answers by **Keane**:

1. necessary condition

$$\pi\{1,2,\ldots,k\} \neq \{1,2,\ldots,k\} \quad \text{for all } k = 1,2,\ldots,r-1$$

2a. sufficient condition

$$|I_1|,|I_2|,\ldots,|I_r| \quad \text{are linearly independent over } \mathbb{Q}$$

2b. sufficient condition (weaker), the so-called i.d.o.c. condition

orbits of all discontinuity points of T are disjoint

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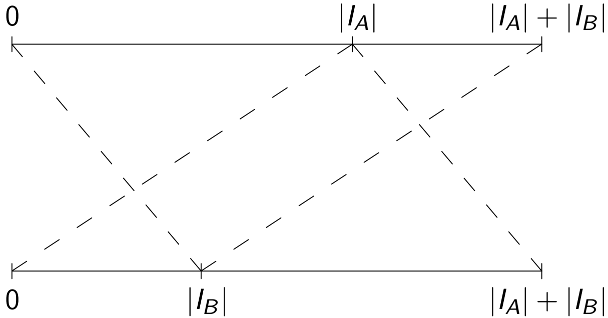
Interval exchange transformation

Example — 2 interval exchange transformation

Only one permutation fulfilling the necessary condition $\pi\{1,2\} = \{2,1\}$



The sufficient condition: $\frac{|I_1|}{|I_2|}$ is irrational
In this case it is also necessary

Interval exchange transformation

Example — 3 interval exchange transformation

Three possible permutations

$\pi\{1,2,3\} = \{2,3,1\}$ $\pi\{1,2,3\} = \{3,1,2\}$ $\pi\{1,2,3\} = \{3,2,1\}$

Interval exchange transformation

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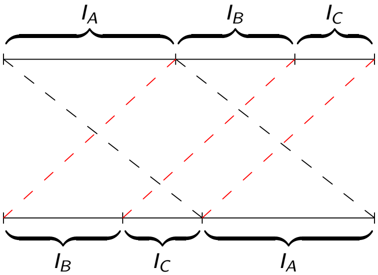

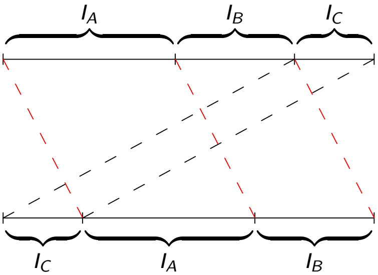

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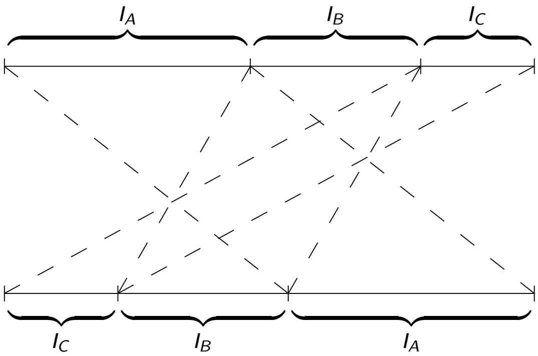

The necessary and sufficient condition: $\frac{|I_1|+|I_2|}{|I_2|+|I_3|}$ is irrational

Note that this does not mean i.d.o.c.

Infinite words associated with $r$iet

Definition

Let T be a r interval exchange transformation, $I = \cup I_X$ its domain and $x_0 \in I$. Biinfinite word $u_T(x_0) = (u_n)_{n\in\mathbb{Z}} = \cdots u_{-2}u_{-1}|u_0u_1\cdots$ associated with T is defined by

$$u_n := X \quad \text{if } T^n(x_0) \in I_X. \tag{1}$$

Specially for $r = 3$: $I = I_A \cup I_B \cup I_C$ and $(u_n) \in \{A, B, C\}^{\mathbb{Z}}$ is given by

$$u_n = \begin{cases} A & \text{if } T^n(x_0) \in I_A \\ B & \text{if } T^n(x_0) \in I_B \\ C & \text{if } T^n(x_0) \in I_C \end{cases}$$

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Infinite words associated with *riet*

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Definition

An aperiodic word $u_T(x_0) = (u_n)_{n\in\mathbb{Z}}$ given by (1) is called *riet* word.

Infinite words associated with $r$iet

Properties

- In the case $r = 2$, 2iet words are Sturmian words
- **Keane**: complexity is $\mathcal{C}(n) \leq (r-1)n + 1$
- A $r$iet word with $\mathcal{C}(n) = (r-1)n + 1$ for all n is called non-degenerated, otherwise it is degenerated

Proposition

Let T be a 3iet transformation with $|I_A| = \alpha$, $|I_B| = \beta$ and $|I_C| = \gamma$.

- *The infinite word $u_T(x_0)$ is aperiodic if and only if*

$$\alpha + \beta \text{ and } \beta + \gamma \text{ are linearly independent over } \mathbb{Q}.$$

- *If the word $u_T(x_0)$ is aperiodic then it is degenerated if and only if*

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Morphisms and incidence matrices

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A mapping $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ is said to be a morphism over $\mathcal{A}$ if

$$\varphi(w\widehat{w}) = \varphi(w)\varphi(\widehat{w}) \quad \text{for } \forall w, \widehat{w} \in \mathcal{A}^*$$

The action of a morphism φ can be naturally extended to biinfinite words

$$\varphi(u) = \varphi(\cdots u_{-2}u_{-1} | u_0 u_1 \cdots) := \cdots \varphi(u_{-2})\varphi(u_{-1}) | \varphi(u_0)\varphi(u_1) \cdots .$$

The mapping $\varphi : u \mapsto \varphi(u)$ is continuous on $\mathcal{A}^{\mathbb{Z}}$

Incidence matrix of a morphism over a k-letter alphabet $\{a_1, \ldots, a_k\}$ is $\mathbf{M}_\varphi \in \mathbb{N}^{k \times k}$

$$(\mathbf{M}_\varphi)_{ij} = \text{number of letters } a_j \text{ in the word } \varphi(a_i)$$

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Let φ and ψ be morphisms over $\mathcal{A}$, the matrix of their composition, that is, of the morphism $u \mapsto (\varphi \circ \psi)(u) = \varphi\big(\psi(u)\big)$ is obtained by

$$\mathbf{M}_{\varphi\circ\psi} = \mathbf{M}_\psi \mathbf{M}_\varphi \,.$$

Let $u \in \mathcal{A}^{\mathbb{Z}}$ has well defined densities of letters

$$\vec{\rho}_u = \big(\rho(a_1), \ldots, \rho(a_k)\big) \,.$$

Then for the infinite word $\varphi(u)$

$$\vec{\rho}_{\varphi(u)} = \frac{\vec{\rho}_u \mathbf{M}_\varphi}{\vec{\rho}_u \mathbf{M}_\varphi \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}} \,,$$

If u is a fixed point of φ then $\vec{\rho}_u \mathbf{M}_\varphi = \Lambda \vec{\rho}_u$.

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Morphisms preserving sturmian words

Definitions

Two ways to define

- A morphism φ over the binary alphabet $\{0, 1\}$ is said to be locally sturmian if there is a sturmian word u such that $\varphi(u)$ is also sturmian.
- A morphism φ over the binary alphabet $\{0, 1\}$ is said to be sturmian if $\varphi(u)$ is sturmian for all sturmian words u.

Monoid of Sturm, denoted St, finitely generated monoid of morphisms with generators

$$\psi_1 : \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 1 \end{array}, \qquad \psi_2 : \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 1 \end{array}, \qquad \psi_3 : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}.$$

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Morphisms preserving sturmian words

Properties

Theorem (Berstel, Mignosi and Séébold)

Let φ be a morphism, the following three conditions are equivalent

(i) $\varphi \in St$

(ii) φ *is sturmian*

(iii) φ *is locally sturmian*

Corollary

- A matrix $\mathbf{M} \in \mathbb{N}^{2\times 2}$ is the incidence matrix of a sturmian morphism if and only if $\det \mathbf{M} = \pm 1$
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Results in the 3iet case

A morphism φ over $\{A, B, C\}$ is 3iet preserving if $\varphi(u)$ is a 3iet word for every 3iet word u.

Theorem A

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Let φ be a 3iet preserving morphism and let $\mathbf{M}$ be its incidence matrix. Then one of the following holds

- $\det \mathbf{M} = \pm 1$ *and $\varphi(u)$ is non-degenerated for every non-degenerated 3iet word u,*
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Outline

Theorem A — Ideas behind the proof

Lemma

Let $u = (u_n)_{n\in\mathbb{Z}}$ be a 3iet word, $\sigma : \{A, B, C\}^* \to \{0, 1\}^*$ be a morphism given by

$$A \mapsto 0\,, \qquad B \mapsto 01\,, \qquad C \mapsto 1\,.$$

Then $\sigma(u)$ is sturmian.

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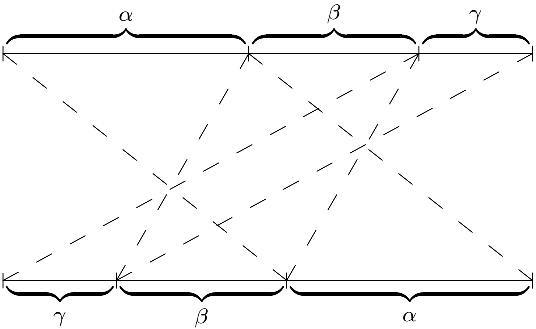

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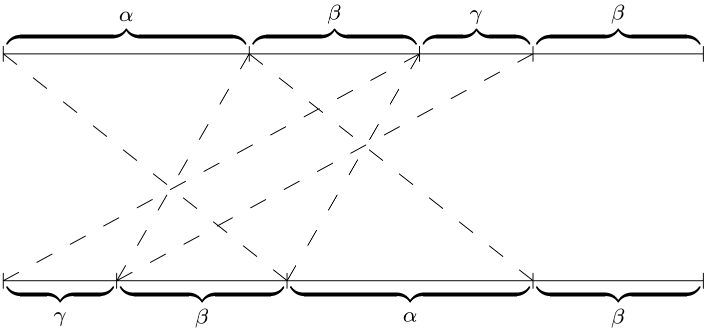

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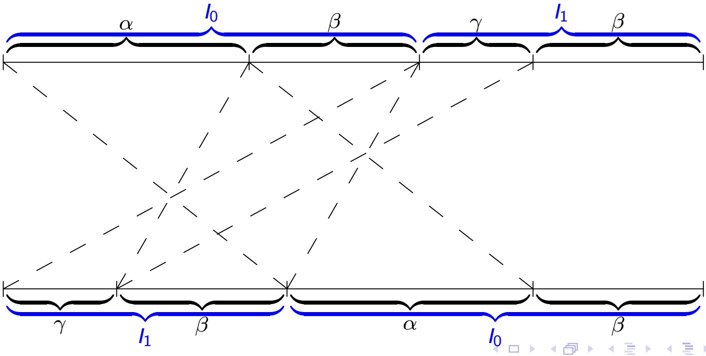

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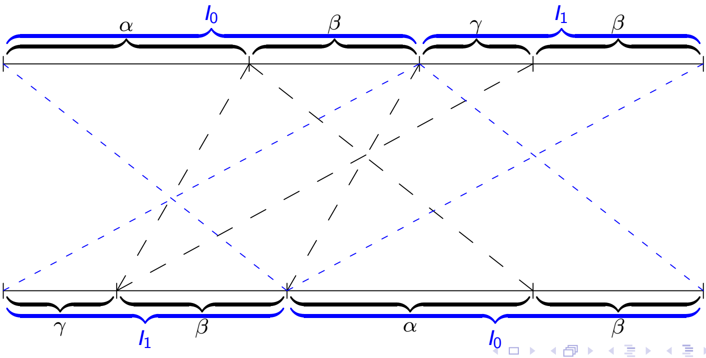

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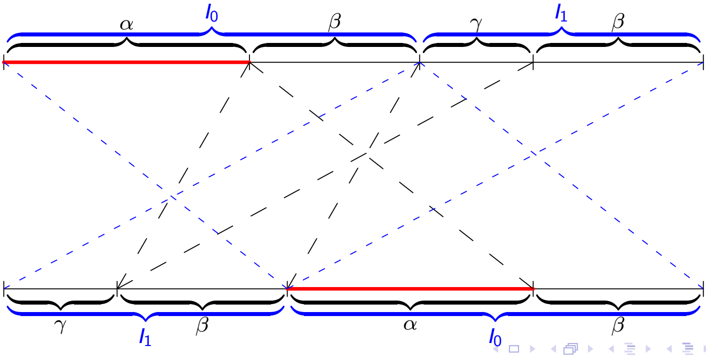

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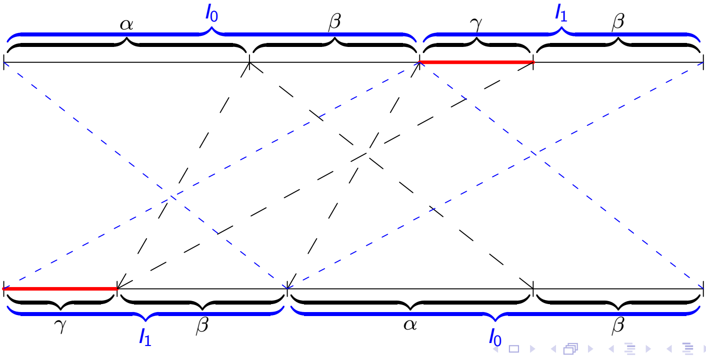

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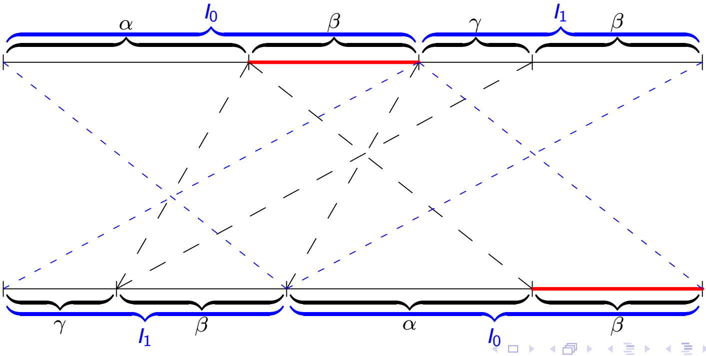

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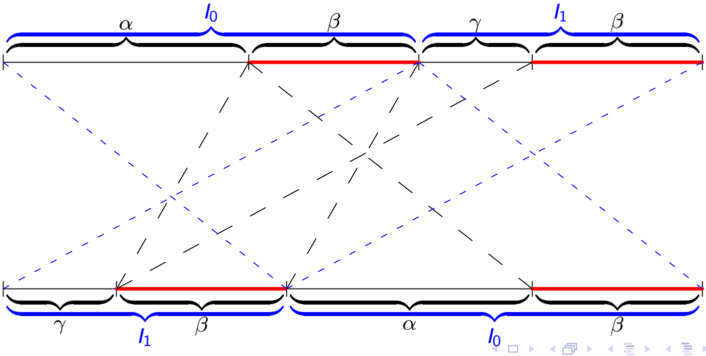

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- For any sturmian word u, there exist a sequence $(u^{(m)})$ of 3iet words such that
$$u = \lim_{m\to\infty} u^{(m)} .$$
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- Any morphism on $\{A, B, C\}^{\mathbb{Z}}$ is continuous, thus
$$(\sigma \circ \varphi)(u^{(m)}) \to (\sigma \circ \varphi)(u)$$
- By previous Lemma, $(\sigma \circ \varphi)(u^{(m)})$ are sturmian and so their limit $(\sigma \circ \varphi)(u)$ is sturmian (or has rational densities of letters).
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Theorem A — corollaries

Corollary

Let φ be a 3iet preserving, $\mathbf{M}$ its matrix. Then the vector $(1,-1,1)$ is a left eigenvector of $\mathbf{M}$, associated with the eigenvalue $\det \mathbf{M}$ or $-\det \mathbf{M}$.

The other eigenvalues λ_1 and λ_2 of the matrix $\mathbf{M}$ are either quadratic mutually conjugate algebraic units, or $\lambda_1, \lambda_2 \in \{1,-1\}$.

Corollary

The sum of the first and the third row of $\mathbf{M}$ differs from the sum of its second row by $\pm\det \mathbf{M}$. Formally,

$$(1,0,1)\mathbf{M}\begin{pmatrix}1\\1\\1\end{pmatrix} - (0,1,0)\mathbf{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = \pm\det \mathbf{M}\,.$$

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Theorem B — Ideas behind the proof

Geometric representation of a fixed point of φ

Let φ be a morphism over $\{a_1, \ldots, a_k\}$, u its fixed point.

Let $\mathbf{M}_\varphi$ have a positive right eigenvector $\vec{x} = (x_1, \ldots, x_k)$.

Let Λ be the eigenvalue corresponding to $\vec{x}$, $\mathbf{M}_\varphi \vec{x} = \Lambda \vec{x}$.

- Λ is the spectral radius of $\mathbf{M}_\varphi$
- Since $\mathbf{M}_\varphi$ is integral matrix one has $\Lambda \geq 1$.

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$$\Sigma := \left\{ \sum_{i=1}^{k} |w|_{a_i} x_i \;\middle|\; w \text{ is an arbitrary prefix of } u_0u_1u_2\cdots \right\}$$

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The set Σ can be equivalently defined as

$$\Sigma = \{t_n \mid n \in \mathbb{Z}\}, \quad \text{where} \quad t_0 = 0 \text{ and } t_{n+1} - t_n = x_i \Leftrightarrow u_n = a_i\,.$$

Properties

- Since u is fixed point of φ, we have $\Lambda\Sigma \subset \Sigma$ (self-similar set)
- if $u_n = a_i$ then $\#\big((\Lambda t_n, \Lambda t_{n+1}] \cap \Sigma\big) = |\varphi(a_i)|$

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Geometric representation of a fixed point of φ — illustration

Morphism $\varphi : 0 \mapsto 10,\ 1 \mapsto 110$, fixed point $u = \lim_{n\to\infty} \varphi^n(0)|\varphi^n(1)$

$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\Lambda = \tau^2$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

The corresponding right eigenvector of M is $\begin{pmatrix} 1 \\ \tau \end{pmatrix}$.

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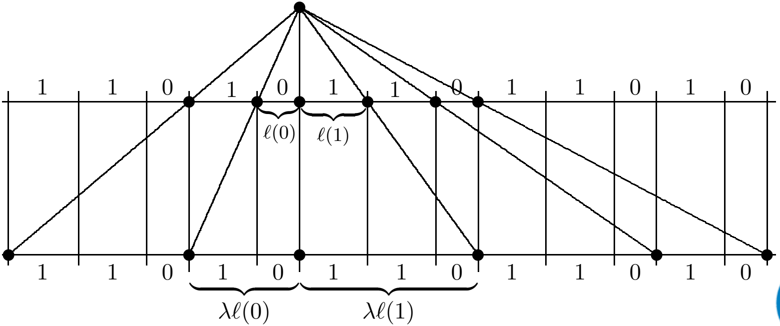

Theorem B — Ideas behind the proof

Two discrete sets associated with infinite words

- Fixed point of a morphism $\mapsto$ self-similar set Σ
- 3iet word $\mapsto$ C&P set Σ_C

For a 3iet word being a fixed point of φ: Σ and Σ_C coincide.
Then

$$\#\big((\Lambda t_n, \Lambda t_{n+1}] \cap \Sigma_C\big) = \#\big((\Lambda t_n, \Lambda t_{n+1}] \cap \Sigma\big) = \begin{cases} |\varphi(A)| \\ |\varphi(B)| \\ |\varphi(C)| \end{cases}$$

depending on $t_{n+1} - t_n$.
Fixed point of a morphism is also fixed point of its arbitrary power, hence $\exists k, m \in \mathbb{N}$ such that

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$$|\varphi^n(AC)| = \#\big((\Lambda t_k, \Lambda t_{k+2}] \cap \Sigma_C\big)$$
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difference : $\pm(\det \mathbf{M})^n \leq R$

Theorem B — Ideas behind the proof

Two discrete sets associated with infinite words

- Fixed point of a morphism $\mapsto$ self-similar set Σ
- 3iet word $\mapsto$ C&P set Σ_C

For a 3iet word being a fixed point of φ: Σ and Σ_C coincide.
Then

$$\#\big((\Lambda t_n, \Lambda t_{n+1}] \cap \Sigma_C\big) = \#\big((\Lambda t_n, \Lambda t_{n+1}] \cap \Sigma\big) = \begin{cases} |\varphi(A)| \\ |\varphi(B)| \\ |\varphi(C)| \end{cases}$$

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Theorem B — Ideas behind the proof

Theorem is proved provided that the 3iet word u is a fixed point of a morphism φ.

Proposition

Let φ be a primitive 3iet preserving morphism. Then there exists $p \in \mathbb{N}$, $p \geq 1$, such that φ^p has a fixed point, and this fixed point is a 3iet word.

Let φ_0 be a primitive 3iet preserving morphism, $\mathbf{M}_0$ a positive power of its matrix

Let φ be a non-primitive 3iet preserving morphism, $\mathbf{M}$ its matrix

Hence $\mathbf{M}\mathbf{M}_0$ is positive, thus the matrix of a primitive 3iet preserving morphism.

$$1 \geq |\det(\mathbf{M}\mathbf{M}_0)| = |\det \mathbf{M}| \underbrace{|\det \mathbf{M}_0|}_{=1} = |\det \mathbf{M}|$$

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Outline

Comments and open problems

Incompleteness of the result

Matrices of 3iet preserving morphisms belong to $\mathrm{E}(3,\mathbb{N}) := \{\mathbf{M} \in \mathbb{N}^{3\times 3} \mid \mathbf{MEM}^T = \pm\mathbf{E} \text{ and } \det\mathbf{M} = \pm 1\}$, where $\mathbf{E} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.

In contrast to the sturmian case the opposite is not true. $\mathrm{E}(3,\mathbb{N})$ contains matrices associated with morphisms, which are not 3iet preserving.

Example. Consider $\mathbf{M} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 0 & 5 \end{pmatrix}$. Up to permutation of letters we have

$$\varphi(A) = B^2C\,, \quad \varphi(B) = A^2B^3C^5\,, \quad \varphi(C) = A^3C^5\,.$$

Hence CC is a factor of $\varphi(C)$ and $\varphi(A) = BCB$.

Take a 3iet word u containing AA. As $\varphi(AA) = BCBBCB$, u contains both BB and CC.

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Comments and open problems

The number of morphisms

The mapping $\varphi \to \mathbf{M}_\varphi$, where φ is a morphism and $\mathbf{M}_\varphi$ is its incidence matrix is not one-to-one.

In sturmian case for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{N}^{2\times 2}$ with $ad - bc = \pm 1$ there exist $a + b + c + d - 1$ different sturmian morphisms.

The same question for matrices of 3iet preserving morphisms is not solved.

The number of generators

The monoid of sturmian morphisms is finitely generated and so is the monoid of their matrix representations.

Let Φ_{3iet} be the monoid of 3iet preserving morphisms, and $\mathcal{R}(\Phi_{\mathrm{3iet}})$ the monoid of their matrix representations. None of Φ_{3iet}, $\mathcal{R}(\Phi_{\mathrm{3iet}})$ or $\mathrm{E}(3, \mathbb{N})$ is finitely generated.

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