Fully Coupled Pauli-Fierz Hamiltonians at Zero and Positive Temperature

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1 Introduction

The purpose of these notes is to give a fairly narrow but thorough introduction to the spectral analysis of Hamiltonians and Liouvilleans describing finite dimensional small systems linearly coupled to a scalar massless field or reservoir. The Hamiltonians describe the system at zero temperature, and the standard Liouvillean implements unitarily the dynamics of the system at positive temperature.

We focus our attention on results valid at arbitrary coupling strength and whose proofs are purely operator theoretic, i.e. for the standard Liouvillean, does not make use of the underlying modular structure. For the Liouvillean this means that important structure results that does not seem to have a purely operator theoretic proof, like Jadczyk's theorem, will only be mentioned in passing.

2 Construction of Operators

2.1 The Hamiltonian

As a small quantum system we take a finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^\nu$ with Hamiltonian $K \in M_\nu(\mathbb{C})$, a self-adjoint $\nu \times \nu$ matrix $K^* = K$. In fact we can, without loss of generality, choose K to be diagonal with its real eigenvalues sitting on the diagonal.

The dispersion relation for the field is the massless relativistic relation $k \to |k|$ considered as a multiplication operator on $\mathfrak{h} = L^2(\mathbb{R}^3)$. This gives rise to the second quantized free field energy $H_{\mathrm{ph}} = \mathrm{d}\Gamma(|k|)$, as a self-adjoint operator on the bosonic Fock-space $\mathcal{F} = \Gamma(\mathfrak{h}) = \oplus_{\ell=0}^\infty \mathfrak{h}^{\otimes_s \ell}$. We write $|0\rangle = (1, 0, 0, \dots)$ for the vacuum state in $\mathcal{F}$.

We define a class of admissible coupling operators/functions

$$G \in \mathcal{B}(\mathcal{K}; \mathcal{K} \otimes \mathfrak{h}) = L^2\big(\mathbb{R}^3; M_\nu(\mathbb{C})\big).$$

That the two spaces above can be identified can be seen as follows: If $G\colon \mathbb{R}^3 \to M_\nu(\mathbb{C})$ is square integrable one can define a bounded operator $B_G \in \mathcal{B}(\mathcal{K}; \mathcal{K} \otimes \mathfrak{h})$ by

$$(B_G v)(k) = G(k)v,$$

where we identified $\mathcal{K} \otimes \mathfrak{h}$ isometrically with $L^2(\mathbb{R}^3; \mathbb{C}^\nu)$. Then

$$\|B_G\|^2 = \sup_{|v|\le 1} \|B_G v\|^2 = \sup_{|v|\le 1} \int_{\mathbb{R}^3} |G(k)v|^2 \, \mathrm{d}k \le \|G\|^2.$$

and the linear map $G \to B_G$ is a contraction, but it is not an isometry. To see that it is surjective with a bounded inverse, let $B \in \mathcal{B}(\mathcal{K}; \mathcal{K} \otimes \mathfrak{h})$ and define the candidate for an inverse G by $G_{ij}(k) = \langle (Be_i)(k), e_j\rangle$, where $e_1, \ldots, e_\nu$ is the standard basis for $\mathbb{C}^\nu$. Then

$$\sum_{j=1}^{\nu} \int_{\mathbb{R}^3} |G_{ij}(k)|^2 \mathrm{d}k = \int_{\mathbb{R}^3} |(Be_i)(k)|^2 \mathrm{d}k = \|Be_i\|^2_{\mathcal{K}\otimes\mathfrak{h}} \leq \|B\|^2.$$

Hence

$$\|G\|^2 = \int_{\mathbb{R}^3} \|G(k)\|^2 \mathrm{d}k \leq \sum_{1\leq i,j\leq \nu} \int_{\mathbb{R}^3} |G_{ij}(k)|^2 \mathrm{d}k \leq \nu \|B\|^2.$$

From now on we will identify couplings G with elements of $L^2(\mathbb{R}^3; M_\nu(\mathbb{C}))$, and norms of couplings will be L^2-norms. We remark that the identification of coupling operators as $\mathcal{B}(\mathcal{K})$-valued functions above is particular to finite dimensional small systems, cf. [16, Remark 5.1]. Let $\mu > 0$ be arbitrary, but fixed. For the coupling G we assume the existence of a constant $C > 0$ such that

$$(\mathbf{HGn}) \quad \begin{array}{lll} \forall k \in \mathbb{R}^3, |k| \leq 1, & \text{and} \quad |\alpha| \leq n: & \|\partial_k^\alpha G(k)\| \leq C|k|^{n-\frac{3}{2}+\mu-|\alpha|} \\ \forall k \in \mathbb{R}^3, |k| \geq 1, & \text{and} \quad |\alpha| \leq n: & \|\partial_k^\alpha G(k)\| \leq C|k|^{-\frac{3}{2}-\mu}. \end{array}$$

The derivatives are distributional derivatives. We will make use of the condition **(HGn)** on G with $n = 0, 1$.

The above conditions reflect that $|k|^{|\alpha|-n}\partial_k^\alpha G$ is slightly better than square integrable near zero, and the $\partial_k^\alpha G$'s are slightly better than square integrable at infinity. For our commutator estimates in Sect. 4 it will not suffice to demand just square integrability. We remark that there is nothing special about three dimensions or the dispersion $|k|$. For some results we could deal with infinite dimensional small system $\mathcal{K}$, and more singular G's. The above special case however captures the essentials, and permits us to formulate simple - yet pertinent - conditions that can be used for all our results at zero temperature.

We now define the free and coupled Hamiltonians as

$$\begin{aligned} H_0 &= K \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{K}} \otimes H_{\mathrm{ph}} \\ H &= H_0 + \phi(G), \end{aligned}$$

where

$$\phi(G) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} \{G(k)^* a(k) + G(k) a^*(k)\} \, \mathrm{d}k.$$

We remark that H_0 is self-adjoint on $\mathcal{D}(H_0) = \mathcal{D}(\mathbb{1}_{\mathcal{K}} \otimes H_{\mathrm{ph}})$ and that

$$\mathcal{C} = \mathcal{K} \otimes \Gamma_{\mathrm{fin}}(C_0^\infty(\mathbb{R}^3)) \tag{2.1}$$

is a core for H_0. Note the form bound on $\mathcal{C}$

$$\pm\phi(G) \le \sigma N + 2\sigma^{-1}\|G\|^2 \tag{2.2}$$

$$\pm\phi(G) \le \sigma H_{\mathrm{ph}} + 2\sigma^{-1}\|G/\sqrt{|k|}\|^2 \tag{2.3}$$

valid for all $\sigma > 0$ (and G). By (2.3) and Kato-Rellich's theorem, [48, Thm. X.12], H is essentially self-adjoint on $\mathcal{C}$ and $\mathcal{D}(H) = \mathcal{D}(H_0)$. In particular, the domain of H does not depend on G. The notation $\Gamma_{\mathrm{fin}}(V)$, with $V \subset \mathfrak{h}$ a subspace, denotes the algebraic direct sum of $V^{\otimes_{\mathrm{s}} n}$, with tensor products of subspaces of Hilbert spaces always being algebraic, whereas tensor products of Hilbert spaces always denote Hilbert space tensor products, i.e. completion of algebraic tensor products.

The bound (2.3) in particular implies that H is bounded from below. We furthermore observe that if we equip the space of G's satisfying **(HG0)** with the norm $\|G\|_0^2 = \int_{\mathbb{R}^3}(1+|k|^{-1})\|G(k)\|^2\mathrm{d}k$, then the resolvent map $(z,G) \to (H-z)^{-1}$ is norm continuous. Here $\mathrm{Im} z \neq 0$. We define

$$\Sigma = \inf \sigma(H) > -\infty. \tag{2.4}$$

The spectrum of H is in fact a half-line starting at Σ as we now prove, using an argument from [23]. Before we provide the proof we establish a version of the so called pull through formula

Proposition 2.1. *Suppose* **(HG0)**. *For any $z \in \mathbb{C}\backslash[\Sigma,\infty)$ and $\psi \in \mathcal{D}(\sqrt{N})$ we have as an $L^2(\mathbb{R}^3;\mathcal{H})$-identity*

$$a(k)(H-z)^{-1}\psi = (H+|k|-z)^{-1}a(k)\psi - \frac{1}{\sqrt{2}}(H+|k|-z)^{-1}(G(k)\otimes \mathbb{1}_{\mathcal{F}})\psi.$$

Remark 2.2. It is an immediate consequence of [43, Prop. II.1] that H is of class $C^1_{\mathrm{Mo}}(N)$, implying in particular that $\mathcal{D}(N)$, and hence by interpolation $\mathcal{D}(\sqrt{N})$, is preserved by resolvents of H. Hence both sides of the pull through formula defines elements of $L^2(\mathbb{R}^3;\mathcal{H})$. The commutator is $[H,N]^\circ = \mathrm{i}\phi(\mathrm{i}G)$. We note that it also holds true that N is of class $C^1(H)$, something we will not make any use of.

Proof. Let $\tilde\psi \in \mathcal{C}$ and compute

$$a(k)(H-z)\tilde\psi = (H+|k|-z)a(k)\tilde\psi + \frac{1}{\sqrt{2}}(G(k)\otimes \mathbb{1}_{\mathcal{F}})\tilde\psi$$

as an $L^2(\mathbb{R}^3;\mathcal{H})$-identity, where the only possibly irregular contribution is G near zero. Since $z-|k| \in \rho(H)$ - the resolvent set for H - we obtain the $L^2(\mathbb{R}^3;\mathcal{H})$-identity

$$a(k)\tilde\psi = (H+|k|-z)^{-1}a(k)(H-z)\tilde\psi - \frac{1}{\sqrt{2}}(H+|k|-z)^{-1}(G(k)\otimes \mathbb{1}_{\mathcal{F}})\tilde\psi.$$

Let $h \in L^2(\mathbb{R}^3)$ and $\varphi \in \mathcal{C}$. Then

$$\langle a^*(h)\varphi, \tilde\psi\rangle = \langle \tilde\varphi, (H-z)\tilde\psi\rangle - \int_{\mathbb{R}^3} \frac{\overline{h(k)}}{\sqrt{2}} \langle \varphi, (H+|k|-z)^{-1}(G(k)\otimes \mathbb{1}_{\mathcal{F}})\tilde\psi\rangle \,\mathrm{d}k,$$

where

$$\tilde\varphi = \int_{\mathbb{R}^3} h(k) a^*(k)(H+|k|-z)^{-1}\varphi \,\mathrm{d}k \in \mathcal{H}.$$

From this expression, and H being essentially self-adjoint on $\mathcal{C}$, we observe that the above identity remains true for $\tilde\psi \in \mathcal{D}(H)$. Inserting $\tilde\psi = (H-z)^{-1}\psi$, where $\psi \in \mathcal{D}(\sqrt{N})$ yields the proposition. Here we used that $L^2(\mathbb{R}^3)\otimes\mathcal{C}$ (algebraic tensor product) is dense in $L^2(\mathbb{R}^3;\mathcal{H})$. □

For stronger versions of the pull through formula see [10, 24]. We are now ready to show that the spectrum is a half-axis. The argument goes back to [23], cf. also [10].

Theorem 2.3. *Suppose* **(HG0)***. Then* $\sigma(H) = [\Sigma,\infty)$.

Proof. It suffices to show that $\sigma((H-\Sigma+1)^{-1}) \supset (0,1]$. To see this, let $\lambda > 0$, $\epsilon > 0$ and choose $\tilde\psi \in \mathbb{1}[H \le \Sigma + \epsilon/2]\mathcal{H}$ to be normalized. Since $\mathcal{C}$ is dense in $\mathcal{D}(H)$, we can pick a $\psi \in \mathcal{C}$ such that $\|(H-\Sigma)(\tilde\psi - \psi)\| \le \epsilon/2$ and hence we must have $\|(H-\Sigma)\psi\| \le \epsilon$.

Choose a function $h \in C_0^\infty(\mathbb{R})$ real-valued with $\|h\| = 1$ and $\operatorname{supp} h \subset [-1,1]$. Put $h_n(k) = n^{3/2}h(n(|k|-\lambda))$. Form $\psi_n = a^*(h_n)\psi$ and compute for $\varphi \in \mathcal{D}(\sqrt{N})$ using the pull through formula Proposition 2.1

$$\begin{aligned}
&\langle \varphi, \big((H-\Sigma+1)^{-1} - (\lambda+1)^{-1}\big)\psi_n\rangle \\
&= \int_{\mathbb{R}^3} h_n(k)\langle a(k)\big((H-\Sigma+1)^{-1} - (\lambda+1)^{-1}\big)\varphi, \psi\rangle \mathrm{d}k \\
&= \int_{\mathbb{R}^3} h_n(k)\langle \big((H+|k|-\Sigma+1)^{-1} - (\lambda+1)^{-1}\big)a(k)\varphi, \psi\rangle \mathrm{d}k \\
&\quad + \langle \int_{\mathbb{R}^3} \frac{h_n(k)}{\sqrt{2}} G(k)\mathrm{d}k \otimes \mathbb{1}_{\mathcal{F}}\varphi, \big((H+|k|-\Sigma+1)^{-1}\psi\rangle.
\end{aligned}$$

Since h_n goes to zero weakly in $L^2(\mathbb{R}^3)$, the last term is $o(1)\|\varphi\|\|\psi\|$ in the limit of large n. To deal with the first term on the right-hand side we estimate using the support properties of h_n and the choice of ψ:

$$\begin{aligned}
&\big|h_n(k)\big\langle a(k)\big((H+|k|-\Sigma+1)^{-1}-(\lambda+1)^{-1}\big)\varphi,\psi\big\rangle\big| \\
&\leq \frac{|h_n(k)|}{\sqrt{|k|}}\big\|(H_{\mathrm{ph}}+2)^{-1}\sqrt{|k|}a(k)\varphi\big\| \\
&\quad\times\big\|(H_{\mathrm{ph}}+2)(H+|k|-\Sigma+1)^{-1}\big((H-\Sigma)+(|k|-\lambda)\big)\psi\big\| \\
&\leq C\left(\epsilon+\frac{1}{n}\right)\frac{|h_n(k)|}{\sqrt{|k|}}\big\|\sqrt{|k|}a(k)(H_{\mathrm{ph}}+1)^{-1}\varphi\big\|.
\end{aligned}$$

Noting that $\|h_n/\sqrt{|k|}\| \leq (\lambda-1/n)^{-1/2}\|h_n\| = (\lambda-1/n)^{-1/2}$ we conclude from Cauchy-Schwartz that

$$\big|\big\langle\varphi,\big((H-\Sigma+1)^{-1}-(\lambda+1)^{-1}\big)\psi_n\big\rangle\big| \leq C(\epsilon+o(1))\|\varphi\|,$$

where $o(1)$ refers to the large n limit. It now remains to prove that $\|a^*(h_n)\psi\|$ is bounded away from zero. But this follows from the computation

$$\|a^*(h_n)\psi\|^2 = \|\psi\|^2 + \|a(h_n)\psi\|^2.$$

Recall that when h_n goes to zero weakly, we have $a(h_n)\psi \to 0$ in norm, whenever $\psi \in \mathcal{D}(\sqrt{N})$. $\square$

We end this subsection introducing some extra structure that will be used to construct the standard Liouvillean in the next subsection.

We will need a conjugate linear involution operator $\mathbf{C}$ on $\mathcal{H}$ defined as follows. It is a tensor product of two conjugate linear involutions, one on $\mathcal{K}$ and one on $\mathcal{F}$. On $\mathcal{K}$ we simply take coordinate wise complex conjugation $(\mathbf{c}v)_j = \bar{v}_j$, and on $\mathcal{F}$ we take second quantized complex conjugation $\Gamma(\mathbf{c})$, acting on an n-particle state by complex conjugation, or equivalently described by the intertwining $\Gamma(\mathbf{c})a^\#(g)\Gamma(\mathbf{c}) = a^\#(\bar{g})$. In conclusion $\mathbf{C} = \mathbf{c}\otimes\Gamma(\mathbf{c})$. Note that $\langle\mathbf{C}\psi,\varphi\rangle = \langle\mathbf{C}\varphi,\psi\rangle$.

With this choice of conjugation we can define $H^{\mathbf{c}} = \mathbf{C}H\mathbf{C} = H_0 + \phi(\overline{G})$. Note that $H_0^{\mathbf{c}} = \mathbf{C}H_0\mathbf{C} = H_0$. Clearly, the spectrum, pure point spectrum and absolutely/singular continuous spectrum of the two operators coincide. Eigenvectors are related by $\psi^{\mathbf{c}} = \mathbf{C}\psi$, where $H\psi = \lambda\psi$ and $H^{\mathbf{c}}\psi^{\mathbf{c}} = \lambda\psi^{\mathbf{c}}$. Finally we observe that the spectral resolutions E and $E^{\mathbf{c}}$ of the operators H and $H^{\mathbf{c}}$ are related by $E^{\mathbf{c}}_\psi = E_{\mathbf{C}\psi}$.

We will use the notation N for the number operator $\mathrm{d}\Gamma(\mathbb{1}_{\mathfrak{h}})$ as an operator on $\mathcal{F}$, and we recycle the same notation on $\mathcal{H}$ instead of the more cumbersome $\mathbb{1}_{\mathcal{K}}\otimes N$.

2.2 The Standard Liouvillean

The Liouvillean, at inverse temperature $\beta > 0$, is a self-adjoint operator on the doubled Hilbert space $\mathcal{H}^{\mathrm{L}} := \mathcal{H} \otimes \mathcal{H}$. The zero temperature Liouvillean, corresponding to $\beta = \infty$, is given by

$$L_\infty = H \otimes \mathbb{1}_{\mathcal{H}} - \mathbb{1}_{\mathcal{H}} \otimes H^{\mathbf{c}},$$

which is essentially self-adjoint on algebraic tensor products $\mathcal{D} \otimes \mathcal{D}$, where $\mathcal{D} \subset \mathcal{H}$ is a core for H. See [47, Thm. VIII.33]. As a choice of core we take

$$\mathcal{C}^{\mathrm{L}} = \mathcal{C} \otimes \mathcal{C}, \tag{2.5}$$

where $\mathcal{C}$ was defined in (2.1). Observe that L_∞ is unbounded from below and indeed $\sigma(L_\infty) = \mathbb{R}$.

We furthermore write

$$L_0 = H_0 \otimes \mathbb{1}_{\mathcal{H}} - \mathbb{1}_{\mathcal{H}} \otimes H_0$$

for the uncoupled Liouvillean. Recall that $H_0^{\mathbf{c}} = H_0$. With this notation, at least formally, the zero temperature ($\beta = \infty$) Liouvillean can be written as the operator sum $L_\infty = L_0 + \phi(G) \otimes \mathbb{1}_{\mathcal{H}} - \mathbb{1}_{\mathcal{H}} \otimes \phi(\overline{G})$.

We will need stronger conditions than **(HGn)** on the coupling G when dealing with the Liouvillean. Let $n \in \mathbb{N}_0$. We assume that G admits n distributional derivatives in $L^1_{\mathrm{loc}}(\mathbb{R}^3; M_\nu(\mathbb{C}))$ and the existence of a constant $C > 0$ such that

$$(\mathbf{LGn}) \quad \begin{array}{lll} \forall k \in \mathbb{R}^3, |k| \leq 1, \quad \text{and} \quad |\alpha| \leq n: & & \|\partial_k^\alpha G(k)\| \leq C|k|^{n-1+\mu-|\alpha|} \\ \forall k \in \mathbb{R}^3, |k| \geq 1, \quad \text{and} \quad |\alpha| \leq n: & & \|\partial_k^\alpha G(k)\| \leq C|k|^{-\frac{3}{2}-\delta_{\alpha,0}-\mu}. \end{array}$$

We will make use of the condition **(LGn)** on G with $n = 0, 1, 2$. Note that **(LGn)** implies **(HGn)**. As for the Hamiltonian, there is nothing particular about dimension 3. The difference between **(HGn)** and **(LGn)** comes from having to absorb an infrared singularity from the Boltzmann density

$$\rho_\beta(k) = \frac{1}{\mathrm{e}^{\beta|k|} - 1},$$

which mix the left and right field components at positive temperature, $\beta < \infty$. One could you use a different density, modifying **(LGn)** accordingly. See also Remark 2.9.

We write

$$G_{\mathrm{l}}(k) = G(k) \otimes \mathbb{1}_{\mathcal{K}} \quad \text{and} \quad G_{\mathrm{r}}(k) = \mathbb{1}_{\mathcal{K}} \otimes \overline{G(k)}. \tag{2.6}$$

At finite inverse temperature β we abbreviate

$$G_{\mathrm{l}}^{\beta} = \sqrt{1+\rho_\beta}\, G_{\mathrm{l}} - \sqrt{\rho_\beta}\, G_{\mathrm{r}}^{*} \quad \text{and} \quad G_{\mathrm{r}}^{\beta} = \sqrt{1+\rho_\beta}\, G_{\mathrm{r}} - \sqrt{\rho_\beta}\, G_{\mathrm{l}}^{*}. \tag{2.7}$$

It will also be convenient to write $a_{\mathrm{l}}^{\#}(k) = a^{\#}(k) \otimes \mathbb{1}_{\mathcal{F}}$ and $a_{\mathrm{r}}^{\#}(k) = \mathbb{1}_{\mathcal{F}} \otimes a^{\#}(k)$. The interaction at positive temperature is $W_\beta(G)$ where

$$W_\beta(G) := \phi_{\mathrm{l}}\big(G_{\mathrm{l}}^{\beta}\big) - \phi_{\mathrm{r}}\big(G_{\mathrm{r}}^{\beta}\big).$$

Here the left and right fields are defined the obvious way. The zero and positive temperature Liouvillean is thus densely defined, a priori on $\mathcal{C}^{\mathrm{L}}$, as the operator sum

$$L_\beta = L_0 + W_\beta(G) = L_\infty + I_\beta(G),$$

where

$$I_\beta(G) = \phi_{\mathrm{l}}\big(G_{\mathrm{l}}^{\beta} - G_{\mathrm{l}}\big) - \phi_{\mathrm{r}}\big(G_{\mathrm{r}}^{\beta} - G_{\mathrm{r}}\big). \tag{2.8}$$

That the Liouvilleans L_β, $0 < \beta < \infty$ are essentially self-adjoint on $\mathcal{C}^{\mathrm{L}}$ was proved in [32, Lemma 3.2], cf. also [8, 13, 40], using Nelson's commutator theorem [48, Thm. X.37]. This requires that G can absorb a power of the dispersion $|k|$, which is the source of the $\delta_{\alpha,0}$ term in the ultraviolet part of condition **(LGn)**. We warn the reader that the domain of L_β may depend on both β and G, an issue that complicates the analysis of the operator. Proposition 2.4 2 below remedies this issue somewhat.

We write $N^{\mathrm{L}} = N \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes N$ for the number operator on $\mathcal{F} \otimes \mathcal{F}$, and as for N we use the same notation to denote $\mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes N^{\mathrm{L}}$. Note that $\mathbf{C} N^{\mathrm{L}} \mathbf{C} = N^{\mathrm{L}}$.

We introduce the so called *modular conjugation* J, which is a conjugate linear involution on $\mathcal{H}^{\mathrm{L}}$ given by

$$J := (\mathbf{C} \otimes \mathbf{C})\mathcal{E}, \tag{2.9}$$

where $\mathcal{E}$ is the exchange operator defined on simple tensors by $\mathcal{E}(\psi \otimes \varphi) = \varphi \otimes \psi$. Here $\psi, \varphi \in \mathcal{H}$. Clearly $J L_\infty J = -L_\infty$. Indeed, the identity holds on $\mathcal{C}^{\mathrm{L}}$ and extends to $\mathcal{D}(L_\infty)$ since $\mathcal{C}^{\mathrm{L}}$ is an operator core for L_∞.

Computing as an identity first on $\mathcal{C}^{\mathrm{L}}$ we find

$$J \phi_{\mathrm{l}}\big(G_{\mathrm{l}}^{\beta}\big) J = \phi_{\mathrm{r}}\big(G_{\mathrm{r}}^{\beta}\big),$$

and hence

$$J L_\beta J = -L_\beta.$$

As above one should first verify the identity on $\mathcal{C}^{\mathrm{L}}$ and extend by continuity to $\mathcal{D}(L_\beta)$. Consequently, we observe that the spectrum and pure point spectrum of

L_β is reflection symmetric around 0. Furthermore the spectral resolution E^β associated with L_β satisfies $E^\beta(B) = JE^\beta(-B)J$ and hence the absolutely and singular continuous spectra of L_β are also reflection symmetric.

We remark that there is a different way of representing the interaction $W_\beta(G)$ which is more natural from the point of view of the underlying operator algebraic framework. This representation makes use of the left and right Araki-Woods fields. To construct these we introduce annihilation operators

$$a_{\mathrm{l}}^{\mathrm{AW}}(k) = \sqrt{1+\rho_\beta(k)}\, a_{\mathrm{l}}(k) + \sqrt{\rho_\beta(k)}\, a_{\mathrm{r}}^*(k)$$
$$a_{\mathrm{r}}^{\mathrm{AW}}(k) = \sqrt{1+\rho_\beta(k)}\, a_{\mathrm{r}}(k) + \sqrt{\rho_\beta(k)}\, a_{\mathrm{l}}^*(k)$$

and the Araki-Woods creation operators $a_{\mathrm{l/r}}^{\mathrm{AW}*}(k)$ are now obtained by "taking adjoints". We can express $W_\beta(G)$ in terms of left and right Araki-Woods fields $\phi_{\mathrm{l/r}}^{\mathrm{AW}}$ as follows

$$W_\beta(G) = \phi_{\mathrm{l}}^{\mathrm{AW}}(G_{\mathrm{l}}) - \phi_{\mathrm{r}}^{\mathrm{AW}}(G_{\mathrm{r}}), \tag{2.10}$$

where $G_{\mathrm{l/r}}$, where introduced in (2.6).

We end the subsection with a proposition that permits us to work effectively with standard Liouvilleans, despite domain problems. Its proof follows closely arguments from [16], establishing similar statements for technically related operators.

Proposition 2.4. *Suppose* **(LG0)**. *The following holds*

1. $N^{\mathrm{L}} \in C^1_{\mathrm{Mo}}(L_\beta)$ *and the operator* $[N^{\mathrm{L}}, L_\beta]^\circ$ *extends from* $\mathcal{D}(N^{\mathrm{L}})$ *by continuity to an element of* $\mathcal{B}(\mathcal{D}(\sqrt{N^{\mathrm{L}}}); \mathcal{H}^{\mathrm{L}})$.
2. $\mathcal{D}(N^{\mathrm{L}}) \cap \mathcal{D}(L_\beta)$ *does not depend on* β, *nor on* G.
3. $\mathcal{C}^{\mathrm{L}}$ *is dense in* $\mathcal{D}(N^{\mathrm{L}}) \cap \mathcal{D}(L_\beta)$ *with respect to the intersection topology.*

Proof. To establish 1 we argue as in the verification of [16, Cond. 2.1 (2), cf. Sect. 5.5].

First observe that N^{L} and L_0 commute such that we can compute as a form on the core $\mathcal{C}^{\mathrm{L}}$

$$\big[(N^{\mathrm{L}}+1)^{-1}, L_\beta\big] = (N^{\mathrm{L}}+1)^{-1} W_\beta(G) - W_\beta(G)(N^{\mathrm{L}}+1)^{-1}.$$

The right-hand side extends to a bounded operator, and since $\mathcal{C}^{\mathrm{L}}$ is dense in $\mathcal{D}(L_\beta)$, the form $[(N^{\mathrm{L}}+1)^{-1}, L_\beta]$ defined on $\mathcal{D}(L_\beta)$ extends by continuity to a bounded form on $\mathcal{H}^{\mathrm{L}}$, coinciding with the closure of the right-hand side as a form on $\mathcal{C}^{\mathrm{L}}$. Hence $N^{\mathrm{L}} \in C^1(L_\beta)$.

Having established that N^{L} is of class $C^1(L_\beta)$ we know that $[N^{\mathrm{L}}, L_\beta]$ extends from the intersection domain $\mathcal{D}(N^{\mathrm{L}}) \cap \mathcal{D}(L_\beta)$ to a bounded form on $\mathcal{D}(N^{\mathrm{L}})$.

Hence, to compute this form it suffices to compute it on a core of N^{L}. Compute as a form on $\mathcal{C}^{\mathrm{L}}$

$$\mathrm{i}[N^{\mathrm{L}}, L_\beta] = W_\beta(\mathrm{i}G), \tag{2.11}$$

which extends from $\mathcal{C}^{\mathrm{L}}$ to $\mathcal{B}(\mathcal{D}(\sqrt{N^{\mathrm{L}}}); \mathcal{H}^{\mathrm{L}})$. This proves 1.

As for 2 we follow the proof of [16, Lemma 5.15]. Let $T_0 = L_0 + \mathrm{i}(N^{\mathrm{L}} + 1)$. Since L_0 and N^{L} commute we clearly have $\mathcal{D}(T_0) = \mathcal{D}(L_0) \cap \mathcal{D}(N^{\mathrm{L}}) =: \mathcal{D}_0$.

We now construct $L_\beta + \mathrm{i}(N^{\mathrm{L}}+1)$ in two different ways. First define $\widehat{L} = L_0 + W_\beta(G)$ as a symmetric operator on $\mathcal{D}_0$. Then $T_0 + W_\beta(G) = \widehat{L} + \mathrm{i}(N^{\mathrm{L}}+1) =: T_1$ is by [48, Corollary to Thm. X.48] a closed operator on $\mathcal{D}_0$. Conversely we can use [22, Thm. 2.25] to construct $T_2^\pm = L_\beta \pm \mathrm{i}(N^{\mathrm{L}} + 1)$ as closed operators on $\mathcal{D}_\beta = \mathcal{D}(L_\beta) \cap \mathcal{D}(N^{\mathrm{L}})$ with $T_2^{+*} = T_2^-$. Since $\mathcal{C}^{\mathrm{L}} \subset \mathcal{D}_0$ we find that $\widehat{L}$ is a symmetric extension of $L_{\beta_{|\mathcal{C}^{\mathrm{L}}}}$. Hence, $\mathcal{C}^{\mathrm{L}}$ being a core for L_β, we find that $\widehat{L} \subset L_\beta$. This implies that $T_1 \subset T_2^+ =: T_2$. Since $T_2^\pm$ are both accretive we find that T_2 generates a contraction semigroup. To conclude the proof we only need to show that the $\rho(T_1) \cap \rho(T_2) \neq \emptyset$. But this follows from the Hille-Yosida theorem, cf. [48, Thm. X.47a].

Finally we turn to 3. From what was just proved, together with the closed graph theorem, we conclude that it suffices to prove that $\mathcal{C}^{\mathrm{L}}$ is dense in $\mathcal{D}_0$ with respect to the norm

$$\|\psi\|_0 = \|N^{\mathrm{L}}\psi\|_{\mathcal{H}^{\mathrm{L}}} + \|L_0\psi\|_{\mathcal{H}^{\mathrm{L}}} + \|\psi\|_{\mathcal{H}^{\mathrm{L}}}.$$

Since L_0 and N^{L} commute it suffices to show that one can approximate $\psi \in \mathcal{D}_0$ with $\psi = \mathbb{1}[N^{\mathrm{L}} \leq n]\psi$, for some n. Similarly, since L_0 and N^{L} commute with $\Gamma_R := \mathbb{1}_{\mathcal{K}} \otimes \Gamma(\mathbb{1}[|k| \leq R]) \otimes \mathbb{1}_{\mathcal{K}} \otimes \Gamma(\mathbb{1}[|k| \leq R])$, it suffices to approximate states ψ, non-zero in finitely many particle sectors, and satisfying $\Gamma_R\psi = \psi$, for some $R > 0$.

Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{C}^{\mathrm{L}}$ be a sequence with $\|\psi - \varphi_n\|_{\mathcal{H}^{\mathrm{L}}} \to 0$ for $n \to \infty$. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ satisfy $0 \leq \chi \leq 1$, $\chi(k) = 1$ for $|k| \leq R$, and $\chi(k) = 0$ for $|k| \geq R + 1$. Then $\Gamma_\chi := \mathbb{1}_{\mathcal{K}} \otimes \Gamma(\chi) \otimes \mathbb{1}_{\mathcal{K}} \otimes \Gamma(\chi)$ preserves $\mathcal{C}^{\mathrm{L}}$ and $\|\psi - \Gamma_\chi\varphi_n\|_{\mathcal{H}^{\mathrm{L}}} \to 0$ for $n \to \infty$.

Now that both ψ and $\Gamma_\chi\varphi_n$ only have finitely many non-zero components all supported inside a box of side length $R + 1$, one can easily verify that

$$\|\psi - \Gamma_\chi\varphi_n\|_0 \to 0, \quad \text{for} \quad n \to \infty.$$

This completes the proof. □

We remark that L_β is presumably *not* of class $C^1(N^{\mathrm{L}})$, cf. Remark 2.2.

2.3 Jakšić-Pillet Gluing

We proceed to discuss a unitarily equivalent form of the Liouvillean obtained by so called the Jakšić-Pillet gluing procedure, cf. [13, 32]

But first we pass to polar coordinates on the Hamiltonian level. Define a unitary transform $T_{\mathrm{l}}\colon \mathfrak{h} \to \tilde{\mathfrak{h}}_{\mathrm{l}} := L^2([0,\infty)) \otimes L^2(S^2)$ by the prescription

$$(T_{\mathrm{l}} f)(\omega, \Theta) = \omega f(\omega\Theta).$$

Denote by $\widetilde{\mathcal{F}}_{\mathrm{l}} = \Gamma(\tilde{\mathfrak{h}}_{\mathrm{l}})$ the Fock space in polar coordinates. The subscripts l is for later use and refers to the left component in the tensor product $\mathcal{H}^{\mathrm{L}} = \mathcal{H} \otimes \mathcal{H}$. The twiddle indicates an object represented in polar coordinates for the Hamiltonian, and after gluing for the Liouvillean.

The coupling in polar coordinates becomes

$$\widetilde{G}(\omega, \Theta) := \omega G(\omega\Theta)$$

and the Hamiltonian takes the form

$$\widetilde{H} = \big(\mathbb{1}_{\mathcal{K}} \otimes \Gamma(T_{\mathrm{l}})\big) H \big(\mathbb{1}_{\mathcal{K}} \otimes \Gamma(T_{\mathrm{l}})\big) = K \otimes \mathbb{1}_{\widetilde{\mathcal{F}}_{\mathrm{l}}} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega) + \phi(\widetilde{G}),$$

a priori as an identity on $C_0^\infty([0,\infty) \otimes C^\infty(S^2)$ and extended to $\mathcal{D}(\widetilde{H}) = \mathcal{D}(\mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega))$ by continuity.

To deal with the Liouvillean we similarly need a map $T_{\mathrm{r}}\colon \mathfrak{h} \to \tilde{\mathfrak{h}}_{\mathrm{r}} := L^2((-\infty, 0]) \otimes L^2(S^2)$, defined by $(T_{\mathrm{r}} f)(\omega, \Theta) = (T_{\mathrm{l}} f)(-\omega, \Theta)$. Put $\widetilde{\mathcal{F}}_{\mathrm{r}} = \Gamma(\tilde{\mathfrak{h}}_{\mathrm{r}})$. This sets up a unitary transformation

$$T\colon \mathfrak{h} \oplus \mathfrak{h} \to \tilde{\mathfrak{h}} := L^2(\mathbb{R}) \otimes L^2(S^2),$$

by the construction

$$(T(f,g))(\omega,\Theta) = \mathbb{1}[\omega \geq 0](T_{\mathrm{l}} f)(\omega,\Theta) + \mathbb{1}[\omega \leq 0](T_{\mathrm{r}} g)(\omega, \Theta).$$

Using the canonical identification $I\colon \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \to \mathcal{F} \otimes \mathcal{F}$ we construct a unitary map

$$\mathcal{U}\colon \mathcal{H}^{\mathrm{L}} \to \widetilde{\mathcal{H}}^{\mathrm{L}} := \mathcal{K} \otimes \mathcal{K} \otimes \widetilde{\mathcal{F}},$$

where $\widetilde{\mathcal{F}} = \Gamma(L^2(\mathbb{R}) \otimes L^2(S^2))$. The map $\mathcal{U}$ is defined on simple tensors by

$$\mathcal{U}(v \otimes \eta \otimes w \otimes \xi) = v \otimes w \otimes \Gamma(T) I^*(\eta \otimes \xi)$$

and extended to $\mathcal{H}^{\mathrm{L}}$ by linearity. Here $v, w \in \mathcal{K}$ and $\xi, \eta \in \mathcal{F}$. As an alternative core we take

$$\widetilde{\mathcal{C}}^{\mathrm{L}} = \mathcal{K} \otimes \mathcal{K} \otimes \Gamma_{\mathrm{fin}}\big(C_0^\infty(\mathbb{R}) \otimes C^\infty(S^2)\big).$$

Let $L_K = K \otimes \mathbb{1}_{\mathcal{K}} - \mathbb{1}_{\mathcal{K}} \otimes K$ as an operator on $\mathcal{K} \otimes \mathcal{K}$.

In the new coordinate system, we can write the interaction $W_\beta(G)$ as a field operator as follows. First, the zero temperature interaction is

$$\widetilde{G}_\infty := \mathbb{1}[|\omega \geq 0]\widetilde{G}_{\mathrm{l}}(\omega, \Theta) - \mathbb{1}[\omega \leq 0]\widetilde{G}_{\mathrm{r}}(-\omega, \Theta), \tag{2.12}$$

where $\widetilde{G}_{\mathrm{l/r}}(\omega, \Theta) = \omega G_{\mathrm{l/r}}(\omega\Theta)$, cf. (2.6). With this construction we have $\mathcal{U}(\phi_{\mathrm{l}}(G_{\mathrm{l}}) - \phi_{\mathrm{r}}(G_{\mathrm{r}}))\mathcal{U}^* = \phi(\widetilde{G}_\infty)$. The computation is easily done on $\widetilde{\mathcal{C}}^{\mathrm{L}}$ and extended by continuity to $\mathcal{D}(\phi(\widetilde{G}))$. At finite temperature, the interaction reads

$$\widetilde{G}_\beta(\omega, \Theta) := \sqrt{1 + \widetilde{\rho}_\beta}\widetilde{G}_\infty + \sqrt{\widetilde{\rho}_\beta}\widetilde{G}^*_{\infty,\mathcal{R}}, \tag{2.13}$$

where $\widetilde{G}_{\infty,\mathcal{R}}(\omega, \Theta) = \widetilde{G}_\infty(-\omega, \Theta)$ is the reflected glued coupling, and

$$\widetilde{\rho}_\beta(\omega, \Theta) = \rho_\beta(\omega\Theta) = \frac{1}{\mathrm{e}^{\beta|\omega|} - 1}.$$

Recalling (2.7), we observe that we similarly have $\mathcal{U}(\phi_{\mathrm{l}}(G_{\mathrm{l}}^\beta) - \phi_{\mathrm{r}}(G_{\mathrm{r}}^\beta))\mathcal{U}^* = \phi(\widetilde{G}_\beta)$.

As observed in [32] we have the following alternative representation of $\widetilde{G}_\beta$

$$\widetilde{G}_\beta = \left(\frac{\omega}{1 - \mathrm{e}^{-\beta\omega}}\right)^{\frac{1}{2}} \widehat{G}_{\mathrm{l}} - \left(\frac{\omega}{\mathrm{e}^{\beta\omega} - 1}\right)^{\frac{1}{2}} \widehat{G}_{\mathrm{r}}, \tag{2.14}$$

where

$$\begin{aligned} \widehat{G}_{\mathrm{l}}(\omega, \Theta) &= \left(\mathbb{1}[\omega \geq 0]\sqrt{\omega}G(\omega\Theta) + \mathbb{1}[\omega \leq 0]\sqrt{-\omega}G(-\omega\Theta)^*\right) \otimes \mathbb{1}_{\mathcal{K}} \\ \widehat{G}_{\mathrm{r}}(\omega, \Theta) &= \mathbb{1}_{\mathcal{K}} \otimes \left(\mathbb{1}[\omega \geq 0]\sqrt{\omega}\overline{G(\omega\Theta)} + \mathbb{1}[\omega \leq 0]\sqrt{-\omega}\overline{G(-\omega\Theta)^*}\right). \end{aligned} \tag{2.15}$$

This form of the interaction mirrors the Araki-Woods representation (2.10).

Remark 2.5. The representation (2.13) allows us to easily observe that under the assumption **(LGn)**, the ultraviolet part of $\widetilde{G}_\beta$ and its first n derivatives are square integrable, whereas (2.14) allows us to conclude the same for the infrared region. $\Diamond$

We can now write down the standard Liouvillean in the new coordinate system as

$$\widetilde{L}_\beta = L_K \otimes \mathbb{1}_{\widetilde{\mathcal{F}}} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{H}_{\mathrm{ph}} + \phi(\widetilde{G}_\beta),$$

with $\widetilde{H}_{\mathrm{ph}} = \mathrm{d}\Gamma(\omega)$. Note that ω denotes both a real number and multiplication by the identity function in $L^2(\mathbb{R})$.

Again, by Nelson's commutator theorem, $\widetilde{L}_\beta$ is essentially self-adjoint on $\widetilde{\mathcal{C}}^{\mathrm{L}}$. We observe that L_β and $\widetilde{L}_\beta$ are unitarily equivalent through $\mathcal{U}$. As an identity on $\widetilde{\mathcal{C}}^{\mathrm{L}}$ we have $L_\beta = L_\infty + \phi(\widetilde{G}_\beta - \widetilde{G}_\infty)$, with $\widetilde{L}_\infty = \widetilde{L}_0 + \phi(\widetilde{G}_\infty)$ and $\widetilde{L}_0 = L_K \otimes \mathbb{1}_{\widetilde{\mathcal{F}}} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{H}_{\mathrm{ph}}$. These operators are also essentially self-adjoint on $\widetilde{\mathcal{C}}^{\mathrm{L}}$ and their closures are unitarily equivalent with the appropriate untwitled objects.

In the glued coordinate system we write $\widetilde{N} = \mathcal{U}N^{\mathrm{L}}\mathcal{U}^* = \mathrm{d}\Gamma(\mathbb{1}_{\tilde{\mathfrak{h}}})$, where the second quantization is here performed in $\widetilde{\mathcal{F}}$.

The statements 1 and 2 in the following corollary to Proposition 2.4 are an immediate consequence of Proposition 2.4 1 and 2. The item 3 however is not, but it can be proved by an argument identical to the one employed at the end of the proof above.

Corollary 2.6. *Suppose* **(LG0)**. *The following holds*

1. $\widetilde{N} \in C^1_{\mathrm{Mo}}(\widetilde{L}_\beta)$ *and the operator* $[\widetilde{N}, \widetilde{L}_\beta]^\circ$ *extends from* $\mathcal{D}(\widetilde{N})$ *by continuity to an element of* $\mathcal{B}(\mathcal{D}(\sqrt{\widetilde{N}}); \widetilde{\mathcal{H}}^{\mathrm{L}})$.
2. $\mathcal{D}(\widetilde{N}) \cap \mathcal{D}(\widetilde{L}_\beta)$ *does not depend on* β, *nor on* G.
3. $\widetilde{\mathcal{C}}^{\mathrm{L}}$ *is dense in* $\mathcal{D}(\widetilde{N}) \cap \mathcal{D}(\widetilde{L}_\beta)$ *with respect to the intersection topology.*

We remark that it is a consequence of Proposition 2.4 and the above corollary that, supposing **(LG0)**, the resolvents of L_β and $\widetilde{L}_\beta$ are strongly continuous in $\beta \in (0,\infty]$ and G, using the norm $\|G\|_0' = \|(1+|k|^{-1/2})G\|$. Indeed, it suffices to prove strong convergence on $\mathcal{D}(\sqrt{\widetilde{N}})$ where we compute

$$(L_\beta(G)-z)^{-1} - (L_{\beta'}(G')-z)^{-1} = (L_{\beta'}(G')-z)^{-1}\phi(\widetilde{G}_\beta - \widetilde{G}'_{\beta'})(L_\beta(G)-z)^{-1}.$$

Here we used [16, Lemma 3.3] to observe that $(L_\beta(G)-z)^{-1} \colon \mathcal{D}(\sqrt{\widetilde{N}}) \to \mathcal{D}(\sqrt{\widetilde{N}})$. The result now follows by observing that

$$\lim_{(\beta',G')\to(\beta,G)} \big\|\widetilde{G}_\beta - \widetilde{G}'_{\beta'}\big\| = 0.$$

Norm continuity, even in β, of the resolvent is probably false but an argument is lacking. This means that while $\sigma(L_\infty) = \mathbb{R}$, cf. [47, Thm. VIII.33], we cannot a priori exclude that the spectrum of L_β could collapse for $\beta < \infty$ to become a proper subset of $\mathbb{R}$, cf. the discussion around [47, Thm. VIII.24].

Instead we proceed as for the Hamiltonian, via a pull through formula:

Proposition 2.7. *Suppose* **(LG0)**. *For any* $\beta > 0$, $z \in \mathbb{C}\backslash\mathbb{R}$ *and* $\psi \in \mathcal{D}(\sqrt{\widetilde{N}})$ *we have as an* $L^2(\mathbb{R}\times S^2; \widetilde{\mathcal{H}}^{\mathrm{L}})$*-identity*

$$a(\omega,\Theta)(\widetilde{L}_\beta - z)^{-1}\psi = (\widetilde{L}_\beta + \omega - z)^{-1}a(\omega,\Theta)\psi - (\widetilde{L}_\beta + \omega - z)^{-1}(\widetilde{G}_\beta(\omega,\Theta)\otimes\mathbb{1}_{\widetilde{\mathcal{F}}})\psi.$$

Using this pull through formula we obtain

Theorem 2.8. *Suppose* **(LG0)***. For any* $\beta > 0$ *we have* $\sigma(L_\beta) = \sigma(\widetilde{L}_\beta) = \mathbb{R}$.

We omit proofs of Proposition 2.7 and Theorem 2.8 since they are verbatim the same as for Proposition 2.1 and Theorem 2.3, keeping in mind Corollary 2.6, in particular the consequence that resolvents of $\widetilde{L}_\beta$ preserves $\mathcal{D}(\sqrt{\widetilde{N}})$, and that $\widetilde{\mathcal{C}}^{\mathrm{L}}$ is a an operator core for $\widetilde{L}_\beta$.

The two HVZ-type theorems, Theorems 2.3 and 2.8, will play no role in the notes apart from clarifying the general spectral picture.

Remark 2.9. Our results on the standard Liouvillean will mostly be proved in the Jakšić-Pillet glued coordinates. Since only ω-derivatives will play a role, this allows us to formulate slightly weaker assumptions using $\widetilde{G}_\beta$ instead of G. This improvement is in general largely irrelevant, hence the present formulation with **(LGn)**.

More importantly, for couplings G on a special form, one can due to the representation (2.14) allow for interactions at positive temperature far more singular than what is permitted by **(LGn)**. To make this precise assume G takes the form

$$G(k) = |k|^{-1/2} g(k) G_0,$$

where $G_0 \in M_\nu(\mathbb{C})$ is self-adjoint $G_0^* = G_0$, and $g\colon \mathbb{R}^3 \to \mathbb{R}$. Define

$$\hat{g}(\omega,\Theta) = \mathbb{1}[\omega \geq 0] g(\omega\Theta) + \mathbb{1}[\omega \leq 0] g(-\omega\Theta).$$

Then we can represent $\widehat{G}_{\mathrm{l}} = \hat{g} G_0$ and $\widehat{G}_{\mathrm{r}} = \hat{g}\overline{G}_0$, cf. (2.15). Hence we see that differentiability of $\widetilde{G}_\beta$ is governed by that of $\hat{g}$. For the spin-boson model g is a form factor (or ultraviolet cutoff), e.g. constant near 0 or perhaps of the form $\mathrm{e}^{-k^2/\Lambda^2}$ to take some popular choices. Here $\hat{g}$ will be constant across the singularity at $\omega = 0$, or equal to $\mathrm{e}^{-\omega^2/\Lambda^2}$ for the other choice.

For this class of models we can reformulate replacements for **(LGn)**. Let $n \in \mathbb{N}_0$. There exists $\hat{g}\colon \mathbb{R} \times S^2 \to \mathbb{C}$ and $G_0 \in M_\nu(\mathbb{C})$ self-adjoint such that $\hat{g}$ admits n distributional ω derivatives in $L^1_{\mathrm{loc}}(\mathbb{R} \times S^2)$ and such that

$$(\mathbf{LGn}') \quad \begin{array}{l} G(\omega,\Theta) = |\omega|^{-\frac{1}{2}}\, \hat{g}(\omega,\Theta)\, G_0 \\ \forall \omega\Theta \in \mathbb{R}^3, |\omega| \leq 1, \quad \text{and} \quad j \leq n: \quad |\partial_\omega^j \hat{g}(\omega,\Theta)| \leq C|\omega|^{n-1+\mu-j} \\ \forall \omega\Theta \in \mathbb{R}^3, |\omega| \geq 1, \quad \text{and} \quad j \leq n: \quad |\partial_\omega^j \hat{g}(\omega,\Theta)| \leq C|\omega|^{-1-\delta_{j,0}-\mu}. \end{array}$$

This type of condition was used in [14, 18].

Finally we remark that it was observed and utilized in [18], that the Jakšić-Pillet gluing is not canonical in that one can glue the two reservoirs together at $\omega = 0$, twisting one of them with a phase. This allows one to consider $\hat{g}$ as complex valued and then pick the gluing phase such that $\hat{g}(\omega\Theta)$ and $\overline{\hat{g}(-\omega\Theta)}$ fit together seamlessly across $\omega = 0$. In fact, one can in this way also allow for singular behavior of the form $|k|^{1/2}$ at zero, and not just $|k|^{-1/2}$. This would require an extra twist by the angle π corresponding to a sign change across zero. $\diamondsuit$

2.4 Multiple Reservoirs

We have made a choice, in the name of concreteness to focus on finite dimensional quantum systems coupled to a massless scalar field (in three dimensions) and their thermal Liouvilleans.

Our methods and indeed theorems however have validity beyond this particular choice. We single out here the case of multiple reservoirs at possibly different inverse temperatures $\vec{\beta} = (\beta_1, \ldots, \beta_q)$.

The easiest way to observe that the results of these notes carry over to the case of multiple reservoirs is to replace $\mathfrak{h} = L^2(\mathbb{R}^3)$ by $L^2(\mathbb{R}^3 \times \{1, \ldots, q\}) \sim \mathfrak{h}^q$, q being the number of reservoirs. The dispersion becomes $\omega(k, j) = |k|$ (or $|k|\mathbb{1}_{\mathbb{C}^q}$). Given q couplings $G_1, \ldots G_q$ all satisfying the same sets of conditions, one can construct a coupling for the multi-reservoir system by setting $G(k, j) = G_j(k)$.

As for the standard Liouvillean one should replace $G^{\beta}_{\mathrm{l/r}}$ by functions $G^{\vec{\beta}}_{\mathrm{l/r}}(\cdot, j) = G^{\beta_j}_{\mathrm{l/r}}(\cdot, j)$. Similarly for $\widetilde{G}_{\vec{\beta}}$.

Weak-coupling as well as high and low-temperature results remain valid if all coupling, respectively temperatures, are taken into the same regime.

Only one type of result here does not extend to the case of multiple reservoirs, and that is the existence/non-existence results for eigenvalues of L_β discussed in Subsect. 3.4, which make critical use of the modular structure of the thermal Liouvilleans. In fact, if two inverse temperatures are distinct, at weak coupling and under a suitable non-triviality condition on G one has $\sigma_{\mathrm{pp}}(L_{\vec{\beta}}) = \emptyset$, cf. [14, Thm. 7.17].

One could also replace the thermal density ρ_β by other densities and a number of our results (except for low temperature statements) remain valid. However, the reader doing that would have to reformulate the condition **(LGn)** where the $1/|k|$ singularity of ρ_β is built in. The reader can consult [14, 15] for discussions of other models.

Finally we remark that essentially what we exploit is the Jakšić-Pillet glued representation of the standard Liouvillean and presumably one could rephrase everything in this abstract setup. See also [14, Sect. 8].

The reader can consult [34, 35, 41, 42] for papers devoted to multiple reservoirs. Three of these papers consider also the non-selfadjoint C-Liovillean, which seems to be a more natural object when considering non-stationary steady states.

2.5 Open Problems I

There are not that many serious problems pertaining to the material from this section. We did mention two related conjectures regarding the standard Liouvillean, while not in itself of great interest, resolving them would serve to clarify the picture:

Problem 2.1. Clarify to what extend the domain of the standard Liouvillean L_β is β and G dependent.

Problem 2.2. Verify that, as conjectured, the resolvents of the Liouvillean are *not* norm continuous in β and G.

As a final topic, we discuss the ultraviolet singularity of the models. For e.g. the spin-boson model, the coupling G goes as $1/\sqrt{|k|}$ for large momenta, which is more singular than what we can deal with. It is well-known that the Nelson (and the polaron) model is renormalizable, but this is due to a regularizing effect stemming from the small system, in that the Laplacian allows for control of the ultraviolet contributions [3, 38, 44]. Indeed, we do not expect that the model has a meaningful ultraviolet limit and it should not be a relevant question since it is a model describing low energy/momentum phenomena only. Having said that, it would still be undesirable if the choice of (a reasonable) cutoff would influence whether or not the Liouvillean admits non-zero eigenstates, and if it does, will the point spectrum, being related to energy differences, have an ultraviolet limit. This is an underlying issue that will not play a role in these notes since our focus will be on the more physically relevant infrared regime.

3 Bound States

In this section we study the basic properties of bound states. The key is the following formal computation

$$\langle \psi, \mathrm{i}[H,A]\psi\rangle = 0,$$

whenever ψ is a bound state for H and A is some auxiliary operator. Choosing A such that the commutator $\mathrm{i}[H,A]$ contains a positive operator N and a remainder controllable either by H or some fractional power of N, will imply - at least formally - that ψ is in the form domain of N. In our case, the operator N will be the number operator N (or N^{L} for the Liouvillean).

It turns out to be a surprisingly delicate question to establish such a bound rigorously for the standard Liouvillean, but for the Hamiltonian it is fairly straightforward. The first argument of this type is for the Hamiltonian and is due to Skibsted

[49], and for the Liouvillean it goes back to Fröhlich and Merkli [19], cf. also [20]. The result we present here for the Liouvillean improves on the theorem of Fröhlich and Merkli.

As a consequence of such number bounds we will be able to establish virial theorems for the Hamiltonian and the Liouvillean.

3.1 Number Bounds at Zero Temperature

As for A we make the choice

$$A = \mathrm{d}\Gamma(a),$$

with

$$a = \frac{\mathrm{i}}{2}\left\{\frac{k}{|k|}\cdot\nabla_k + \nabla_k\cdot\frac{k}{|k|}\right\}, \tag{3.16}$$

the generator of radial translations. Note that a should be viewed as the closure of a restricted to $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ and that a is a maximally symmetric operator, but not self-adjoint. Since H is not of class $C^1(A)$, we cannot directly make sense out of the formal computation above.

Instead we introduce a family of regularized conjugate operators $A_n = \mathrm{d}\Gamma(a_n)$ with

$$a_n = \frac{\mathrm{i}}{2}\left\{\frac{k}{\sqrt{|k|^2+n^{-1}}}\cdot\nabla_k + \nabla_k\cdot\frac{k}{\sqrt{|k|^2+n^{-1}}}\right\}.$$

The a_n's, constructed as closures from $C_0^\infty(\mathbb{R}^3)$, are self-adjoint and $H\in C^1(A_n)$ for all n, provide **(HG1)** is assumed. This construction goes back to [49] and was used also in [23].

Let ψ be a bound state for H, i.e. $H\psi = E\psi$ for some $E\in\mathbb{R}$. It is now a consequence of the standard Virial Theorem, cf. [12, 21], that $\langle\psi, \mathrm{i}[H, A_n]^\circ\psi\rangle = 0$. Computing the commutator we find

$$\mathrm{i}[H, A_n]^\circ = \mathrm{d}\Gamma\big(\frac{|k|}{\sqrt{|k|^2+n^{-1}}}\big) - \phi(\mathrm{i}a_n G).$$

Note that assuming **(HG1)** we have $a_n G\in L^2(\mathbb{R}^3; M_\nu(\mathbb{C})))$. From the estimate (2.3), applied with $|k|/\sqrt{|k|^2+n^{-1}}$ in place of $|k|$, we get

$$\mathrm{i}[H, A_n]^\circ \geq \frac{1}{2}\mathrm{d}\Gamma\big(\frac{|k|}{\sqrt{|k|^2+n^{-1}}}\big) - C\big\|(|k|^2+n^{-1})^{\frac{1}{4}}|k|^{-\frac{1}{2}}a_n G\big\|^2.$$

To check for the finiteness and uniform boundedness of the norm on the right-hand side we write $a_n = \frac{k}{\sqrt{|k|^2+n^{-1}}}\cdot \mathrm{i}\nabla_k + \frac{\mathrm{i}}{2}\,\mathrm{div}(k/\sqrt{k^2+n^{-1}})$ and estimate

$$\big\|(|k|^2+n^{-1})^{\frac{1}{4}}|k|^{-\frac{1}{2}}a_n G\big\| \leq \big\|\nabla G\big\| + C\big\|G/|k|\big\|, \tag{3.17}$$

for some n-independent constant C. Since $n \to |k|/\sqrt{|k|^2 + n^{-1}}$ is monotonously increasing towards 1 we conclude from Lebesgue's theorem on monotone convergence the following:

Theorem 3.1. *Suppose* **(HG1)**. *There exists a $C > 0$ such that for any normalized bound state $\psi \in \mathcal{H}$ of H we have $\psi \in \mathcal{D}(\sqrt{N})$ and*

$$\big\|\sqrt{N}\psi\big\| \le C\big(\|\nabla G\| + \|G/|k|\|\big).$$

Since N commutes with the conjugation $\mathbf{C}$, we observe that the same theorem holds for bound states of $H^{\mathbf{c}}$.

That the constant C in the theorem above can be chosen uniformly in E, is a consequence of $\mathcal{K}$ being finite dimensional. For e.g. the confined Nelson model, this would be false since one will need a resolvent of K to bound the relevant aG. This is however a mute point, since we in Subsect. 4.3 will prove that H does not have high energy bound states!

In fact if one assumes in addition **(HG2)** one can do better and get $\psi \in \mathcal{D}(N)$ using [16]. This is however a much deeper result and will not play a role in these notes.

3.2 Number Bounds at Positive Temperature

For the standard Liouvillean L_∞ at zero temperature we observe that since eigenstates are on the form $\psi \otimes \varphi$, with ψ, φ eigenstates of H and $H^{\mathbf{c}}$ respectively, they are due to Theorem 3.1 automatically in the domain of $\mathcal{D}(\sqrt{N^{\mathrm{L}}})$. Hence, eigenstates of $\widetilde{L}_\infty$ are in the domain of $\sqrt{\widetilde{N}}$. The situation at positive temperature is a good deal more subtle.

We begin with a key technical lemma, which enables us to compute commutators. The proof follows closely a similar argument from [16, Proof of Cond. 2.1 (3)].

Before stating the lemma we need some notation. Let $m \in C^1(\mathbb{R})$ be real-valued and bounded with bounded derivative. Put

$$\tilde{a}_m = \frac{\mathrm{i}}{2}\left\{m\frac{\mathrm{d}}{\mathrm{d}\omega} + \frac{\mathrm{d}}{\mathrm{d}\omega}m\right\} \otimes 1\!\!1_{L^2(S^2)} \tag{3.18}$$

and

$$\widetilde{A}_m = 1\!\!1_{\mathcal{K}\otimes\mathcal{K}} \otimes \mathrm{d}\Gamma(\tilde{a}_m). \tag{3.19}$$

We leave it to the reader to argue that $\tilde{a}_m$ and $\widetilde{A}_m$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}) \otimes C^\infty(L^2(S^2))$ and $\widetilde{\mathcal{C}}^{\mathrm{L}}$ respectively.

Lemma 3.2. *Suppose* **(LG1)**. *Then*

$$\langle \psi, \mathrm{i}[\widetilde{L}_\beta, (\widetilde{A}_m - z)^{-1}]\varphi \rangle = -\langle \psi, (\widetilde{A}_m - z)^{-1}\widetilde{L}'_\beta(\widetilde{A}_m - z)^{-1} \rangle,$$

for all $\psi, \varphi \in \mathcal{D}(\sqrt{\widetilde{N}}) \cap \mathcal{D}(\widetilde{L}_\beta)$ *and* $z \in \mathbb{C}$ *with* $\mathrm{Im}z \neq 0$. *Here*

$$\widetilde{L}'_\beta = \mathbb{1}_\mathcal{K} \otimes \mathrm{d}\Gamma(m) - \phi(\mathrm{i}\tilde{a}_m \widetilde{G}_\beta)$$

defined as a form on $\mathcal{D}(\sqrt{\widetilde{N}})$.

Remark 3.3. We first observe that the expression in the lemma makes sense. Since $\widetilde{A}_m$ and $\widetilde{N}$ commute, $(\widetilde{A}_m - z)^{-1}$ preserves the domain of $\sqrt{\widetilde{N}}$. By boundedness of m and m' together with Remark 2.5 we see that $\widetilde{L}'_\beta$ is well-defined as a form on $\mathcal{D}(\sqrt{\widetilde{N}})$.

Proof. By [16, Remark 3.5], it suffices to check the identity for $\psi, \varphi \in \mathcal{D}(\widetilde{L}_\beta) \cap \mathcal{D}(\widetilde{N})$, which by Corollary 2.6 equals $\mathcal{D}(\widetilde{L}_0) \cap \mathcal{D}(\widetilde{N})$. On this domain $\widetilde{L}_\beta$ can be written as the operator sum $\widetilde{L}_0 + \phi(\widetilde{G}_\beta)$. Hence it suffices to prove that

$$\begin{aligned} \langle \psi, \mathrm{i}[\widetilde{L}_0, (\widetilde{A}_m - z)^{-1}]\varphi \rangle &= -\langle \psi, (\widetilde{A}_m - z)^{-1} \mathbb{1}_\mathcal{K} \otimes \mathrm{d}\Gamma(m)(\widetilde{A}_m - z)^{-1}\varphi \rangle, \\ \langle \psi, \mathrm{i}[\phi(\widetilde{G}_\beta), (\widetilde{A}_m - z)^{-1}]\varphi \rangle &= \langle \psi, (\widetilde{A}_m - z)^{-1}\phi(\mathrm{i}\tilde{a}_m\widetilde{G}_\beta)(\widetilde{A}_m - z)^{-1}\varphi \rangle, \end{aligned} \tag{3.20}$$

for all $\psi, \varphi \in \mathcal{D}(\widetilde{L}_0) \cap \mathcal{D}(\widetilde{N})$ and $z \in \mathbb{C}$ with $\mathrm{Im}z \neq 0$. The second identity in (3.20) can easily be verified for $\psi, \varphi \in \widetilde{\mathcal{C}}^{\mathrm{L}}$ from which it extends by density since $\phi(\widetilde{G}_\beta)$ and $\phi(\mathrm{i}\tilde{a}_m\widetilde{G}_\beta)$ are $\sqrt{\widetilde{N}}$-bounded, cf. Remark 2.5.

As for the first identity in (3.20) one should first observe that all objects preserve particle sectors, i.e. sectors with $\widetilde{N} = n$ for some n. Hence it suffices to establish the identity for ψ, φ being n-particle states. Observe that $\mathrm{d}\Gamma^{(n)}(\omega)$ is of class $C^1_{\mathrm{Mo}}(\mathrm{d}\Gamma^{(n)}(\tilde{a}_m))$, indeed; $\mathrm{i}[\mathrm{d}\Gamma^{(n)}(\omega), \mathrm{d}\Gamma^{(n)}(\tilde{a}_m)]^\circ = \mathrm{d}\Gamma^{(n)}(m)$ is a bounded operator on $\mathcal{F}^{(n)}$, the n-particle sector. Hence the identity

$$\begin{aligned} &\langle \psi, \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \mathrm{i}[\mathrm{d}\Gamma^{(n)}(\omega), (\mathrm{d}\Gamma^{(n)}(\tilde{a}_m) - z)^{-1}]\varphi \rangle \\ &= -\langle \psi, \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes (\mathrm{d}\Gamma^{(n)}(\tilde{a}_m) - z)^{-1}\mathrm{d}\Gamma^{(n)}(m)(\mathrm{d}\Gamma^{(n)}(\tilde{a}_m) - z)^{-1}\varphi \rangle \end{aligned}$$

holds for z with $|\mathrm{Im}z| \geq \sigma_n$ for some σ_n chosen such that $(\mathrm{d}\Gamma^{(n)}(\tilde{a}_m) - z)^{-1}$ preserves $\mathcal{D}(\mathrm{d}\Gamma^{(n)}(\omega))$ inside the n-particle sector, cf. [43, Prop. II.3]. By the unique continuation theorem, the identity then holds for all z with $\mathrm{Im}z \neq 0$. □

We are now ready to state and prove our improvement of the Fröhlich-Merkli number bound. For comparison, we require one less commutator reflected in an improvement by one power of $|k|$ in the infrared behavior of G. It is however still one commutator more than what was needed for the Hamiltonian. It is unclear if this is just a technical issue.

Theorem 3.4. *Suppose* **(LG2)***. Let* ψ *be an eigenstate of* L_β*. Then* $\psi \in \mathcal{D}(\sqrt{N^{\mathrm{L}}})$.

Proof. Let $\psi \in \mathcal{D}(\widetilde{L}_\beta)$ be an eigenstate for $\widetilde{L}_\beta$. It suffices to prove that $\psi \in \mathcal{D}(\widetilde{N}^{1/2})$. We can assume without loss of generality that the eigenvalue is zero, i.e. $\widetilde{L}_\beta\psi = 0$.

Denote by $\tilde{a} = \tilde{a}_{m\equiv 1}$ the generator of translations, cf. (3.18). Similarly we abbreviate $\widetilde{A} = \widetilde{A}_{m\equiv 1} = \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \mathrm{d}\Gamma(\tilde{a})$. Note that $\widetilde{A}$ commutes with $\widetilde{N}$.

Since $\widetilde{N}$ by Corollary 2.6 is of class $C^1(\widetilde{L}_\beta)$ we have $I_n(\widetilde{N})\psi \in \mathcal{D}(\widetilde{L}_\beta) \cap \mathcal{D}(\widetilde{N})$, cf. [43, Prop. II.2]. Here

$$I_n(\widetilde{N}) = n(\widetilde{N}+n)^{-1} \quad \text{satisfies } \mathrm{s}-\lim_{n\to\infty} I_n(\widetilde{N}) = \mathbb{1}_{\widetilde{\mathcal{H}}^{\mathrm{L}}}. \tag{3.21}$$

Put $\psi_n = I_n(\widetilde{N})\psi$. With the choice $m = 1$ we get

$$\widetilde{L}'_\beta = \widetilde{N} - \phi(\mathrm{i}\tilde{a}\widetilde{G}_\beta),$$

as a self-adjoint operator with domain $\mathcal{D}(\widetilde{N})$. We can thus compute using Lemma 3.2 for $m \in \mathbb{N}$ and $z \in \mathbb{C}$, with $\mathrm{Im} z \neq 0$,

$$\begin{aligned}
&\langle \psi_n, \mathrm{i}[\widetilde{L}_\beta, (\widetilde{A}/m - z)^{-1}]\psi_n\rangle \\
&= -\frac{1}{m}\langle \psi_n, (A/m-z)^{-1}\widetilde{L}'_\beta(\widetilde{A}/m-z)^{-1}\psi_n\rangle \\
&= \frac{1}{m}\langle \psi_n, \widetilde{L}'_\beta(\widetilde{A}/m-z)^{-2}\psi_n\rangle \\
&\quad - \frac{\mathrm{i}}{m^2}\langle \psi_n, (\widetilde{A}/m-z)^{-1}\phi(\tilde{a}^2\widetilde{G}_\beta)(\widetilde{A}/m-z)^{-2}\psi_n\rangle.
\end{aligned}$$

In the last equality we used that $\widetilde{L}'_\beta$ is of class $C^1_{\mathrm{Mo}}(\widetilde{A})$ with $[\widetilde{L}'_\beta, \widetilde{A}]^\circ = \mathrm{i}\phi(\tilde{a}^2\widetilde{G}_\beta)$. On the other hand we can undo the commutator on the left-hand side and commute $\widetilde{L}_\beta$ through $I_n(\widetilde{N})$ to get

$$\begin{aligned}
&\langle \psi_n, \mathrm{i}[\widetilde{L}_\beta, (\widetilde{A}/m-z)^{-1}]\psi_n\rangle \\
&\quad = -\langle \psi_n, \{\phi(\mathrm{i}\widetilde{G}_\beta)(\widetilde{N}+n)^{-1}(\widetilde{A}/m-z)^{-1} + (\widetilde{A}/m-z)^{-1}(\widetilde{N}+n)^{-1}\phi(\mathrm{i}\widetilde{G}_\beta)\}\psi_n\rangle.
\end{aligned}$$

Here we used Corollary 2.6 and a twidled version of (2.11).

Let $g \in C^\infty(\mathbb{R})$ be identical to t for $|t| \leq 1$, monotonously increasing and constant outside a ball of radius 2. Suppose in addition that $\sqrt{g'}$ is smooth. We will furthermore require that

$$\forall t \in \mathbb{R}: \quad t g''(t) \leq 0. \tag{3.22}$$

Let $\tilde{g}$ denote an almost analytic extension of g. Abbreviating $g_m(t) = mg(t/m)$ we get

$$\begin{aligned}
&- \big\langle \psi_n, \big\{\phi(\mathrm{i}\widetilde{G}_\beta)(\widetilde{N}+n)^{-1} g_m(\widetilde{A}) + g_m(\widetilde{A})(\widetilde{N}+n)^{-1}\phi(\mathrm{i}\widetilde{G}_\beta)\big\}\psi_n\big\rangle \\
&\quad = \big\langle \psi_n, \widetilde{L}_\beta' g_m'(\widetilde{A})\psi_n\big\rangle \\
&\qquad - \frac{\mathrm{i}}{m\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{g}(z)\big\langle \psi_n, (\widetilde{A}/m - z)^{-1}\phi(\tilde{a}^2\widetilde{G}_\beta)(\widetilde{A}/m-z)^{-2}\psi_n\big\rangle \mathrm{d}z.
\end{aligned}$$

We estimate the left-hand side to be $O(m/\sqrt{n})$ and the second term on the right-hand side is $O(\sqrt{n}/m)$, cf. Remark 2.5. Hence we get

$$\big|\big\langle \psi_n, \widetilde{L}_\beta' g_m'(\widetilde{A})\psi_n\big\rangle\big| \le C\Big(\frac{m}{\sqrt{n}} + \frac{\sqrt{n}}{m}\Big) \tag{3.23}$$

for some $C > 0$. Put $h(t) = \sqrt{g'(t)}$. Then $h \in C_0^\infty(\mathbb{R})$ and defining $h_m(t) = h(t/m)$ we find $h_m(\widetilde{A})^2 = g_m'(\widetilde{A})$. Let $\tilde{h}$ be an almost analytic extension of h. Then

$$\big\langle \psi_n, \widetilde{L}_\beta' g_m'(\widetilde{A})\psi_n\big\rangle = \big\langle \psi_n, h_m(\widetilde{A})\widetilde{L}_\beta' h_m(\widetilde{A})\psi_n\big\rangle + \big\langle \psi_n, [\widetilde{L}_\beta', h_m(\widetilde{A})] h_m(\widetilde{A})\psi_n\big\rangle.$$

Observe that $[h_m(\widetilde{A}), \widetilde{L}_\beta'](\widetilde{N}+1)^{-1/2} = [h_m(\widetilde{A}), \phi(\mathrm{i}\tilde{a}\widetilde{G}_\beta)](\widetilde{N}+1)^{-1/2}$ is of the order $1/m$ and hence

$$\big\langle \psi_n, \widetilde{L}_\beta' g_m'(\widetilde{A})\psi_n\big\rangle = \big\langle \psi_n, h_m(\widetilde{A})\widetilde{L}_\beta' h_m(\widetilde{A})\psi_n\big\rangle + O\left(\frac{\sqrt{n}}{m}\right). \tag{3.24}$$

Finally using that $\widetilde{L}_\beta' \ge \widetilde{N}/2 - C'$, for some $C' > 0$, we get from (3.23) and (3.24) that

$$\big\langle \psi_n, h_m(\widetilde{A})\widetilde{N} h_m(\widetilde{A})\psi_n\big\rangle \le C\left(\frac{m}{\sqrt{n}} + \frac{\sqrt{n}}{m}\right)$$

for some $C > 0$ and all $n, m \ge 1$.

We now pick $n = m^2$ such that we obtain the bound

$$\big\langle \psi, h_m(\widetilde{A}) I_{m^2}(\widetilde{N})^2 \widetilde{N} h_m(\widetilde{A})\psi\big\rangle \le 2C$$

uniformly in m.

Let $E^{(\widetilde{N},\widetilde{A})}$ be the joint spectral resolution on $\mathbb{N}_0 \times \mathbb{R}$ induced by the two commuting operators $\widetilde{N}$ and $\widetilde{A}$. Then

$$\big\langle \psi, h_m(\widetilde{A}) I_{m^2}(\widetilde{N})^2 \widetilde{N} h_m(\widetilde{A})\psi\big\rangle = \int_{\mathbb{N}_0\times\mathbb{R}} h_m(t)^2 \frac{nm^4}{(n+m^2)^2}\, \mathrm{d}E_\psi^{(\widetilde{N},\widetilde{A})}(n,t).$$

Since $h_m(t)^2 \frac{nm^4}{(n+m^2)^2} \to n$ monotonously as $m \to \infty$ we conclude using the monotone convergence theorem that $\int_{\mathbb{N}_0\times\mathbb{R}} n \, \mathrm{d}E_\psi^{(\widetilde{N},\widetilde{A})}(n,t) < \infty$. Here we used (3.22) to ensure that $m \to h_m(t)$ is monotonously increasing towards 1. Being a joint spectral resolution we have

$$\int_{\mathbb{N}_0\times\mathbb{R}} n \, \mathrm{d}E_\psi^{(\widetilde{N},\widetilde{A})}(n,t) = \int_{\mathbb{N}_0} n \, \mathrm{d}E_\psi^{\widetilde{N}}(n),$$

where $E^{\widetilde{N}}$ is the spectral resolution for $\widetilde{N}$. Hence $\langle \psi, \widetilde{N}\psi\rangle < \infty$ and we are done. □

3.3 Virial Theorem's

Having established the number bounds we can now formulate and prove two virial theorems

Let $m \in C^1(\mathbb{R})$ be real-valued and bounded with bounded derivative as in the previous subsection. Given such a function m we can construct a maximally symmetric operator by the prescription

$$a_m = \frac{\mathrm{i}}{2}\left\{\frac{m(|k|)k}{|k|} \cdot \nabla_k + \nabla_k \cdot \frac{m(|k|)k}{|k|}\right\}$$

on $C_0^\infty(\mathbb{R}^3\backslash\{0\})$. This gives rise to a maximally symmetric operator $A_m = \mathrm{d}\Gamma(a_m)$ on $\mathcal{H}$. We write, supposing **(HG1)**,

$$H_m' = \mathrm{d}\Gamma(m(|k|)) - \phi(\mathrm{i}a_m G),$$

as a form on $\mathcal{D}(\sqrt{N})$, cf. (2.2). If $\inf m(\omega) > 0$, then H' is self-adjoint on $\mathcal{D}(N)$. We have

Theorem 3.5. *Suppose* **(HG1)**. *Let $\psi \in \mathcal{H}$ be a bound state for the Hamiltonian H. Then $\langle \psi, H_m'\psi\rangle = 0$.*

Proof. Note that the expectation value is meaningful due to the number bound in Theorem 3.1.

Replace m by a regularizing function $m_n(r) = m(r)r/\sqrt{r^2+n^{-1}}$ as in the proof of the number bound in Subsect. 3.1. Then the associated a_{m_n} is self-adjoint and so is $A_{m_n} = \mathrm{d}\Gamma(a_{m_n})$. Furthermore, H is of class $C^1(A_{m_n})$ for all n. We can compute the commutator $\mathrm{i}[H, A_{m_n}]^\circ = \mathrm{d}\Gamma(m_n(|k|)) - \phi(\mathrm{i}a_{m_n}G)$. By the usual virial theorem, cf. [12, 21], together with (3.17), Theorem 3.1 and Lebesgue's dominated convergence theorem we conclude the proof. □

To deal with the standard Liouvillean we use the observables $\tilde{a}_m$ and $\widetilde{A}_m$ from (3.18) and (3.19). Write, supposing now **(LG1)**,

$$\widetilde{L}'_\beta = 1\!\!1_{\mathcal{K}\otimes\mathcal{K}} \otimes \,\mathrm{d}\Gamma(m(\omega)) - \phi(\mathrm{i}\tilde{a}_m \widetilde{G}_\beta),$$

which by Remark 2.5 is a well-defined form on $\mathcal{D}(\sqrt{\widetilde{N}})$. Again, if $\inf m(\omega) > 0$, then $\widetilde{L}'_\beta$ is self-adjoint on $\mathcal{D}(\widetilde{N})$. Define

$$L'_\beta = \mathcal{U}^* \widetilde{L}'_\beta \mathcal{U},$$

which is a well-defined form on $\mathcal{D}(\sqrt{N^{\mathrm{L}}})$. Under the **(LG1)** assumption we can compute

$$L'_\beta = 1\!\!1_{\mathcal{K}\otimes\mathcal{K}} \otimes \big(\mathrm{d}\Gamma(m(|k|)\otimes 1\!\!1_{\mathcal{F}} + 1\!\!1_{\mathcal{F}} \otimes \mathrm{d}\Gamma(m(|k|)\big) - \phi_{\mathrm{l}}(\mathrm{i}a_m G^\beta_{\mathrm{l}}) + \phi_{\mathrm{r}}(\mathrm{i}a_m G^\beta_{\mathrm{r}}).$$

We warn the reader that if one follows Remark 2.9 and imposes an **(LG1')** assumption instead of **(LG1)**, then $a_m G^\beta_{\mathrm{l/r}}$ may not be well-defined.

Theorem 3.6. *Suppose* **(LG2)**. *Let $\psi \in \mathcal{H}^{\mathrm{L}}$ be a bound state for the standard Liouvillean L_β, at inverse temperature $0 < \beta \leq \infty$. Then $\langle \psi, L'_\beta \psi\rangle = 0$.*

Proof. First of all we note that the expectation value is meaningful due to Theorem 3.4. Secondly, it suffices to prove the theorem in the glued coordinates where L'_β is replaced by $\widetilde{L}'_\beta$ and ψ is an eigenstate for $\widetilde{L}_\beta$.

Let ψ be a bound state for $\widetilde{L}_\beta$. Using the notation $I_n(\widetilde{A}_m) = \mathrm{i}n(\widetilde{A}_m + \mathrm{i}n)^{-1}$, cf. also (3.21), we write

$$B_n = \widetilde{A}_m I_n(\widetilde{A}_m) = \mathrm{i}n 1\!\!1_{\widetilde{\mathcal{H}}^{\mathrm{L}}} + n^2(\widetilde{A}_m + \mathrm{i}n)^{-1}.$$

Then B_n is bounded for all n and $\lim_{n\to\infty} I_n(\widetilde{A}_m) = 1\!\!1_{\widetilde{\mathcal{H}}^{\mathrm{L}}}$. We compute using Lemma 3.2 as a form on $\mathcal{D}(\sqrt{\widetilde{N}})$

$$0 = \big\langle \psi, \mathrm{i}[\widetilde{L}_\beta, B_n]\psi\big\rangle = \big\langle \psi, I_n(\widetilde{A}_m)\widetilde{L}'_\beta I_n(\widetilde{A}_m)\psi\big\rangle.$$

Since $\widetilde{N}$ commutes with $I_n(\widetilde{A}_m)$, we can - keeping Theorem 3.4 in mind - take the limit $n \to \infty$ and conclude the theorem. □

The theorem of course remains true if we pass to the Jakšić-Pillet glued operator $\widetilde{L}_\beta$. While the proof given above is at least formally identical to a standard proof of the usual virial theorem, the reader should keep in mind that it relies on the non-trivial Lemma 3.2 and Theorem 3.4.

The virial theorem's are the tools that will allow us to deduce statements about non-existence, local finiteness, and finite multiplicity, for eigenvalues given a so called positive commutator estimate. This is the subject of Sect. 4

3.4 A Review of Existence and Non-existence Results

The first theorem we highlight is due to Gérard [24, Thm. 1] and establishes existence of a ground state for the Hamiltonian H under an **(HG1)** condition. Subsequently some improvements appeared in [10, 45].

Theorem 3.7. *Assume* **(HG1)***. Then the bottom of the spectrum* Σ *of* H *is an eigenvalue.*

In a somewhat surprising recent development Hasler and Herbst proved that the Spin-Boson model, cf. Remark 2.9, admits a ground state if the coupling is sufficiently weak [28]. They used the renormalization group method of Bach, Fröhlich and Sigal [7]. See also Problems 3.3 and 3.4 in the following subsection.

The following beautiful theorem, due to Derezinski, Jaksic and Pillet establishes the existence of a β-KMS vector, which is in particular an eigenvector of L_β with eigenvalue zero. See [14, Thm. 7.3] and [15, Appendix B]. This improves on an earlier result of Bach, Fröhlich and Sigal [8, Thm. IV.3], who required more infrared regularity.

Theorem 3.8. *Suppose* **(LG0)***. Then for any inverse temperature* $0 < \beta < \infty$ *the standard Liouvillean* L_β *has a* β*-KMS vector sitting in the kernel.*

It is worth noting that although the above theorem mirrors Gérard's result for the Hamiltonian, it holds true for more singular interactions. In particular, one can not rule out a situation where H has no ground state, but L_β has a β-KMS vector in its kernel. Indeed, this situation actually occurs in the $\nu = 1$ case. Here the Pauli-Fierz Hamiltonian is of the type considered by Dereziński in [11], where it is referred to as a van Hove Hamiltonian. If we consider

$$G(k) = |k|^{-\frac{1}{2}}\hat{g}(k),$$

with $\hat{g} \in C_0^\infty(\mathbb{R}^3)$ real-valued playing the role of an ultraviolet cutoff. We put $\hat{g}(0) = 1$ such that the infrared behavior is captured by $|k|^{-1/2}$. It satisfies **(LG0)** needed for Theorem 3.8, but not **(HG1)** needed for Theorem 3.7.

With this coupling the Hamiltonian becomes of infrared type II, again referring to the terminology of [11], and does not admit a ground state. The ground state should be the coherent state $\mathrm{e}^{\mathrm{i}\phi(\mathrm{i}|k|^{-3/2}\hat{g})}|0\rangle$, but this is not in the Fock-space since $|k|^{-3/2}\hat{g} \notin L^2(\mathbb{R}^3)$. To see what happens with the standard Liouvillean we observe that for $\nu = 1$ (and real $\hat{g}$) we have

$$G_\mathrm{l}^\beta = G_\mathrm{r}^\beta = \big(\sqrt{1+\rho_\beta} - \sqrt{\rho_\beta}\big)|k|^{-\frac{1}{2}}\hat{g}.$$

Expanding ρ_β around $k = 0$ we see that $\sqrt{1+\rho_\beta} - \sqrt{\rho_\beta} \sim \sqrt{\beta|k|}/2$. Hence

$$G^\beta_{\mathrm{l/r}} \sim \frac{\sqrt{\beta}}{2} \tag{3.25}$$

at $k = 0$. Hence we can diagonalize the Liouvillean with a tensor product of Weyl operators as follows. Put

$$V = \mathrm{e}^{\mathrm{i}\phi(\mathrm{i}|k|^{-1}G^\beta_\mathrm{l})} \otimes \mathrm{e}^{\mathrm{i}\phi(\mathrm{i}|k|^{-1}G^\beta_\mathrm{r})},$$

which due to (3.25) is a well-defined unitary operator. Then $V^* L_\beta V = L_0$ and $V(|0\rangle \otimes |0\rangle)$ is the only eigenstate and in particular the β-KMS state. Note that the energy shift one gets for the Hamiltonian does not occur here, since the shift from the left and right components cancel each other out.

The final result we discuss in this subsection is a consequence of Theorem 3.8 and a theorem of Jadczyk [31], which has as a consequence that existence and simplicity of the 0 eigenvalue for the standard Liouvillean implies non-existence of non-zero eigenvalues! We refer the reader to the short and very elegant paper [33] for details, which are entirely operator algebraic in nature.

Theorem 3.9. *Suppose* **(LG0)**. *Let $0 < \beta < \infty$ and suppose that 0 is a simple eigenvalue for L_β. Then $\sigma_{\mathrm{pp}}(L_\beta) = \{0\}$.*

3.5 Open Problems II

As the reader may have observed, the bottleneck for applying the virial theorem to the standard Liouvillean is the number bound Theorem 3.4, where we compared with the Hamiltonian case Theorem 3.1 need much stronger assumptions. This is in particular unfortunate since the positive commutator estimates we establish in the following section hold under an **(LG1)** assumption, not the **(LG2)** assumption needed for the number bound.

Problem 3.1. Can the number bound in Theorem 3.4 be established under an **(LG1)** condition, or some other condition truly weaker than **(LG2)**.

The author does not know one way or the other what the answer may be this problem. We remark that although, the number bound is a bottleneck viz a viz the structure of the point spectrum, the **(LG2)** condition is what one would expect for a limiting absorption principle to hold, given a positive commutator estimate. Hence, from a broader picture the **(LG2)** condition will appear anyway.

The proof of the number bound Theorem 3.4 did not make essential use of the small system being finite dimensional. Hence we expect the theorem to remain true also for confined small systems, like the standard Liouvillean for the confined Nelson model.

Problem 3.2. Extend Theorem 3.4 to the case where the small system $\mathcal{K}$ is not necessarily finite dimensional.

As mentioned in Subsect. 3.4, Hasler and Herbst established in [28] the existence of an interacting ground state for the spin-boson model with physical infrared singularity $|k|^{-1/2}$, provided the coupling is sufficiently weak. This result came as a complete surprise to the author, since it is contrary to the solvable model with $\mathcal{K} = \mathbb{C}$ and the confined Nelson model [10, 11, 30, 39, 46]. Furthermore it goes beyond what was considered the natural borderline established in [24], cf. also [6, 10, 45]. In fact there has been speculation that gauge invariance of the minimally coupled model was responsible for the existence result of Griesemer-Lieb-Loss [26, 37], something that was however debunked by Hasler-Herbst [27, 29] who proved that existence of a ground state, at weak coupling, remains true even after dropping the quadratic term in the minimally coupled model, thus breaking gauge invariance.

The $|k|^{-1/2}$ infrared behavior of G is sometimes called the "ohmic case", a terminology we use below.

Problem 3.3. Does there exists a critical coupling at which the ground state seize to exist for the spin-boson model considered by Hasler and Herbst? Or does a ground state exist for all couplings?

Problem 3.4. Characterize the properties of ohmic G that ensures existence of a ground state for H in the weak coupling regime. As a simpler problem, consider G's on the form $G(k) = |k|^{-1/2}\hat{g}(k)G_0$ as discussed in Remark 2.9.

For the thermal standard Liouvillean, one has existence of a β-KMS vector in the kernel of L_β at all values of β, cf. Theorem 3.8, and furthermore the modular structure ensures that a simple 0-eigenvalue implies absence of non-zero eigenvalues, cf. Theorem 3.9. These results were derived from the underlying algebraic structure of standard Liouvilleans, and may not have natural operator theoretic proofs. It would be natural to ask if it is not possible to extract even more information from the underlying algebraic framework.

Problem 3.5. Can one exploit the underlying algebraic structure to infer more information on the point spectrum and pertaining eigenstates, than what is afforded by Theorem 3.9. In particular, can one use algebraic arguments to conclude that zero is at most a simple eigenvalue of L_β?

It is well known that establishing instability or outright absence of embedded eigenvalues away from zero coupling, or some other explicitly solvable regime, is a daunting task. It is for example not known if embedded (necessarily negative) eigenvalues of N-body Schrödinger operators are unstable under perturbations of pair-potentials. One can only show generic instability under perturbations by external potentials cf. [2, 1]. In [17] a Fermi Golden Rule was established at arbitrary coupling for the Hamiltonian, but to conclude instability one needs better control of eigenstates beyond the ground state (where Perron-Frobenius theory applies). The case of perturbation around zero coupling is far better understood [7, 8, 9, 13, 14, 18, 25, 40]. Hence, whether or not the kernel of the standard Liouvillean is generically one-dimensional beyond the weak-coupling regime is not a question one is likely to answer using perturbation theory of embedded eigenvalues only.

We stress that we consider Problem 3.5 to be the most important problem highlighted in these notes. The reason being that, due to Theorem 3.9, it reduces the question of establishing return to equilibrium beyond the weak coupling regime to positive commutator estimates and limiting absorption principles. Something we see no fundamental obstacle to obtaining, although the picture is not yet entirely clear. See Subsect. 4.5.

Finally, it would be natural, in the spirit of [11], to investigate the types of ultraviolet and infrared behavior of the standard Liouvillean when $\nu = 1$, which is a solvable case. See also the discussion on ground states versus β-KMS states when $\nu = 1$ in the previous subsection, which indicates that the infrared type II property, cf. [11], characterizes existence of β-KMS states.

Problem 3.6. Classify possible types of ultraviolet and infrared behavior of the "van Hove Liouvillean", i.e. when $\nu = 1$.

4 Commutator Estimates

4.1 The Weak Coupling Regime

The weak coupling regime is very well understood. To explore it we replace G by λG, where $\lambda \in \mathbb{R}$ is small in norm. In fact, obtaining positive commutator estimates in this regime is an easy exercise. Indeed, choosing a to be generator of radial translation (3.16) we get using (2.2)

$$H' = N - \lambda\phi(\mathrm{i}aG) \geq \frac{1}{2}N - 4\lambda^2\|aG\|^2.$$

Choosing λ such that $4\lambda^2\|aG\|^2 \leq 1/4$ yields

$$H' \geq \frac{1}{4}\mathbb{1}_{\mathcal{H}} - \frac{1}{4}\mathbb{1}_{\mathcal{K}} \otimes |0\rangle\langle 0|. \tag{4.26}$$

We can now prove

Corollary 4.1. *Suppose* **(HG1)**. *There exists $\lambda_0 > 0$ such that for $\lambda \in [-\lambda_0, \lambda_0]$ the pure point spectrum $\sigma_{\rm pp}(H)$ is finite and all eigenvalues have finite multiplicity. (Here H is defined with G replaced by λG.)*

Proof. Let $\lambda_0 > 0$ be such that (4.26) is satisfied for $|\lambda| \leq \lambda_0$. Assume towards a contradiction that there exists an enumerable sequence ψ_n of normalized eigenstates. From (4.26) and Theorem 3.5 we find that

$$\langle \psi_n, \mathbb{1}_{\mathcal{K}} \otimes |0\rangle\langle 0|\psi_n\rangle \geq 1.$$

This is a contradiction since ψ_n converges weakly to zero and $\mathbb{1}_{\mathcal{K}} \otimes |0\rangle\langle 0|$ is a compact operator. Recall that $\mathcal{K}$ was assumed finite dimensional. ☐

Similarly for the Liouvillean where we can again choose $\tilde{a} = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\omega} \otimes \mathbb{1}_{L^2(S^2)}$ to be the generator of translations in the glued variable. Then

$$\widetilde{L}'_\beta = \widetilde{N}^{\mathrm{L}} - \lambda\phi(\mathrm{i}\tilde{a}\widetilde{G}_\beta) \geq \frac{1}{2}\widetilde{N}^{\mathrm{L}} - 4\lambda^2\big\|\tilde{a}\widetilde{G}_\beta\big\|^2.$$

Hence choosing $|\lambda| \leq \lambda_0 < (2\|\tilde{a}\widetilde{G}_\beta\|)^{-2}$ we arrive at

Corollary 4.2. *Suppose* **(LG2)**. *There exists $\lambda_0 > 0$ such that for $\lambda \in [-\lambda_0, \lambda_0]$ the pure point spectrum $\sigma_{\rm pp}(L_\beta)$ is finite and all eigenvalues have finite multiplicity. (Here L_β is defined with G replaced by λG.)*

Proof. The proof is identical to the proof of Corollary 4.1, except we make use of Theorem 3.6 instead of Theorem 3.5. ☐

This theorem improves on a result of Merkli [40] due to the improvement in the number bound Theorem 3.4. See also [19, 18, 20].

4.2 Conjugate Operators

Let $\chi \in C_0^\infty(\mathbb{R})$ satisfy $\chi(-\omega) = \chi(\omega)$, $0 \leq \chi \leq 1$, $\chi(\omega) = 1$ for $|\omega| \leq 1/2$ and $\chi(\omega) = 0$ for $|\omega| > 1$.

Let $\mu > 0$ be the constant used to define the class of couplings we can treat, cf. **(HGn)** and **(LGn)**. We use it to construct an auxiliary function $d\colon (0,\infty) \to [1,\infty)$ as follows

$$d(\omega) = \chi(\omega)\omega^{-\mu/4} + \chi(\omega/2) - \chi(\omega) + (1 - \chi(\omega/2))\omega^{\mu/4}.$$

We leave it to the reader to verify the following properties of d

(d1) $(\omega - 1)d'(\omega) \geq 0$.

(d2) $\lim_{\omega\to 0_+} d(\omega) = \lim_{\omega\to+\infty} d(\omega) = +\infty$.

(d3) $\exists C > 0$ s.t. $|d'(\omega)| \le C d(\omega)/\omega$ for all $\omega > 0$.

We extend d to $\mathbb{R}\backslash\{0\}$ be setting $d(\omega) = d(-\omega)$ for $\omega < 0$.

For a given

$$\underline{\delta} = (\delta_0, \delta_\infty) \in (0,1] \times [1,\infty) =: \Delta_0 \tag{4.27}$$

we define a smooth positive function $m_\delta \colon \mathbb{R} \to [1,\infty)$ by

$$\begin{aligned} m_{\underline{\delta}}(\omega) = {} & d(\delta_0)\chi(\omega/\delta_0) \\ & + d(\omega)\big(\chi(\omega/(2\delta_\infty)) - \chi(\omega/\delta_0)\big) \\ & + d(\delta_\infty)\big(1 - \chi(\omega/(2\delta_\infty))\big). \end{aligned}$$

Observe that $m_{\underline{\delta}}$ has bounded and compactly supported derivatives.

Our conjugate operator on the one-particle level for the Hamiltonian at zero temperature is defined as the modified generator of radial translations

$$a_{\underline{\delta}} = \frac{\mathrm{i}}{2}\left\{ m_{\underline{\delta}}(k)\frac{k}{|k|}\cdot\nabla_k + \nabla_k \cdot \frac{k}{|k|} m_{\underline{\delta}}(k)\right\}.$$

Note that $a_{\underline{\delta}}$ a priori defined on $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ is closable and its closure is a maximally symmetric operator. The conjugate operator is obtained through second quantization

$$A_{\underline{\delta}} = \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(a_{\underline{\delta}})$$

and is again a maximally symmetric operator closable on $\mathcal{K}\otimes\Gamma_{\mathrm{fin}}(C_0^\infty(\mathbb{R}^3\backslash\{0\}))$.

To get a conjugate operator for the Liouvillean we do the construction after gluing and define the modified generator of translations

$$\tilde a_{\underline{\delta}} := \frac{\mathrm{i}}{2}\left\{ m_{\underline{\delta}}(\omega)\frac{\mathrm{d}}{\mathrm{d}\omega} + \frac{\mathrm{d}}{\mathrm{d}\omega} m_{\underline{\delta}}(\omega)\right\} \otimes \mathbb{1}_{L^2(S^2)},$$

which is essentially self-adjoint on $C_0^\infty(\mathbb{R}) \otimes C^\infty(S^2)$. We second quantize to obtain

$$\widetilde{A}_{\underline{\delta}} := \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \mathrm{d}\Gamma(\tilde a_{\underline{\delta}}),$$

which is essentially self-adjoint on $\widetilde{\mathcal{C}}^{\mathrm{L}}$. Note that we have simplified the notation a bit writing $\tilde a_{\underline{\delta}}$ and $\widetilde{A}_{\underline{\delta}}$ instead of the more cumbersome $\tilde a_{m_{\underline{\delta}}}$ and $\widetilde{A}_{m_{\underline{\delta}}}$, cf. (3.18) and (3.19).

Note that as an identity on $C_0^\infty(\mathbb{R}\backslash\{0\})\otimes C^\infty(S^2)$ we have $T(a_{\underline{\delta}}\otimes \mathbb{1}_{\mathfrak{h}} - \mathbb{1}_{\mathfrak{h}} \otimes a_{\underline{\delta}})T^* = \tilde a_{\underline{\delta}}$ and furthermore

$$\mathcal{U}\widetilde{L}_\infty'\mathcal{U}^* = L_\infty' = H' \otimes \mathbb{1}_{\mathcal{H}} + \mathbb{1}_{\mathcal{H}} \otimes H^{\mathbf{c}\prime}. \tag{4.28}$$

The latter being an operator identity on $\mathcal{D}(N^{\mathrm{L}})$.

4.3 Estimates at Zero Temperature

Throughout this section we will for $\underline{\delta}' \in \Delta_0$, cf. (4.27), use the notation

$$\Delta(\underline{\delta}') := \big\{\underline{\delta} \in \Delta_0 \,\big|\, \delta_0 \le \delta_0', \delta_\infty \ge \delta_\infty'\big\}.$$

We begin with a new high-energy estimate, which is particular to the case of finite dimensional small systems. It will not hold e.g. for (confined) atomic small systems.

For $\underline{\delta} \in \Delta_0$, we will write $N_{\underline{\delta}}$ for $\mathrm{d}\Gamma(m_{\underline{\delta}})$, the modified number operator appearing in $H' = N_{\underline{\delta}} - \phi(\mathrm{i}a_{\underline{\delta}}G)$. The reason for introducing the modified generator of radial translation is that $N_{\underline{\delta}}$ is large in the infrared and ultraviolet regimes, which allows us to handle very soft and very hard photons.

Theorem 4.3. *Suppose* **(HG1)**. *Let $e > 0$ be given. There exists $\delta_\infty' \ge 1$, $c > 0$ and $E_0 \in \mathbb{R}$ such that for all $\underline{\delta} \in \Delta((1, \delta_\infty'))$ we have*

$$H' \ge e\mathbb{1}_{\mathcal{H}} - c\mathbb{1}[H \le E_0]$$

in the sense of forms on $\mathcal{D}(N)$.

Proof. The first step we take is to estimate from below

$$H' \ge \frac{1}{2} N_{\underline{\delta}} - C\mathbb{1}_{\mathcal{H}} \tag{4.29}$$

exploiting the $N^{1/2}$-boundedness of $\phi(\mathrm{i}a_{\underline{\delta}}G)$, cf. (2.2), and the inequality $N \le N_{\underline{\delta}}$. Here C is some positive number.

For $R > 1$ we perform a partition of unity in momentum space as follows. Let

$$F^R = \begin{pmatrix} \mathbb{1}[|k| < R] \\ \mathbb{1}[|k| \ge R] \end{pmatrix} : \mathfrak{h} \to L^2(B(0,R)) \oplus L^2(B(0,R)^c) =: \mathfrak{h}_<^R \oplus \mathfrak{h}_>^R \tag{4.30}$$

and observe that F^R is unitary. We lift to $\mathcal{F}$ and get

$$\check{\Gamma}(F^R) = I\Gamma(F^R) \colon \mathcal{F} \to \Gamma(\mathfrak{h}_<^R) \otimes \Gamma(\mathfrak{h}_>^R) =: \mathcal{F}_<^R \otimes \mathcal{F}_>^R. \tag{4.31}$$

Put $H_0^{\mathrm{x}} = K \otimes \mathbb{1} + \mathbb{1}_{\mathcal{K}} \otimes H_{\mathrm{ph}|\mathcal{F}_<^R} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{ph}|\mathcal{F}_>^R}$ and abbreviate $\lambda_{\max} =$

$\max\sigma(K)$. We now compute for $\widetilde{E} > \lambda_{\max}$

$$
\begin{aligned}
N_{\underline{\delta}} &\geq N_{\underline{\delta}} \mathbb{1}[H_0 > \widetilde{E}] \\
&= \check{\Gamma}(F^R)^* \left\{ \mathbb{1}_{\mathcal{K}} \otimes N_{\underline{\delta}|\mathcal{F}^R_<} \otimes \mathbb{1}_{\mathcal{F}^R_>} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{F}^R_<} \otimes N_{\underline{\delta}|\mathcal{F}^R_>} \right\} \mathbb{1}[H_0^{\mathrm{x}} > \widetilde{E}] \check{\Gamma}(F^R) \\
&\geq \Gamma(\mathbb{1}[|k| < R]) N_{\underline{\delta}} \mathbb{1}[H_0 > \widetilde{E}] + m_\delta(R) \check{\Gamma}(F^R)^* \mathbb{1}_{\mathcal{K}\otimes\mathcal{F}^R_<} \otimes \overline{P}_\Omega \mathbb{1}[H_0^{\mathrm{x}} > \widetilde{E}] \check{\Gamma}(F^R) \\
&\geq \frac{\widetilde{E} - \lambda_{\max}}{R} \Gamma(\mathbb{1}[|k| < R]) \mathbb{1}[H_0 > \widetilde{E}] + m_{\underline{\delta}}(R) \check{\Gamma}(^R)^* \mathbb{1}_{\mathcal{K}\otimes\mathcal{F}^R_<} \otimes \overline{P}_\Omega \mathbb{1}[H_0^{\mathrm{x}} > \widetilde{E}] \check{\Gamma}(F^R) \\
&\geq \min\left\{ \frac{\widetilde{E} - \lambda_{\max}}{R}, m_{\underline{\delta}}(R) \right\} \mathbb{1}[H_0 > \widetilde{E}].
\end{aligned}
$$

We thus get

$$
\frac{1}{2} N_{\underline{\delta}} - C\mathbb{1}_{\mathcal{H}} \geq \frac{1}{2} \min\left\{ \frac{\widetilde{E} - \lambda_{\max}}{R}, m_{\underline{\delta}}(R) \right\} \mathbb{1}[H_0 > \widetilde{E}] - C\mathbb{1}_{\mathcal{H}}. \tag{4.32}
$$

To pass from H_0 to H we estimate, recalling that Σ denotes the bottom of the spectrum of H (2.4),

$$
\begin{aligned}
\mathbb{1}[H_0 \leq \widetilde{E}] &\leq (\widetilde{E} + 1)(H_0 + 1)^{-1} \\
&= (\widetilde{E} + 1)(H - \Sigma + 1)^{-\frac{1}{2}} \\
&\quad \times \left\{ (H - \Sigma + 1)^{\frac{1}{2}} (H_0 + 1)^{-1} (H - \Sigma + 1)^{\frac{1}{2}} \right\} (H - \Sigma + 1)^{-\frac{1}{2}} \\
&\leq (\widetilde{E} + 1)\widetilde{C}(H - \Sigma + 1)^{-1} \\
&\leq (\widetilde{E} + 1)\widetilde{C}\mathbb{1}[H \leq E] + \frac{(\widetilde{E} + 1)\widetilde{C}}{E - \Sigma + 1}.
\end{aligned}
$$

Combining with (4.29) and (4.32) we arrive at the bound

$$
H' \geq \frac{1}{2} \min\left\{ \frac{\widetilde{E} - \lambda_{\max}}{R}, m_{\underline{\delta}}(R) \right\} \left(1 - \frac{\widetilde{C}(\widetilde{E} + 1)}{E - \Sigma + 1} \right) - C - \frac{(\widetilde{E} + 1)\widetilde{C}}{2} \mathbb{1}[H \leq E].
$$

We are now ready to pick our constants. First choose R large enough such that $d(R)/2 \geq e + C + 1$. Then choose $\delta'_\infty \geq R$ such that for $\underline{\delta} \in (0,1) \times (\delta'_\infty, \infty)$ we have $m_{\underline{\delta}}(R) = d(R)$. Subsequently we fix $\widetilde{E}$ large enough such that $(\widetilde{E} - \lambda_{\max})/(2R) \geq e + C + 1$. With these choices of R, δ' and $\widetilde{E}$ (in that order) we get

$$
H' \geq (e + C + 1) \left(1 - \frac{\widetilde{C}(\widetilde{E} + 1)}{E - \Sigma + 1} \right) - C - \frac{(\widetilde{E} + 1)\widetilde{C}}{2} \mathbb{1}[H \leq E].
$$

Finally we can take E_0 large enough such that with $E = E_0$ the right-hand side is bounded from below by $e\mathbb{1}_{\mathcal{H}} - \frac{1}{2}\widetilde{C}(\widetilde{E} + 1)\mathbb{1}[H \leq E_0]$. $\square$

For the purpose of the following we introduce the terminology that H satisfies a Mourre estimate at $E \in \mathbb{R}$:

Definition 4.4. We say that H satisfies a Mourre estimate at $E \in \mathbb{R}$ if there exists $\underline{\delta}' \in \Delta_0$ such that: For all $\epsilon > 0$ there exist $C > 0$, $\kappa > 0$ and a compact operator K such that as a form on $\mathcal{D}(N)$

$$H' \geq (1 - \epsilon)\mathbb{1} - C\mathbb{1}[|H - E| > \kappa] - K,$$

for any $\underline{\delta} \in \Delta(\underline{\delta}')$.

Theorem 4.3 implies that a Mourre estimate is satisfied at any $E > E_0$. Since H' is bounded from below, we also get a Mourre estimate automatically satisfied at any $E < \Sigma$. Note that the Mourre estimate obviously remains true if we replace κ by any smaller positive κ'.

Lemma 4.5. *Suppose* **(HG1)**. *Let $J \subset \mathbb{R}$ be a compact set with a Mourre estimate satisfied at all $E \in J$. Then there exists $\underline{\delta}' \in \Delta((1,1))$ such that: For any $\epsilon > 0$, there exist $\kappa > 0$, $C > 0$, such that for all $E \in J$ and $\underline{\delta} \in \Delta(\underline{\delta}')$ we have*

$$H' \geq -\epsilon\mathbb{1}_{\mathcal{H}} - C\mathbb{1}[|H - E| \geq \kappa],$$

in the sense of forms on $\mathcal{D}(N)$.

Remark 4.6. If a Mourre estimate holds at all $E \in J$ with the same $\underline{\delta}'$, then this $\underline{\delta}'$ can also be used for the uniform bound. This will be evident from the proof below. $\diamondsuit$

Proof. First note that by the virial theorem, the point spectrum in an open neighborhood of J is locally finite and eigenvalues in J have finite multiplicity.

We begin by verifying the estimate for a fixed E, for which the Mourre estimate is satisfied. If $E \notin \sigma_{\rm pp}(H)$ we proceed as follows: First extract a Mourre estimate with the given ϵ. Write for $0 < \kappa' < \kappa$ the compact error as $K = K\mathbb{1}[|H - E| \leq \kappa'] + K\mathbb{1}[|H - E| > \kappa']$. Pick κ' small enough such that $\|K\mathbb{1}[|H - E| \leq \kappa']\| \leq 1/2$. Then

$$H' \geq -\epsilon\mathbb{1} - (C + \|K\|)\mathbb{1}[|H - E| > \kappa'].$$

If on the other hand $E \in \sigma_{\rm pp}(H)$ we proceed differently. Write P_E for the finite rank orthogonal projection on the eigenspace associated with E. Abbreviate

$\overline{P}_E = \mathbb{1} - P_E$. Since $\mathrm{Ran}(P_E) \subset \mathcal{D}(N^{1/2})$ we can compute and estimate

$$\begin{aligned}
H' &= P_E H' P_E + 2\mathrm{Re}\{P_E H' \overline{P}_E\} + \overline{P}_E H' \overline{P}_E \\
&= 2\mathrm{Re}\{P_E H' \overline{P}_E\} + \overline{P}_E H' \overline{P}_E \\
&\geq (1 - \epsilon/5)\overline{P}_E - C\mathbb{1}[|H - E| \geq \kappa] - \overline{P}_E K \overline{P}_E + 2\mathrm{Re}\{P_E H' \overline{P}_E\} \\
&\geq -\epsilon/5\mathbb{1} - C\mathbb{1}[|H - E| \geq \kappa'] + 2\mathrm{Re}\{P_E H' \overline{P}_E\},
\end{aligned} \tag{4.33}$$

where we used Theorem 3.5 in the second equality and for the first inequality we used the assumed to hold Mourre estimate (applied with ϵ replaced by $\epsilon/5$). In the last step we argued as above to get rid of the compact error by passing to a smaller $\kappa' < \kappa$.

As for the cross term (4.33) we write $H' = N_{\underline{\delta}} - \phi(\mathrm{i}a_{\underline{\delta}}G)$ as a form sum on $D(N^{1/2})$. Recalling Theorem 3.1, we decompose for an $r > 0$ to be fixed later

$$P_E H' \overline{P}_E = P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} > r] \overline{P}_E + K \overline{P}_E,$$

with $K = P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} \leq r] - P_E \phi(\mathrm{i}a_{\underline{\delta}}G)$ being compact. Estimate first for $\sigma > 0$

$$\begin{aligned}
2\mathrm{Re}\{P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} > r] \overline{P}_E\} &= 2\mathrm{Re}\{P_E N_{\underline{\delta}} \mathbb{1}[N_\delta > r]\} - 2P_E N_{\underline{\delta}} \mathbb{1}_{N_{\underline{\delta}} > r} P_E \\
&\geq -\sigma N_{\underline{\delta}} - (2 + \sigma^{-1}) P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} > r] P_E.
\end{aligned}$$

fix σ small enough such that

$$8\sigma \|a_{\underline{\delta}} G\|^2 \leq \epsilon/5. \tag{4.34}$$

Fix now r large enough such that $\|P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} > r] P_E\| \leq \epsilon/(5(2 + \sigma^{-1}))$. We then have

$$2\mathrm{Re}\{P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} > r] \overline{P}_E\} \geq -\sigma N_{\underline{\delta}} - \frac{\epsilon}{5} \mathbb{1}_{\mathcal{H}}.$$

Secondly we estimate for $\sigma' > 0$

$$\begin{aligned}
2\mathrm{Re}\{K \overline{P}_E\} &= 2\mathrm{Re}\{K \mathbb{1}[|H - E| < \kappa] \overline{P}_E\} + 2\mathrm{Re}\{K \mathbb{1}[|H - E| > \kappa]\} \\
&\geq -(\epsilon/5 + \sigma')\mathbb{1} - \frac{\|K\|^2}{\sigma'} \mathbb{1}[|H - E| > \kappa].
\end{aligned}$$

Here we chose $\kappa > 0$ small enough such that $\|K \mathbb{1}[|H - E| < \kappa] \overline{P}_E\| \leq \epsilon/5$. Picking $\sigma' = \epsilon/5$ yields

$$H' \geq -\frac{4\epsilon}{5} \mathbb{1}_{\mathcal{H}} - \sigma N_\delta - C\mathbb{1}[|H - E| > \kappa].$$

To get rid of the extra σN_δ we estimate using (2.2) (with $\sigma = 1/2$) and (4.34)

$$\begin{aligned}(1+2\sigma)H' &\geq -\frac{4\epsilon}{5}\mathbb{1} - C\mathbb{1}[|H-E|>\kappa] + \sigma N_\delta - 2\sigma\phi(\mathrm{i}a_{\underline{\delta}}G)\\ &\geq -\epsilon\mathbb{1} - C\mathbb{1}[|H-E|>\kappa].\end{aligned}$$

It remains to establish that one can choose $\underline{\delta}'$, κ and C such that the desired bound holds for all $E \in J$ and $\underline{\delta} \in \Delta(\underline{\delta}')$. We proceed by assuming, aiming for a contradiction, that given $\underline{\delta}'_n = (1/n, n)$, $\kappa_n = 1/n$ and $C_n = n$, there exists an energy E_n and $\underline{\delta}_n \in \Delta(\underline{\delta}'_n)$ such that the desired bound fails. By compactness of J, we can assume that E_n converges to some E_∞. Let $\underline{\delta}'_\infty$, κ_∞ and C_∞ be the constants just established to exist such that the bound holds true at E_∞ for any $\underline{\delta} \in \Delta(\underline{\delta}'_\infty)$. Picking n large enough such that

$$|E_\infty - E_n| < \kappa_\infty/2, \quad \kappa_n < \kappa_\infty/2, \quad C_n \geq C_\infty, \quad \underline{\delta}'_n \in \Delta(\underline{\delta}'_\infty)$$

sets us up with a contradiction since with $\underline{\delta} \in \Delta(\underline{\delta}'_n) \subset \Delta(\underline{\delta}'_\infty)$ we have

$$H' \geq -\epsilon\mathbb{1} - C_\infty\mathbb{1}[|H-E_\infty| \geq \kappa_\infty] \geq (1-\epsilon)\mathbb{1} - C_n\mathbb{1}[|H-E_n| \geq \kappa_n].$$

□

The following theorem, which appeared originally in [23, Thm. 7.12], states that a Mourre estimate holds at any fixed $E \in \mathbb{R}$. It holds also for confined small system, not necessarily finite dimensional, but the proof simplifies slightly here. Furthermore, since we do not need resolvents of H to control $\phi(a_{\underline{\delta}}G)$ but can do with resolvents of N, the version here in fact holds under slightly weaker IR assumptions on G.

Another special feature of finite dimensional small systems is that we can choose δ_0' uniformly in energy. Indeed, we pick $\delta_0' \in (0,1]$ such that

$$d(\delta_0') \geq 2\sup_{\underline{\delta}\in\Delta_0}\|a_{\underline{\delta}}G\|^2 + 1. \tag{4.35}$$

With this choice we have for all $\underline{\delta} \in \Delta((\delta_0', 1))$ and $|\omega| \leq \delta_0$ that

$$m_{\underline{\delta}}(\omega) \geq m_{\underline{\delta}}(\delta_0) = d(\delta_0) \geq d(\delta_0') \geq 2\sup_{\underline{\delta}\in\Delta_0}\|a_{\underline{\delta}}G\|^2 + 1. \tag{4.36}$$

Theorem 4.7. *Suppose* **(HG1)***. Let $\epsilon > 0$ and $E \in \mathbb{R}$. There exist $\kappa > 0$, $C > 0$ and K a compact and self-adjoint operator, such that the form estimate on $\mathcal{D}(N)$*

$$H' \geq (1-\epsilon)\mathbb{1}_{\mathcal{H}} - C\mathbb{1}[|H-E| \geq \kappa] - K$$

holds true for all $\underline{\delta} \in \Delta((\delta_0', 1))$. Here δ_0' is chosen such that (4.35) *is satisfied.*

Proof. Fix $E \in \mathbb{R}$ and $\epsilon < 0$. We only have something to prove if $E \geq \Sigma$. The proof goes by induction in energy and we assume the theorem holds true for all $E' \leq E - \delta_0'$ and $\epsilon' > 0$.

Write $P = |0\rangle\langle 0|$ and $P^\perp = \mathbb{1}_\mathcal{F} - P$ as projection operators on $\mathcal{F}$ or $\mathcal{H}$ (read as e.g. $\mathbb{1}_\mathcal{K} \otimes P$). In order to use geometric localization we need the extended Hilbert space $\mathcal{H}^\mathrm{x} = \mathcal{H} \otimes \mathcal{F}$ and the extended Hamiltonian $H^\mathrm{x} = H \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_\mathcal{H} \otimes H_\mathrm{ph}$. The extended commutator is

$$H^{\mathrm{x}'} = H' \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_\mathcal{H} \otimes N_{\underline{\delta}}$$

as a self-adjoint operator on $\mathcal{D}(N^\mathrm{x})$, where $N^\mathrm{x} = N \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_\mathcal{H} \otimes N$.

Observe that if $S\colon \mathcal{D}(N^{1/2}) \to \mathcal{H}^\mathrm{x}$ is bounded then for any $\sigma > 0$ we have

$$\begin{aligned}
&\mathrm{Re}\big\{\check\Gamma(j^R)^*(\mathbb{1}_\mathcal{H} \otimes P)S\big\} = \mathrm{Re}\big\{\Gamma(j_0^R)S_0\big\} \\
&\quad = \mathrm{Re}\big\{\mathbb{1}[|H-E|>1]\Gamma(j_0^R)S_0(N+\mathbb{1}_\mathcal{H})^{-1/2}(N+\mathbb{1}_\mathcal{H})^{1/2}\big\} - K_1 \\
&\quad \geq -\sigma(N+\mathbb{1}_\mathcal{H}) - \sigma^{-1}\|S_0(N+\mathbb{1}_\mathcal{H})^{-1/2}\|^2\mathbb{1}[|H-E|>1] - K_1.
\end{aligned}$$

Here $S_0\colon \mathcal{D}(N^{1/2}) \to \mathcal{H} \otimes \mathbb{C}$ is the first component of S and K_1 is compact.

The observation above implies that we can pick $R_0 > 0$ large enough such that for $R \geq R_0$ we have for $\sigma > 0$ as a form on $\mathcal{D}(N)$

$$\begin{aligned}
H' &\geq \check\Gamma(j^R)^* H^{\mathrm{x}'}\check\Gamma(j^R) - \sigma(N+\mathbb{1}_\mathcal{H}) \\
&\geq \check\Gamma(j^R)^*(\mathbb{1}_\mathcal{H} \otimes P^\perp)H^{\mathrm{x}'}(\mathbb{1}_\mathcal{H} \otimes P^\perp)\check\Gamma(j^R) \\
&\qquad - \sigma(N+\mathbb{1}_\mathcal{H}) - C_2\mathbb{1}[|H-E|>1] - K_2. \qquad (4.37)
\end{aligned}$$

Here we employed the observation above with an $(R,\underline{\delta})$-dependent operator $S(R,\underline{\delta}) = -(\phi(\mathrm{i}a_{\underline{\delta}}G) \otimes \mathbb{1}_\mathcal{F})\check\Gamma(j^R)$, which has $\|S_0(R,\underline{\delta})(N+\mathbb{1}_\mathcal{H})^{-1/2}\|$ bounded uniformly in $R \geq R_0$ and $\underline{\delta} \in \Delta_0$.

Fix σ such that

$$\sigma \leq \frac{\epsilon}{5}, \quad 8\sigma\|a_{\underline{\delta}}G\|^2 \leq \frac{\epsilon}{5} \quad \text{and} \quad \frac{1-\frac{4\epsilon}{5}}{1+2\sigma} \geq 1-\epsilon. \qquad (4.38)$$

We now employ the momentum partition of unity from the proof of Theorem 4.3, cf. (4.30) and (4.30). Let $F^{\delta_0} = (\mathbb{1}[|k| \geq \delta_0], \mathbb{1}[|k| < \delta_0])$ and recall that $\check\Gamma(F^{\delta_0})\colon \mathcal{F} \to \mathcal{F}_>^{\delta_0} \otimes \mathcal{F}_<^{\delta_0}$ is unitary. (For notational convenience below, we have switched the order of the interior and exterior regions.) Abbreviate $\widehat{\mathcal{H}}^\mathrm{x} = (\mathbb{1}_\mathcal{H} \otimes \check\Gamma(F^{\delta_0}))\mathcal{H}^\mathrm{x} = \mathcal{H} \otimes \mathcal{F}_>^{\delta_0} \otimes \mathcal{F}_<^{\delta_0}$. Compute the intertwining relations

$$\check\Gamma(F^{\delta_0})P^\perp = \big(\mathbb{1}_{\mathcal{F}_>^{\delta_0}} \otimes P_<^\perp + P_>^\perp \otimes P_<\big)\check\Gamma(F^{\delta_0}) \qquad (4.39)$$

$$\big(\mathbb{1}_\mathcal{H} \otimes \check\Gamma(F^{\delta_0})\big)H^{\mathrm{x}'} = \widehat{H}^{\mathrm{x}'}\big(\mathbb{1}_\mathcal{H} \otimes \check\Gamma(F^{\delta_0})\big) \qquad (4.40)$$

$$\big(\mathbb{1}_\mathcal{H} \otimes \check\Gamma(F^{\delta_0})\big)H^{\mathrm{x}} = \widehat{H}^{\mathrm{x}}\big(\mathbb{1}_\mathcal{H} \otimes \check\Gamma(F^{\delta_0})\big). \qquad (4.41)$$

Here $P_{>/<}$ denote the orthogonal projections onto the vacuum sectors inside $\mathcal{F}^{\delta_0}_{>/<}$, and

$$\widehat{H}^{\mathrm{x}'} = H^{\mathrm{x}'}_{>} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_<} + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \otimes N_{\underline{\delta}|\mathcal{F}^{\delta_0}_<},$$
$$H^{\mathrm{x}'}_{>} = H' \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \otimes N_{\underline{\delta}|\mathcal{F}^{\delta_0}_>},$$
$$\widehat{H}^{\mathrm{x}} = H \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_<} + \mathbb{1}_{\mathcal{H}} \otimes H_{\mathrm{ph}|\mathcal{F}^{\delta_0}_>} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_<} + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \otimes H_{\mathrm{ph}|\mathcal{F}^{\delta_0}_<}.$$

Using that $H' \geq -2\|a_{\underline{\delta}} G\|^2$, cf. (2.2), we estimate

$$\begin{aligned}&\big(\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \otimes P^{\perp}_<\big)\big(H^{\mathrm{x}'}_{>} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_<} + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \otimes N_{\underline{\delta}|\mathcal{F}^{\delta_0}_<}\big)\\ &\quad\geq \big(m(\delta_0) - 2\|a_{\underline{\delta}} G\|^2\big)\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \otimes P^{\perp}_<\end{aligned} \tag{4.42}$$

and observe the identity

$$\begin{aligned}&\big(\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}_> \otimes P_<\big)\big(H^{\mathrm{x}'}_{>} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_<} + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \otimes N_{\underline{\delta}|\mathcal{F}^{\delta_0}_<}\big)\\ &\quad= \big((\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}_>)H^{\mathrm{x}'}_{>}(\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}_>)\big) \otimes P_<.\end{aligned} \tag{4.43}$$

Using the intertwining relations (4.39) and (4.40), together with (4.42), (4.43) and the choice of δ_0', cf. (4.36), we get

$$\begin{aligned}&(\mathbb{1}_{\mathcal{H}} \otimes P^{\perp})H^{\mathrm{x}'}(\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}) \qquad (4.44)\\ &\geq \big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0})^*\big)\Big\{\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \otimes P^{\perp}_< + \big((\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}_>)H^{\mathrm{x}'}_{>}(\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}_>)\big) \otimes P_<\Big\}\big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0})\big).\end{aligned}$$

To deal with the term in the brackets we note that

$$(\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}_>)H^{\mathrm{x}'}_{>}(\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}_>) \geq \mathbb{1}_{\mathcal{H}} \otimes P^{\perp}_> + H' \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \tag{4.45}$$

and estimate using Lemma 4.5, with ϵ replaced by $\epsilon/5$, and the induction assumption

$$\begin{aligned}H' \otimes \mathbb{1}_{\mathcal{F}^{\delta_0}_>} \otimes P_< &= \Big\{\bigoplus_{\ell=1}^{\infty}\int^{\oplus}_{(\mathbb{R}^3\setminus B(\delta_0))^\ell} H' \,\mathrm{d}k_1\cdots\mathrm{d}k_\ell\Big\} \otimes P_<\\ &\geq -\Big\{\bigoplus_{\ell=1}^{\infty}\int^{\oplus}_{(\mathbb{R}^3\setminus B(\delta_0))^\ell}\Big(\epsilon\mathbb{1}_{\mathcal{H}} + C\mathbb{1}[|H + \textstyle\sum_{j=1}^{\ell}|k_j| - E| > \kappa]\Big)\,\mathrm{d}k_1\cdots\mathrm{d}k_\ell\Big\} \otimes P_<\\ &= -\frac{\epsilon}{n}\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}_> \otimes P_< - C\mathbb{1}[|\widehat{H}^{\mathrm{x}} - E| > \kappa]\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}_> \otimes P_<\\ &\geq -\frac{\epsilon}{5}\mathbb{1}_{\widehat{\mathcal{H}}^{\mathrm{x}}} - C\mathbb{1}[|\widehat{H}^{\mathrm{x}} - E| > \kappa].\end{aligned} \tag{4.46}$$

Here κ and C are coming from Lemma 4.5. Combining (4.44)–(4.46), cf. also (4.39) and (4.41), we find

$$
\begin{aligned}
&(\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}) H^{\mathrm{x}\prime} (\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}) \\
&\geq \big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0})^*\big) \Big\{ \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}_{>}^{\delta_0}} \otimes P_{<}^{\perp} + \mathbb{1}_{\mathcal{H}} \otimes P_{>}^{\perp} \otimes P_{<} - \frac{\epsilon}{5} \mathbb{1}_{\widehat{\mathcal{H}}^{\mathrm{x}}} - C \mathbb{1}[|\widehat{H}^{\mathrm{x}} - E| > \kappa] \Big\} \big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0})\big) \\
&= \mathbb{1}_{\mathcal{H}} \otimes P^{\perp} - \frac{\epsilon}{5} \mathbb{1}_{\mathcal{H}^{\mathrm{x}}} - C \mathbb{1}[|H^{\mathrm{x}} - E| > \kappa].
\end{aligned}
$$

Pick a non-negative $f \in C_0^{\infty}(\mathbb{R})$ with $\mathrm{supp}(f) \subset [-\kappa, \kappa]$ and $f = 1$ on $[-\kappa/2, \kappa/2]$. We are now in a position to insert into (4.37) and estimate for some $R \geq R_0$, which we can now fix,

$$
\begin{aligned}
H' &\geq \check{\Gamma}(j^R)^* \Big\{ \mathbb{1}_{\mathcal{H}} \otimes P^{\perp} - \epsilon \mathbb{1}_{\mathcal{H}^{\mathrm{x}}} - C \mathbb{1}[|H^{\mathrm{x}} - E| > \kappa] \Big\} \check{\Gamma}(j^R) \\
&\qquad - \sigma(N + \mathbb{1}_{\mathcal{H}}) - C_2 \mathbb{1}[|H - E| > 1] - K_2 \\
&\geq \check{\Gamma}(j^R)^* \Big\{ (1 - \frac{\epsilon}{5}) \mathbb{1}_{\mathcal{H}^{\mathrm{x}}} - C f(H^{\mathrm{x}} - E) \Big\} \check{\Gamma}(j^R) \\
&\qquad - \sigma(N + \mathbb{1}_{\mathcal{H}}) - C_3 \mathbb{1}[|H - E| > 1] - K_3 \\
&\geq (1 - \frac{2\epsilon}{5}) \mathbb{1}_{\mathcal{H}} - \sigma(N + \mathbb{1}_{\mathcal{H}}) - C f(H - E) - C_3 \mathbb{1}[|H - E| > 1] - K_3 \\
&\geq (1 - \frac{2\epsilon}{5} - \sigma) \mathbb{1}_{\mathcal{H}} - \sigma N - C_4 \mathbb{1}[|H - E| > \kappa/2] - K_3.
\end{aligned}
$$

with $C_4 = C + C_3$.

The proof is now completed as in the proof of Lemma 4.5 by the bound

$$
(1 + 2\sigma) H' \geq H' + \sigma N - 4\sigma \|a_{\underline{\delta}} G\|^2 \geq (1 - \frac{4\epsilon}{5}) \mathbb{1}_{\mathcal{H}} - C_4 \mathbb{1}[|H - E| > \kappa/2] - K_3,
$$

where we used the choice of σ, cf. (4.38). This concludes the proof. □

Repeating the proof of Corollary 4.1 we arrive at

Corollary 4.8. *Suppose* **(HG1)**. *The operator H has a finite number of eigenvalues, all of finite multiplicity.*

We denote by P the finite rank projection that projects onto the subspace consisting of eigenstates for H, and we write $\overline{P} = \mathbb{1} - P$.

Corollary 4.9. *Suppose* **(HG1)**. *Let $\epsilon > 0$. There exists $\underline{\delta}' \in \Delta_0$, $\kappa > 0$, $C > 0$, such that the following two estimates holds for all $\underline{\delta} \in \Delta(\underline{\delta}')$ and $E \in \mathbb{R}$*

$$
H' \geq -\epsilon \mathbb{1} - C \mathbb{1}[|H - E| \geq \kappa], \tag{4.47}
$$

$$
H' \geq (1 - \epsilon) \mathbb{1} - C(\mathbb{1}[|H - E| \geq \kappa] + P), \tag{4.48}
$$

in the sense of forms on $\mathcal{D}(N)$.

Proof. The estimate (4.47) is a direct consequence of Theorems 4.3 and 4.7 together with Lemma 4.5.

We proceed to the second bound (4.48). This bound is obviously true for $E > E_0 + 1$ (cf. Theorem 4.3) and for $E < \Sigma - 1$, so what remains is to prove the estimate uniformly in $E \in [\Sigma - 1, E_0 + 1] =: J$, which is a compact interval. We first argue that the estimate is correct for fixed $E \in J$. Let $\epsilon > 0$ and apply Theorem 4.7 with ϵ replaced by $\epsilon/3$. Write

$$K = PKP + \overline{P}K\overline{P} + 2\mathrm{Re}\{PK\overline{P}\} \geq -\|K\|P + \overline{P}K\overline{P} + 2\mathrm{Re}\{PK\overline{P}\}$$

and estimate

$$K\overline{P} = K\overline{P}\mathbb{1}[|H - E| \geq \kappa] + K\overline{P}\mathbb{1}[|H - E| < \kappa],$$

where one can choose κ small enough such that $\|K\overline{P}\mathbb{1}[|H-E| < \kappa]\| \leq \epsilon/3$. For the first term we estimate

$$\begin{aligned} 2\langle\psi, K\overline{P}\mathbb{1}[|H - E| \geq \kappa]\psi\rangle &\leq 2\|K\psi\|\|\mathbb{1}[|H - E| \geq \kappa]\psi\| \\ &\leq \sigma\|K\|^2\|\psi\|^2 + \frac{1}{\sigma}\langle\psi, \mathbb{1}[|H - E| \geq \kappa]\psi\rangle. \end{aligned}$$

Choosing $\sigma > 0$ small enough we get

$$K \geq -\tfrac{2}{3}\epsilon\mathbb{1} - \widetilde{C}\mathbb{1}[|H - E| \geq \kappa].$$

This completes the argument that for a fixed E one can find κ and C such that the commutator estimate (4.48) holds true.

Suppose the estimate (4.48) is not correct uniformly in E. That is, for any $\kappa > 0$ and $C > 0$ there exists $E \in J$ such that estimate fails to hold.

Put $\kappa_n = 1/n$ and $C_n = n$. This gives a sequence $E_n \in J$, for which the estimate (4.48) is false. We can assume due to compactness of J that E_n converges to an energy $E_\infty \in J$. Recalling that we have just verified that (4.48) holds for a fixed $E \in J$, we get a $\kappa_\infty > 0$ and $C_\infty > 0$ such that (4.48) holds true at E_∞. Pick n large enough such that $1/n < \kappa_\infty/2$, $C_n > C_\infty$ and $|E_\infty - E_n| < \kappa_\infty/2$. Then

$$H' \geq (1-\epsilon)\mathbb{1} - C_\infty(\mathbb{1}[|H-E| \geq \kappa_\infty] + P) \geq (1-\epsilon)\mathbb{1} - C_n(\mathbb{1}[|H-E_n| \geq \kappa_n] + P),$$

contradicting the choice of E_n. □

4.4 Estimates at Positive Temperature

In this subsection we use the notation $\widetilde{N}_{\underline{\delta}}$ for $\mathrm{d}\Gamma(m_{\underline{\delta}})$, which is the analogue of $N_{\underline{\delta}}$ from the previous subsection. We can then write $\widetilde{L}'_\beta = \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{N}_{\underline{\delta}} - \phi(\mathrm{i}\tilde{a}_{\underline{\delta}}\widetilde{G}_\beta)$.

Theorem 4.10. *Suppose* **(LG1)**. *Let $e > 0$ be given. There exists $E_0 > 0$, $\delta'_\infty > 0$ and $C > 0$ such that the following form bound holds on $\mathcal{D}(N^{\mathrm{L}})$ for all $E \geq E_0$ and $\underline{\delta} \in \Delta((1, \delta'_\infty))$*

$$L'_\beta \geq e\mathbb{1} - C\mathbb{1}[|L_\beta| < E].$$

Proof. It suffices to prove the theorem, with L_β and N^{L} replaced by $\widetilde{L}_\beta$ and $\widetilde{N}$. The proof is divided into two steps. First we consider the uncoupled glued Liouvillean $\widetilde{L}_0$. The reader should not confuse the subscript 0 with infinite temperature (zero inverse temperature). We proceed as in the proof of Theorem 4.3.

Observe that

$$\widetilde{L}'_0 = \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{N}_{\underline{\delta}}.$$

For $R > 1$ we again perform a partition of unity in momentum space as follows. Let

$$\begin{aligned} j^R &= \begin{pmatrix} \mathbb{1}[|\omega| < R] \\ \mathbb{1}[|\omega| \geq R] \end{pmatrix} : \tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}}_< \oplus \tilde{\mathfrak{h}}_> \\ \tilde{\mathfrak{h}}_< &:= L^2((-R,R)) \otimes L^2(S^2), \quad \tilde{\mathfrak{h}}_> := L^2((-\infty, R] \cup [R,\infty)) \otimes L^2(S^2). \end{aligned}$$

Put $\widetilde{\mathcal{F}}_< = \Gamma(\tilde{\mathfrak{h}}_<)$, $\widetilde{\mathcal{F}}_> = \Gamma(\tilde{\mathfrak{h}}_>)$ and

$$\widetilde{L}_0^{\mathrm{x}} = L_K \otimes \mathbb{1}_{\widetilde{\mathcal{F}}_<\otimes\widetilde{\mathcal{F}}_>} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega)_{|\widetilde{\mathcal{F}}_<} \otimes \mathbb{1}_{\widetilde{\mathcal{F}}_>} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}\otimes\widetilde{\mathcal{F}}_<} \otimes \mathrm{d}\Gamma(\omega)_{|\widetilde{\mathcal{F}}_>}.$$

Abbreviate $\lambda_{\max} = \max\sigma(K)$ and $\lambda_{\min} = \min\sigma(K)$. We now estimate for $\widetilde{E} > 2\lambda_{\max} - \lambda_{\min}$

$$\begin{aligned} &\mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{N}_{\underline{\delta}} \geq \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{N}_{\underline{\delta}}\mathbb{1}[|\widetilde{L}_0| + \widetilde{N} \geq \widetilde{E}] \\ &= \check{\Gamma}(j^R)^* \left\{ \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{N}_{\underline{\delta}_{|\widetilde{\mathcal{F}}_<}} \otimes \mathbb{1}_{\widetilde{\mathcal{F}}_>} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}\otimes\widetilde{\mathcal{F}}_<} \otimes \widetilde{N}_{\underline{\delta}_{|\widetilde{\mathcal{F}}_>}} \right\} \mathbb{1}[|\widetilde{L}_0^{\mathrm{x}}| + \widetilde{N}^{\mathrm{x}} \geq \widetilde{E}]\check{\Gamma}(j^R) \\ &\geq \Gamma(\mathbb{1}[|\omega| < R])\widetilde{N}_{\underline{\delta}}\mathbb{1}[|\widetilde{L}_0| + \widetilde{N} \geq \widetilde{E}] \\ &\quad + m_\delta(R)\check{\Gamma}(j^R)^*\mathbb{1}_{\mathcal{K}\otimes\mathcal{K}\otimes\widetilde{\mathcal{F}}_<} \otimes \overline{P}_\Omega\mathbb{1}[|\widetilde{L}_0^{\mathrm{x}}| + \widetilde{N}^{\mathrm{x}} \geq \widetilde{E}]\check{\Gamma}(j^R) \\ &\geq \frac{\widetilde{E} - \lambda_{\max} + \lambda_{\min}}{R+1}\Gamma(\mathbb{1}[|\omega| < R])\mathbb{1}[|\widetilde{L}_0| + \widetilde{N} \geq \widetilde{E}] \\ &\quad + m_{\underline{\delta}}(R)\check{\Gamma}(j^R)^*\mathbb{1} \otimes \overline{P}_\Omega\mathbb{1}[|\widetilde{L}_0^{\mathrm{x}}| + \widetilde{N}^{\mathrm{x}} \geq \widetilde{E}]\check{\Gamma}(j^R) \\ &\geq \min\left\{ \frac{\widetilde{E} - \lambda_{\max} + \lambda_{\min}}{R+1}, m_{\underline{\delta}}(R) \right\} \mathbb{1}[|\widetilde{L}_0| + \widetilde{N} \geq \widetilde{E}]. \end{aligned}$$

To conclude the proof we estimate

$$\begin{aligned}
1\!\!1[|\widetilde{L}_0| + \widetilde{N} < \widetilde{E}] &\le (\widetilde{E}+1)(|\widetilde{L}_0| + \widetilde{N} + 1)^{-1} \\
&= (\widetilde{E}+1)(|\widetilde{L}_\beta| + 1)^{-\frac{1}{2}} \\
&\quad \times \left\{ (|\widetilde{L}_\beta| + 1)^{\frac{1}{2}} (|\widetilde{L}_0| + \widetilde{N} + 1)^{-1} (|\widetilde{L}_\beta| + 1)^{\frac{1}{2}} \right\} (|\widetilde{L}_\beta| + 1)^{-\frac{1}{2}} \\
&\le C(\widetilde{E}+1) 1\!\!1[|\widetilde{L}_\beta| < E] + C\frac{\widetilde{E}+1}{E+1}.
\end{aligned}$$

Here we used Proposition 2.4 2 to conclude that

$$C = \left\| (|\widetilde{L}_\beta| + 1)^{\frac{1}{2}} (|\widetilde{L}_0| + \widetilde{N} + 1)^{-1} (|\widetilde{L}_\beta| + 1)^{\frac{1}{2}} \right\| < \infty.$$

We can now pick R, δ', $\widetilde{E}$ and E in that order as in the proof of Theorem 4.3 to conclude the proof. □

It is now an immediate consequence of Theorem 3.6 that

Corollary 4.11. *Suppose* **(LG2)**. *The set of eigenvalues $\sigma_{\mathrm{pp}}(L_\beta)$ is bounded.*

From now on we assume at least **(HG1)** and fix $\underline{\delta}'$ such that Corollary 4.9 holds true. Recall that **(LG1)** implies **(HG1)**.

Proposition 4.12. *Suppose* **(HG1)**. *Let $\epsilon > 0$ be given. There exists $\kappa > 0$ and $C > 0$ such that for all $E \in \mathbb{R}$ and $\underline{\delta} \in \Delta(\underline{\delta}')$*

$$L'_\infty \ge (1-\epsilon) 1\!\!1_{\mathcal{H}^{\mathrm{L}}} - C\big(1\!\!1[|L_\infty - E| \ge \kappa] + P \otimes P^{\mathbf{c}}\big)$$

in the sense of forms on $\mathcal{D}(N^{\mathrm{L}})$.

Remark 4.13. Note that at zero temperature we do not need Nelson's commutator theorem to build L_∞, nor do we have any singularities from ρ_β to absorb. Hence we can work under an **(HG1)** condition instead of an **(LG1)** condition.

Proof. The starting point is the identity

$$L'_\infty = H' \otimes 1\!\!1 + 1\!\!1 \otimes H^{\mathbf{c}\prime}.$$

Denote by $P \in \mathcal{B}(\mathcal{H})$ the projection onto the span of all eigenstates of the operator H. This is a finite range projection and hence compact. Put $P^{\mathbf{c}} = \mathbf{C}P\mathbf{C}$ to be the eigen projection onto the span of the eigenstates of $H^{\mathbf{c}}$. We write $\overline{P} = 1\!\!1 - P$ and $\overline{P}^{\mathbf{c}} = 1\!\!1 - P^{\mathbf{c}}$. We deal with $H' \otimes 1\!\!1$ only since bounds on $1\!\!1 \otimes H^{\mathbf{c}\prime}$ can be obtained by conjugation with $\mathcal{E}\mathbf{C}$, where $\mathcal{E}$ is the exchange map that sends $\psi \otimes \varphi$ to $\varphi \otimes \psi$. Here $\psi, \varphi \in \mathcal{H}$.

We write

$$H' = PH'P + 2\mathrm{Re}\{PH'\overline{P}\} + \overline{P}H'\overline{P}, \tag{4.49}$$

which makes sense as forms on $\mathcal{D}(N^{1/2})$ since P maps into $\mathcal{D}(N^{1/2})$ by Theorem 3.1. We estimate each term differently. For the first and last term we use (4.47) and (4.48) from Corollary 4.9 (applied with $\epsilon/9$ instead of ϵ) and find

$$\begin{aligned} PH'P &\geq -\frac{\epsilon}{9}P - C\mathbb{1}[|H-\lambda-E|>\kappa] \geq -\frac{\epsilon}{9}\mathbb{1}_{\mathcal{H}} - C\mathbb{1}[|H-\lambda-E|>\kappa] \\ \overline{P}H'\overline{P} &\geq \big(1-\frac{\epsilon}{9}\big)\overline{P} - C\mathbb{1}[|H-\lambda-E|>\kappa]. \end{aligned} \tag{4.50}$$

As for the cross term $PH'\overline{P}$ we proceed in a fashion similar to what was done in the proof of Lemma 4.5. Write for an $r>0$

$$PH'\overline{P} = PN_{\underline{\delta}}\mathbb{1}[N_\delta > r]\overline{P} + K\overline{P},$$

with $K = PN_\delta\mathbb{1}[N_{\underline{\delta}} \leq r] - P\phi(\mathrm{i}a_{\underline{\delta}}G)$ being compact. We can now fix first σ small enough and subsequently r large enough such that

$$2\mathrm{Re}\{PN_{\underline{\delta}}\mathbb{1}[N_\delta > r]\overline{P}\} \geq -\sigma N_{\underline{\delta}} - \frac{\epsilon}{9}\mathbb{1}_{\mathcal{H}}$$

and

$$8\sigma\|a_{\underline{\delta}}G\| \leq \frac{\epsilon}{9}, \quad \frac{1-\frac{8\epsilon}{9}}{1+2\sigma} > 1-\epsilon. \tag{4.51}$$

To deal with the term $K\overline{P}$ we note that we can choose κ small enough such that $2\|K\overline{P}\mathbb{1}[|H-\lambda|<\kappa]\| \leq \epsilon/18$ uniformly in λ. Indeed, there exists Λ such that $2\|K\overline{P}\mathbb{1}[|H|>\Lambda]\| \leq \epsilon/18$ and hence by a covering argument there exists $\kappa>0$ such that $2\|K\overline{P}\mathbb{1}[|H-\lambda|<\kappa]\| \leq \epsilon/18$ uniformly in $\lambda\in\mathbb{R}$. We thus get for all $\lambda, E\in\mathbb{R}$.

$$\begin{aligned} 2\mathrm{Re}\{K\overline{P}\} &= 2\mathrm{Re}\{K\overline{P}\mathbb{1}[|H-\lambda-E|>\kappa]\} + 2\mathrm{Re}\{K\overline{P}\mathbb{1}[|H-\lambda-E|<\kappa]\} \\ &\geq -\frac{\epsilon}{18}\mathbb{1}_{\mathcal{H}} - \frac{C}{\epsilon}\mathbb{1}[|H-\lambda-E|>\kappa] - \frac{\epsilon}{18}\mathbb{1}_{\mathcal{H}}. \end{aligned}$$

Inserting this together with (4.50) into (4.49) we arrive at the bound

$$H' \geq \big(1-\frac{\epsilon}{9}\big)\overline{P} - \frac{3\epsilon}{9}\mathbb{1}_{\mathcal{H}} - \sigma N_{\underline{\delta}} - C\mathbb{1}[|H-\lambda-E|>\kappa]. \tag{4.52}$$

From the spectral theorem in multiplication operator form, we get a measure space $(\mathcal{M},\Sigma,\mu)$, a measurable real function f on $\mathcal{M}$ and a unitary map $U\colon\mathcal{H}\to L^2(\mathcal{M})$ such that $UHU^* = M_f$, multiplication by f. Put $U^{\mathbf{c}} = UC$ such that $U^{\mathbf{c}}H^{\mathbf{c}}U^{\mathbf{c}*} = M_f$ as well. Here $U^{\mathbf{c}*} = CU^*$. The combined map $U^{\mathrm{L}} = U\otimes$

$U^{\mathbf{c}}\colon \mathcal{H}^{\mathrm{L}} \to L^2(\mathcal{M}\times\mathcal{M})$ (with product σ-algebra and measure) now sets up the correspondence $U^{\mathrm{L}} L_\infty U^{\mathrm{L}*} = M_{f_1-f_2}$, where $f_j(q_1,q_2) = f(q_j)$.

Then under the identification $L^2(\mathcal{M}\times\mathcal{M}) = L^2(\mathcal{M}; L^2(\mathcal{M}))$ we get

$$\mathbb{1}[|M_{f_1-f_2} - E| > \kappa] = \int_{\mathcal{M}}^{\oplus} \mathbb{1}[|M_f - f(q) - E| > \kappa]\, \mathrm{d}\mu(q).$$

Hence we conclude from (4.52) the estimate

$$\begin{aligned} U^{\mathrm{L}} H' \otimes \mathbb{1}_{\mathcal{H}} U^{\mathrm{L}*} &= \int_{\mathcal{M}}^{\oplus} U H' U^* \mathrm{d}\mu(q) \\ &\geq \int_{\mathcal{M}}^{\oplus} (1-\frac{\epsilon}{9}) U\overline{P}U^* - \frac{3\epsilon}{9}\mathbb{1}_{L^2(\mathcal{M})} - \sigma U N_{\underline{\delta}} U^* - C\mathbb{1}[|M_f - f(q) - E| > \kappa]\, \mathrm{d}\mu(q) \\ &= U^{\mathrm{L}} \left((1-\frac{\epsilon}{9})\overline{P}\otimes \mathbb{1}_{\mathcal{H}} - \frac{3\epsilon}{9}\mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - \sigma N_{\underline{\delta}} \otimes \mathbb{1}_{\mathcal{H}} - C\mathbb{1}[|L_\infty - E| > \kappa] \right) U^{\mathrm{L}*} \end{aligned}$$

in the sense of forms on $U^{\mathrm{L}}\mathcal{D}(N^{\mathrm{L}})$. Adding to the above a similar bound for $\mathbb{1}_{\mathcal{H}} \otimes H^{\mathbf{c}\prime}$ yields

$$\begin{aligned} L'_\infty &\geq \big(1-\frac{\epsilon}{9}\big)\big[\overline{P}\otimes\mathbb{1}_{\mathcal{H}} + \mathbb{1}_{\mathcal{H}}\otimes\overline{P}^{\mathbf{c}}\big] - \frac{6\epsilon}{9}\mathbb{1}_{\mathcal{H}^{\mathrm{L}}} \\ &\quad - \sigma N^{\mathrm{L}} - C\big(\mathbb{1}[|L_\infty - E| > \kappa] + P\otimes P^{\mathbf{c}}\big) \\ &\geq (1-\frac{7\epsilon}{9})\mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - \sigma N^{\mathrm{L}}_{\underline{\delta}} - (C+2)\big(\mathbb{1}[|L_\infty - E| > \kappa] + P\otimes P^{\mathbf{c}}\big). \end{aligned}$$

Here we used that

$$\overline{P}\otimes\mathbb{1}_{\mathcal{H}} + \mathbb{1}_{\mathcal{H}}\otimes\overline{P}^{\mathbf{c}} = 2\mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - P\otimes\overline{P}^{\mathbf{c}} - \overline{P}\otimes P^{\mathbf{c}} - 2P\otimes P^{\mathbf{c}} \geq \mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - 2P\otimes P^{\mathbf{c}}.$$

We now complete the proof, cf. (4.51), by estimating

$$(1+2\sigma)L'_\infty \geq \big(1-\frac{8\epsilon}{9}\big)\mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - C\big(\mathbb{1}[|L_\infty - E| > \kappa] + P\otimes P^{\mathbf{c}}\big).$$

as at the end of the proofs of Lemma 4.5 and Theorem 4.7. □

In order to perturb around zero temperature, we first need to control the difference $\widetilde{G}_\beta - \widetilde{G}_\infty$.

Lemma 4.14. *Suppose* **(LGn)**. *For any $\beta_0 > 0$ there exists $C > 0$ such that for all $\beta \geq \beta_0$ we have*

$$\big\|\widetilde{G}_\beta - \widetilde{G}_\infty\big\| \leq C\beta^{-\frac{1}{2}}. \tag{4.53}$$

If $n \geq 1$ we have furthermore that for all $\underline{\delta} \in \Delta_0$

$$\big\|\tilde{a}_{\underline{\delta}}(\widetilde{G}_\beta - \widetilde{G}_\infty)\big\| \leq C\beta^{-\frac{1}{2}}. \tag{4.54}$$

Proof. We begin with (4.53). For simplicity we only consider the contribution $(\sqrt{1+\widetilde{\rho}_\beta}-1)\widetilde{G}_\infty$, the other term $\sqrt{\widetilde{\rho}_\beta}\widetilde{G}_{\infty,\mathcal{R}}$ being similar.

We split into the infrared and ultraviolet regimes and estimate first for $|\omega|\leq 1$:

$$\big(\sqrt{1+\widetilde{\rho}_\beta(\omega)}-1\big)^2\big|\widetilde{G}_\infty(\omega,\Theta)\big|^2\leq C\big(\sqrt{1+\widetilde{\rho}_\beta(\omega)}-1\big)^2|\omega|^{2n+2\mu}.$$

Hence we can bound the L^2-norm squared of the contribution by a multiple of

$$\int_0^1\big(\sqrt{1+\widetilde{\rho}_\beta(\omega)}-1\big)^2\omega^{2n+2\mu}\,\mathrm{d}\omega\leq\beta^{-1}\int_0^1(1+\omega)\omega^{2n-1+2\mu}\,\mathrm{d}\omega,$$

where we simply discarded the -1 term coming from $\widetilde{G}_\infty$. The integral is finite for all $n\geq 0$. In fact, the effect of subtracting $\widetilde{G}_\infty$ sits in the ultraviolet part where $|\omega|\geq 1$. Here we estimate the L^2-norm squared by

$$\int_1^\infty\big(\sqrt{1+\widetilde{\rho}_\beta(\omega)}-1\big)^2\omega^{-1-2\mu}\,\mathrm{d}\omega\leq\frac{\big(\sqrt{1+\widetilde{\rho}_\beta(1)}-1\big)^2}{2\mu}.$$

Since $\sqrt{1+\widetilde{\rho}_\beta(1)}-1=\sqrt{1/(1-\mathrm{e}^{-\beta})}-1\sim\mathrm{e}^{-\beta/2}$ inn the limit of large β, we get for a fixed $\beta_0>0$ a constant $C=C(\beta_0)$ such that for all $\beta>\beta_0$ we have (4.53) satisfied.

To establish (4.54) we observe that

$$\tilde{a}_{\underline{\delta}}\widetilde{G}_\beta=\big(\sqrt{1+\widetilde{\rho}_\beta}-1\big)\tilde{a}_{\underline{\delta}}\widetilde{G}_\infty+m_{\underline{\delta}}\widetilde{G}_\infty\frac{\partial\sqrt{1+\widetilde{\rho}_\beta}}{\partial\omega}.$$

The first contribution can be estimate exactly as above, using that $n\geq 1$, and yields an $O(\beta^{-1/2})$ term. For the second term we compute

$$\frac{\partial\sqrt{1+\widetilde{\rho}_\beta}}{\partial\omega}=-\frac{\beta}{2}\widetilde{\rho}_\beta\sqrt{1+\widetilde{\rho}_\beta}.$$

In the infrared regime this can be dealt with easily since $\beta\widetilde{\rho}_\beta\leq 1/|\omega|$ and the extra inverse power of ω can be absorbed into $\widetilde{G}_\infty$. Recall that we assume $n\geq 1$. For the ultraviolet regime we get exponential decay in β from $\widetilde{\rho}_\beta(1)$ and we are done. □

We remark that a similar bound holds for $\tilde{a}_{\underline{\delta}}^2(\widetilde{G}_\beta-\widetilde{G}_\infty)$ under a **(LG2)** condition but we do not need this.

Theorem 4.15. *Suppose* **(LG1)**. *Let $\epsilon>0$ be given. There exists $\beta_0>0$, $\kappa>0$ and $C>0$ such that for all $E\in\mathbb{R}$, $\underline{\delta}\in\Delta(\underline{\delta}')$, and $\beta\geq\beta_0$*

$$L_\beta'\geq(1-\epsilon)\mathbb{1}-C\big(\mathbb{1}[|L_\beta-E|\geq\kappa]+P\otimes P^{\mathbf{c}}\big)$$

in the sense of forms on $\mathcal{D}(N^{\mathrm{L}})$.

Proof. From (4.28), Proposition 4.12, applied with $\epsilon/4$ instead of ϵ, and Lemma 4.14 we get as a form bound on $\mathcal{D}(\widetilde{N})$

$$\begin{aligned}\widetilde{L}'_\beta &= \widetilde{L}'_\infty - \phi\big(\mathrm{i}\tilde{a}_{\underline{\delta}}(\widetilde{G}_\beta - \widetilde{G}_\infty)\big)\\ &\geq \big(1 - \frac{\epsilon}{4} - \frac{C_1}{\sigma\beta}\big)\mathbb{1}_{\widetilde{\mathcal{H}}^{\mathrm{L}}} - \sigma\widetilde{N} - C\big(\mathbb{1}[|\widetilde{L}_\infty - E| > \kappa] + P_\infty\big).\end{aligned}$$

Here $P_\infty = \mathcal{U}(P\otimes P^{\mathbf{c}})\mathcal{U}^*$ is the projection onto the eigenstates of $\widetilde{L}_\infty$.

Pick a non-negative $f \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(f) \subset [-\kappa,\kappa]$ and $f = 1$ on $[-\kappa/2,\kappa/2]$. Let $\tilde{f}$ be an almost analytic extension of f. Write

$$f(\widetilde{L}_\infty - E) - f(\widetilde{L}_\beta - E) = \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(\eta)\big((\widetilde{L}_\infty - \eta)^{-1} - (\widetilde{L}_\beta - \eta)^{-1}\big)\,\mathrm{d}\eta.$$

Since $\widetilde{N}$ is of class $C^1(\widetilde{L}_\beta)$ with $[\widetilde{N},\widetilde{L}_\beta]^\circ$ being $\sqrt{\widetilde{N}}$-bounded, cf. Corollary 2.6, we conclude from [16, Lemma 3.3] that $(\widetilde{L}_\beta - \eta)^{-1}$ preserves $\mathcal{D}(\sqrt{\widetilde{N}})$ and from [16, (3.9)] we in fact find that there exists n and C such that

$$\big\|(\widetilde{N}+1)^{\frac{1}{2}}(\widetilde{L}_\beta - \eta)^{-1}(\widetilde{N}+1)^{-\frac{1}{2}}\big\| \leq C\big(|\mathrm{Im}\eta|^{-1} + |\mathrm{Im}\eta|^{-n}\big).$$

It follows that

$$\begin{aligned}&\big\|\big(f(\widetilde{L}_\infty - E) - f(\widetilde{L}_\beta - E)\big)(\widetilde{N}+1)^{-\frac{1}{2}}\big\|\\ &\quad\leq \frac{1}{\pi}\int_{\mathbb{C}} |\bar{\partial}\tilde{f}(\eta)|\,|\mathrm{Im}\eta|^{-1}\big\|\phi\big(\widetilde{G}_\beta - \widetilde{G}_\infty\big)(\widetilde{L}_\beta - \eta)^{-1}(\widetilde{N}+1)^{-\frac{1}{2}}\big\|\,\mathrm{d}\eta\\ &\quad\leq C\big\|\phi\big(\widetilde{G}_\beta - \widetilde{G}_\infty\big)(\widetilde{N}+1)^{-\frac{1}{2}}\big\|.\end{aligned}$$

Appealing to Lemma 4.14 we thus get

$$\begin{aligned}\mathbb{1}[|\widetilde{L}_\infty - E| > \kappa] &\leq \mathbb{1}_{\widetilde{\mathcal{H}}^{\mathrm{L}}} - f(\widetilde{L}_\infty - E)\\ &\leq \mathbb{1}_{\widetilde{\mathcal{H}}^{\mathrm{L}}} - f(\widetilde{L}_\beta - E) + \sigma\widetilde{N} + \frac{C_2}{\sigma\beta}\\ &\leq \mathbb{1}[|\widetilde{L}_\beta - E| > \kappa/2] + \sigma\widetilde{N} + \frac{C_2}{\sigma\beta}.\end{aligned}$$

Choose first $\sigma > 0$ small enough such that

$$12\sigma \sup_{\beta\geq 1,\underline{\delta}\in\Delta_0}\big\|\tilde{a}_{\underline{\delta}}\widetilde{G}_\beta\big\| < \frac{\epsilon}{4} \quad \text{and} \quad \frac{1 - 3\epsilon/4}{1+3\sigma} > 1 - \epsilon, \tag{4.55}$$

and subsequently $\beta_0 \geq 1$ large enough such that

$$\frac{C_1}{\sigma\beta_0} + \frac{C_2}{\sigma\beta_0} < \frac{\epsilon}{4}.$$

With these choices we arrive at the bound

$$\widetilde{L}'_\beta \geq \big(1 - \frac{\epsilon}{2}\big)\mathbb{1}_{\widetilde{\mathcal{H}}^{\mathrm{L}}} - 2\sigma\widetilde{N} - C\big(\mathbb{1}[|\widetilde{L}_\beta - E| \geq \kappa/2] + P_\infty\big).$$

We conclude the proof by the usual argument, i.e. bounding $(1+3\sigma)\widetilde{L}'_\beta$ from below, cf. (4.55) and the previous proof. □

We conclude

Corollary 4.16. *Suppose* **(LG2)***. There exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$, the Liouvillean L_β has finitely many eigenvalues, all of finite multiplicity.*

We remark that in a (β, G)-regime where a positive commutator estimate holds we can under the **(LG2)** condition conclude that eigenstates ψ of the standard Liouvillean L_β satisfy that $\psi \in \mathcal{D}(N^{\mathrm{L}})$. This is a consequence of [16] and improves the basic number bound Theorem 3.4, without imposing further conditions on G.

4.5 Open Problems III

The by far most central open question relevant for this section is whether or not one can establish a positive commutator estimate for the standard Liouvillean for arbitrary inverse temperature β and coupling G. We have an unsubstantiated inkling that it should be possible to use $\widetilde{A}_{\underline{\delta}}$.

Problem 4.1. Establish, for arbitrary β and G, a positive commutator estimate for the Jakšić-Pillet glued standard Liouvillean $\widetilde{L}_\beta$, possibly making use of the conjugate operator $\widetilde{A}_{\underline{\delta}}$.

We remark that we have not in this section made use of the modular conjugation J, cf. (2.9), which takes L_β to $-L_\beta$. This may be an extra ingredient to make use of.

While one can establish positive commutator estimates for the Hamiltonian also for infinite dimensional small systems, cf. [23], the situation is fundamentally different for standard Liouvilleans. To see this consider as the small system a one-dimensional harmonic oscillator. Here the uncoupled Liouvillean L_0 will have point spectrum (a multiple of) $\mathbb{Z}$, with each eigenvalue having infinite multiplicity. Hence, one should not expect a positive commutator estimate with compact error terms, barring some mechanism to lift the infinite degeneracy by other means. However, in the dipole approximation this model is explicitly solvable [4, 5] and Könenberg in his thesis managed to handle perturbations of the Harmonic oscillator potential [36]. Note that one can construct a small system where the Hamiltonian K has compact resolvent and L_0 has point spectrum which is dense in $\mathbb{R}$! See also [19, 20] where an atomic small system is considered, and positive commutator methods are applied in the weak coupling regime.

Problem 4.2. What can be said about the general structure of point spectrum without the assumption of small coupling or a finite dimensional small system. Are positive commutator estimates useful at all?

We emphasize that all the proofs from Subsect. 4.4 make essential use of $\mathcal{K}$ being finite dimensional.

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