Asymptotics of the Poincaré Functions

Gregory Derfel
derfel@cs.bgu.ac.il
Department of Mathematics
Ben Gurion University
Be'er Sheva 84105
Israël

Abstract

In 1890 H. Poincaré has studied the equation

$$f(\lambda z) = R\big(f(z)\big), \quad z \in \mathbf{C}, \tag{1}$$

where $R(z)$ is a rational function and $\lambda \in \mathbf{C}$. He proved that, if $R(0) = 0$, $R'(0) = \lambda$ and $|\lambda| > 1$, then there exists a meromorphic or entire solution of (1). After Poincaré, solutions of (1) are called *the Poincaré functions, satisfying the multiplication theorem*. Later on, Valiron elaborated the case, where $R(z) = P(z)$ is a polynomial, i.e.

$$f(\lambda z) = P\big(f(z)\big), \quad z \in \mathbf{C}, \tag{2}$$

and obtained conditions for the existence of an entire solution $f(z)$. Furthermore, he derived for $M(r) = \max_{|z|\le r} |f(z)|$ the following asymptotic formula:

$$\ln M(r) \sim r^{\rho} Q\left(\frac{\ln r}{\ln|\lambda|}\right), \quad r \to \infty. \tag{3}$$

Here $Q(z)$ is a 1-periodic function bounded between two positive constants, $\rho = \frac{\ln m}{\ln|\lambda|}$ and $m = \deg P(z)$.

An interesting example of such equations, which stems from the description of Brownian motion on Sierpinski's gasket, has been studied by Barlow & Perkins, Grabner & Woess and others. This is the functional equation

$$f(5z) = 4f^2(z) - 3f(z). \tag{4}$$

Similar equations arise also in the theory of brancing processes.

In our talk we develope the above results further. Namely, in addition to (3) we obtain asymptotics of entire solutions $f(z)$ on various rays $re^{i\vartheta}$ of the complex plane . It turns out that this heavily depends on the arithmetic nature of λ.

Further refinements are possible when $\lambda > 1$ is real and $P(z)$ is a polynomial with real positive coefficients.

Joint work with Fritz Vogl.