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Workshop: Combinatorial Problems Raised by Statistical Mechanics

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MAYER'S GRAPH WEIGHTS FOR THE HARD-CORE CONTINUUM GAS IN ONE DIMENSION

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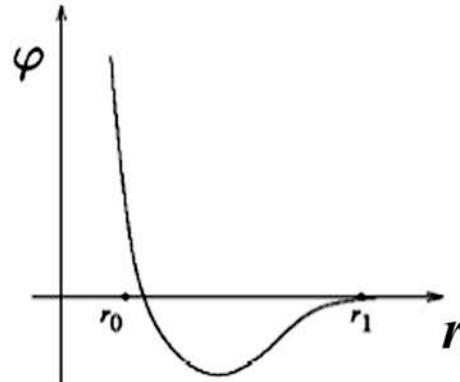
Short review of Mayer's theory of cluster integrals

In the context of a non-ideal gas with N particles in a vessel V included in \mathbb{R}^d , we represent the particles' positions by vectors $\vec{x}_1, \dots, \vec{x}_N$.
The system is free from external influences.

The *partition function* is defined as

$$Z(V, T, N) = \frac{1}{N! \lambda^{dN}} \int_V \cdots \int_V \exp \left(-\beta \sum_{i < j} \varphi(|\vec{x}_i - \vec{x}_j|) \right) d\vec{x}_1 \dots d\vec{x}_N,$$

where λ and β depend on the temperature T and where the interaction between two particles at distance r is expressed by a potential function $\varphi(r)$ as illustrated in



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$$Z(V, N, T) = \frac{1}{N! \lambda^{3N}} \int_V \cdots \int_V \exp \left(-\beta \sum_{i < j} \varphi(|\vec{x}_i - \vec{x}_j|) \right) d\vec{x}_1 \cdots d\vec{x}_N,$$

(partition function)

The *grand canonical partition function*

is the generating function for the partition functions,

$$Z_{\text{gr}}(V, T, z) = \sum_{N=0}^{\infty} Z(V, N, T) (\lambda^3 z)^N,$$

where the variable z is called the *fugacity* or the *activity*.

Macroscopic parameters :

$$\frac{P}{kT} = \frac{1}{V} \log Z_{\text{gr}}(V, T, z), \quad \bar{N} = z \frac{\partial}{\partial z} \log Z_{\text{gr}}(V, T, z), \quad \rho := \frac{\bar{N}}{V}.$$

pressure P , the average number of particles \bar{N} , and the *density* ρ .

$$Z(V, N, T) = \frac{1}{N! \lambda^{3N}} \int_V \cdots \int_V \exp \left(-\beta \sum_{i < j} \varphi(|\vec{x}_i - \vec{x}_j|) \right) d\vec{x}_1 \cdots d\vec{x}_N$$

$$Z_{\text{gr}}(V, T, z) = \sum_{N=0}^{\infty} Z(V, N, T) (\lambda^3 z)^N \quad \frac{P}{kT} = \frac{1}{V} \log Z_{\text{gr}}(V, T, z) \quad \rho := \frac{N}{V}$$

The virial expansion :

$$\frac{P}{kT} = \frac{N}{V} + \gamma_2(T) \left(\frac{N}{V} \right)^2 + \gamma_3(T) \left(\frac{N}{V} \right)^3 + \dots$$

(proposed by Kamerlingh Onnes, 1901)

It is the starting point of Mayer's theory of "cluster integrals":

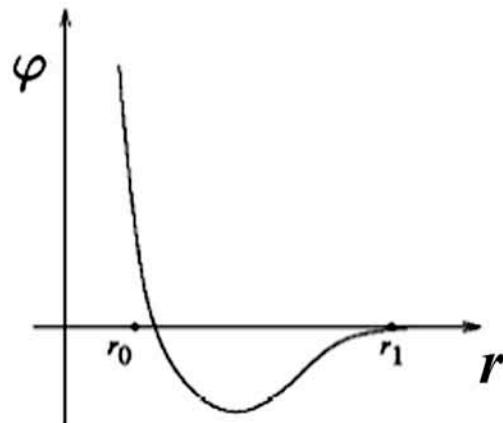


$$Z(V, N, T) = \frac{1}{N! \lambda^{3N}} \int_V \cdots \int_V \exp \left(-\beta \sum_{i < j} \varphi(|\vec{x}_i - \vec{x}_j|) \right) d\vec{x}_1 \cdots d\vec{x}_N$$

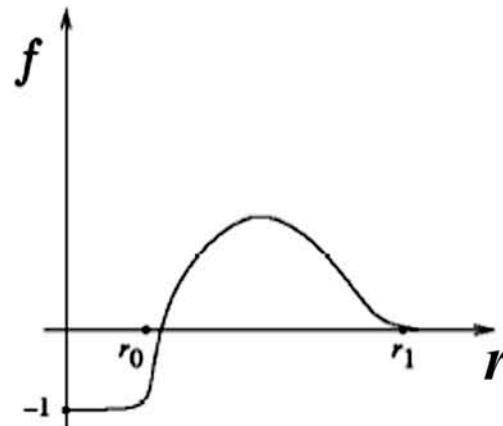
Mayer's theory of “cluster integrals”

Mayer's idea consists of setting

$$1 + f_{ij} = \exp(-\beta \varphi(|\vec{x}_i - \vec{x}_j|)), \quad f_{ij} = f(r_{ij}).$$



a) the function $\varphi(r)$



b) the function $f(r) = \exp(-\beta \varphi(r)) - 1$

$$\begin{aligned}
Z(V, N, T) &= \frac{1}{N! \lambda^{3N}} \int_V \cdots \int_V \exp \left(-\beta \sum_{i < j} \varphi(|\vec{x}_i - \vec{x}_j|) \right) d\vec{x}_1 \cdots d\vec{x}_N \\
&= \frac{1}{N! \lambda^{3N}} \int_V \cdots \int_V \prod_{1 \leq i < j \leq N} (1 + f_{ij}) d\vec{x}_1 \cdots d\vec{x}_N. \\
&= \frac{1}{N! \lambda^{3N}} \sum_{g \in \mathcal{G}[N]} \int_V \cdots \int_V \prod_{\{i,j\} \in g} f_{ij} d\vec{x}_1 \cdots d\vec{x}_N \\
&= \frac{1}{N! \lambda^{3N}} \sum_{g \in \mathcal{G}[N]} W(g), \quad \mathcal{G}[N] = \text{the set of all simple graphs on } \{1, 2, 3, \dots, N\}
\end{aligned}$$

where the weight $W(g)$ of a graph g is given by the integral

$$W(g) = \int_V \cdots \int_V \prod_{\{i,j\} \in g} f_{ij} d\vec{x}_1 \cdots d\vec{x}_N.$$

(this is the *first Mayer weight* of a graph g)

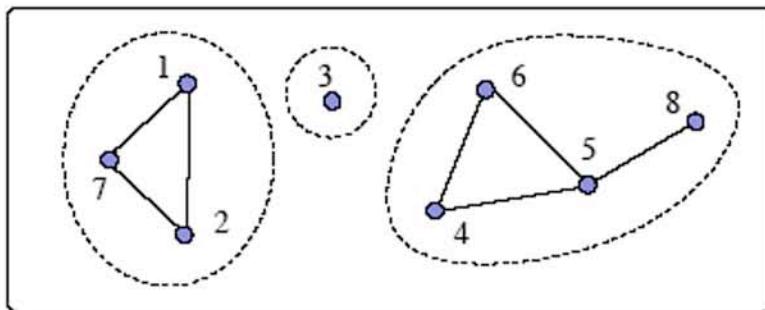
In terms of $W(g)$, the grand-canonical function becomes

$$\begin{aligned} Z_{\text{gr}}(V, T, z) &= \sum_{N=0}^{\infty} Z(V, N, T) (\lambda^d z)^N \\ &= \sum_{N=0}^{\infty} \frac{1}{N! \lambda^{dN}} \sum_{g \in \mathcal{G}[N]} W(g) (\lambda^d z)^N \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{g \in \mathcal{G}[N]} W(g) z^N \\ &= \mathcal{G}_W(z), \end{aligned}$$

the exponential generating series of graphs weighted by the function W .

Proposition *$W(g)$, the First Mayer Weight of a simple graph g , is multiplicative on the connected components of g . In other words, for c_1, c_2, \dots, c_m the m connected components of g , we have*

$$W(g) = W(c_1)W(c_2)\dots W(c_m).$$



$$W(g) = W(c_1)W(c_2)\dots W(c_m)$$

Since W is multiplicative on connected components, the exponential formula can be used:

$$\mathcal{G}_W(z) = \exp(\mathcal{C}_W(z)),$$

where \mathcal{C} denotes the species (class) of connected graphs, so that

$$\log \mathcal{G}_W(z) = \mathcal{C}_W(z) = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{c \in \mathcal{C}[N]} W(c) z^N.$$

Corollary 2 *The pressure of the system can be expressed in terms of the exponential generating function of connected graphs weighted by W . More precisely, we have*

$$\frac{P}{kT} = \frac{1}{V} \log Z_{\text{gr}}(V, T, z) = \frac{1}{V} \mathcal{C}_W(z).$$

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The thermodynamic limit $w(c)$

Let c be a connected graph over $\{1, 2, \dots, N\}$. The **Second Mayer weight $w(c)$** is defined as the limit (which exists under certain integrability conditions)

$$\begin{aligned} w(c) &= \lim_{V \rightarrow \infty} \frac{1}{V} W(c) \\ &= \lim_{V \rightarrow \infty} \frac{1}{V} \int_{V^N} \prod_{\{i,j\} \in c} f_{ij} d\vec{x}_1 \dots d\vec{x}_N. \\ &= \int_{(\mathbb{R}^d)^{N-1}} \prod_{\{i,j\} \in c; \vec{x}_N = \vec{0}} f_{ij} d\vec{x}_1 \dots d\vec{x}_{N-1}. \end{aligned}$$

Here, V going to infinity has the following meaning. The vessel $V \in \mathbb{R}^d$ must contain a ball $B(0, R)$ centered at the origin, with radius $R \in]0, \infty)$. V goes to infinity means that R goes to infinity.

Conditions for the existence of

$$w(c) = \lim_{V \rightarrow \infty} \frac{1}{V} W(c) = \lim_{V \rightarrow \infty} \frac{1}{V} \int_{V^N} \prod_{\{i,j\} \in c} f_{ij} d\vec{x}_1 \dots d\vec{x}_N \quad (*)$$

Proposition *If the function $f : [0, \infty) \rightarrow \mathbb{R}$ is integrable and bounded and if*

$$\int_0^\infty r^{d-1} |f(r)| dr < \infty,$$

(for example if $|f(r)| = O(1/r^{d+\epsilon})$, $r \rightarrow \infty$)

then for any fixed $\vec{x}_N \in \mathbb{R}^d$, the function $F_{\vec{x}_N} : \mathbb{R}^{d(N-1)} \rightarrow \mathbb{R}$, defined by

$$F_{\vec{x}_N}(\vec{x}_1, \dots, \vec{x}_{N-1}) = \prod_{\{i,j\} \in c} f(|\vec{x}_i - \vec{x}_j|) = \prod_{\{i,j\} \in c} f_{ij}$$

is integrable over $(\mathbb{R}^d)^{N-1}$ and its integral is independent of \vec{x}_N . Moreover the limit $()$ exists and is equal to*

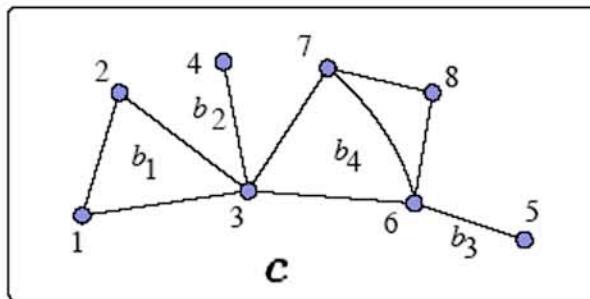
$$w(c) = \int_{(\mathbb{R}^d)^{N-1}} \prod_{\{i,j\} \in c; \vec{x}_N = \vec{0}} f_{ij} d\vec{x}_1 \dots d\vec{x}_{N-1}.$$

In the thermodynamic limit, the pressure is given by

$$\begin{aligned}\frac{P}{kT} &= \lim_{V \rightarrow \infty} \frac{1}{V} \log Z_{\text{gr}}(V, T, z) \\ &= \lim_{V \rightarrow \infty} \frac{1}{V} \mathcal{C}_W(z) \\ &= \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{c \in \mathcal{C}[N]} \lim_{V \rightarrow \infty} \frac{1}{V} W(c) z^N \\ &= \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{c \in \mathcal{C}[N]} w(c) z^N \\ &= \mathcal{C}_w(z).\end{aligned}$$

Proposition *In the thermodynamic limit, the pressure of the system is given directly in terms of the exponential generating function of connected graphs weighted by the Second Mayer Weight $w(c)$, according to formula*

$$\frac{P}{kT} = \mathcal{C}_w(z).$$



block = *maximal 2-connected subgraph*

A connected graph c with blocks b_1, b_2, b_3, b_4

Proposition *The second Mayer weight w is block-multiplicative. More precisely, for any connected graph c whose blocks are b_1, b_2, \dots, b_m , we have*

$$w(c) = w(b_1)w(b_2)\dots w(b_m).$$

Proof.

$$\begin{aligned} w(c) &= \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\} \in c; \vec{x}_N = \vec{0}} f_{ij} d\vec{x}_1 d\vec{x}_2 \dots d\vec{x}_{N-1} \\ &= \int_{\mathbb{R}^{d(N-1)}} \prod_{\{i,j\} \in b; \vec{x}_N = \vec{0}} f_{ij} \prod_{\{i,j\} \in c \setminus b; \vec{x}_N = \vec{0}} f_{ij} d\vec{x}_1 d\vec{x}_2 \dots d\vec{x}_{N-1} \\ &= \int_{\mathbb{R}^{dk}} \prod_{\{i,j\} \in b; \vec{x}_N = \vec{0}} f_{ij} d\vec{x}_1 d\vec{x}_2 \dots d\vec{x}_k \times \int_{\mathbb{R}^{d(N-k-1)}} \prod_{\{i,j\} \in c \setminus b; \vec{x}_N = \vec{0}} f_{ij} d\vec{x}_{k+1} d\vec{x}_{k+2} \dots d\vec{x}_{N-1} \\ &= w(b) \cdot w(c \setminus b). \end{aligned}$$

Block-multiplicativity of the second Mayer weight w implies :

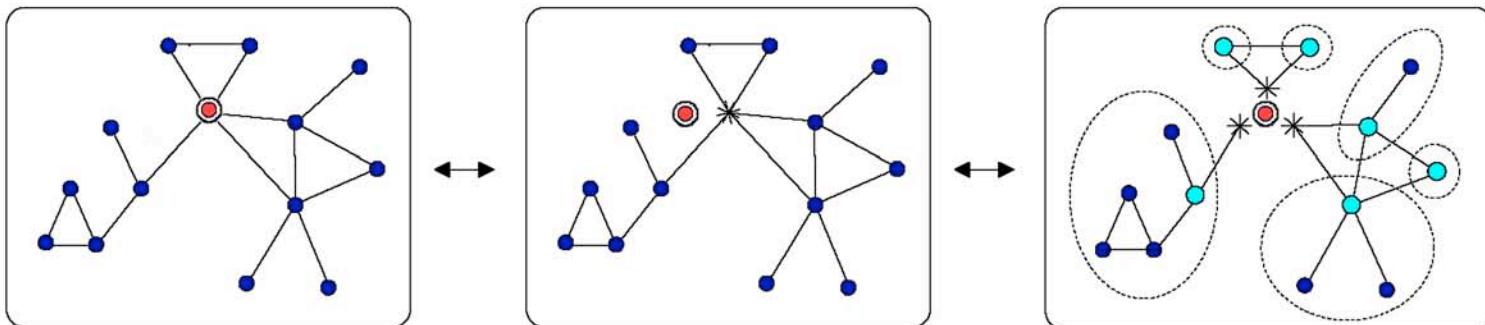
Proposition Let \mathcal{B} and \mathcal{C}_w^* be the species of blocks and of pointed connected graphs, then

$$\mathcal{C}_w^* = X \cdot E(\mathcal{B}'_w(\mathcal{C}_w^*)).$$

For the corresponding generating series, we have

Note: $\mathbf{F}^* = X \cdot \mathbf{F}'$,
 $\mathbf{F}^*(z) = z\mathbf{F}'(z)$.

$$\mathcal{C}_w^*(z) = z \exp(\mathcal{B}'_w(\mathcal{C}_w^*(z))).$$



More generally, we have :

Theorem Let w be a block-multiplicative weight function on connected graphs with all blocks in a particular species \mathcal{B} . Then we have

$$\mathcal{C}_{\mathcal{B},w}^* = X \cdot E(\mathcal{B}'_w(\mathcal{C}_{\mathcal{B},w}^*)), \quad \mathcal{C}_{\mathcal{B},w}^*(z) = z \exp(\mathcal{B}'_w(\mathcal{C}_{\mathcal{B},w}^*(z))).$$

Computation of the virial expansion (following Uhlenbeck and Ford)

We have, for the density $\rho(z) = z \frac{\partial}{\partial z} \frac{1}{V} \log Z_{\text{gr}}(V, T, z) = z \frac{\partial}{\partial z} \mathcal{C}_w(z) = \mathcal{C}_w^*(z)$.

Since $\mathcal{C}_w^*(z) = z \exp(\mathcal{B}'_w(\mathcal{C}_w^*(z)))$ it follows that $\rho(z) = z \exp \mathcal{B}'_w(\rho(z))$.

$$\frac{P}{kT} = \frac{1}{V} \log Z_{\text{gr}}(V, T, z) = \mathcal{C}_w(z) = \int_0^z \mathcal{C}'_w(t) dt = \int_0^z \frac{\rho(t)}{t} dt.$$

Let us make the change of variable $t = t(r) = r \exp(-\mathcal{B}'_w(r))$, inverse function of $r = \rho(t)$.

Note that $\rho(0) = 0$ and $\rho(z) = \rho$, and also that $dt = [\exp(-\mathcal{B}'_w(r)) - r \exp(-\mathcal{B}'_w(r)) \cdot \mathcal{B}''_w(r)] dr$.

$$\begin{aligned} \frac{P}{kT} &= \int_0^z \frac{\rho(t)}{t} dt = \int_0^\rho (1 - r \mathcal{B}''_w(r)) dr = \rho - \int_0^\rho r \mathcal{B}''_w(r) dr \\ &= \rho - \int_0^\rho \sum_{n \geq 1} n \beta_{n+1} \frac{r^n}{n!} dr = \rho - \sum_{n \geq 2} (n-1) \beta_n \frac{\rho^n}{n!}, \end{aligned}$$

where we have set $\mathcal{B}_w(r) = \sum_{n \geq 2} \beta_n \frac{r^n}{n!}$. This is precisely the virial expansion, with $\rho = \frac{N}{V}$.

Hence the n^{th} virial coefficient, for $n \geq 2$, is given by $\gamma_n(T) = -\frac{(n-1)}{n!} \beta_n = -\frac{(n-1)}{n!} |\mathcal{B}[n]|_w$.

Virial expansion (combinatorial form)

$$\frac{P}{kT} = \mathcal{C}_w(z) = \frac{\bar{N}}{V} + \gamma_2(T) \left(\frac{\bar{N}}{V} \right)^2 + \gamma_3(T) \left(\frac{\bar{N}}{V} \right)^3 + \dots$$

where

$$\gamma_n(T) = -\frac{(n-1)}{n!} |\mathcal{B}[n]|_w.$$

and \mathcal{B} is the species of all 2-connected graphs weighted by

$$w(c) = \int_{(\mathbb{R}^d)^{n-1}} \prod_{\{i,j\} \in c; \vec{x}_n = \vec{0}} f_{ij} d\vec{x}_1 \dots d\vec{x}_{n-1}. \quad (\text{Second Mayer weight})$$

The following combinatorial equations hold

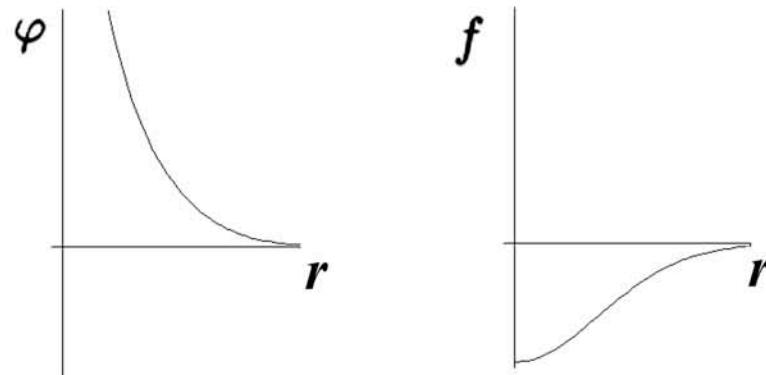
$$\mathcal{C}_w^\bullet = X \cdot E(\mathcal{B}'_w(\mathcal{C}_w^\bullet)), \quad \mathcal{G}_w = E(\mathcal{C}_w)$$

where \mathcal{G}_w is the species of all graphs

and \mathcal{C}_w is the species of all connected graphs.

Gaussian model

$$f(r) = -\exp(-\alpha r^2)$$



It is interesting, mathematically, to consider a Gaussian model, where

$$f_{ij} = -\exp(-\alpha||\vec{x}_i - \vec{x}_j||^2),$$

which corresponds to a soft repulsive potential, at constant temperature. In this case, all cluster integrals can be explicitly computed :

The weight $w(c)$ of a connected graph c has value

$$w(c) = (-1)^{e(c)} \left(\frac{\pi}{\alpha}\right)^{\frac{d(n-1)}{2}} \gamma(c)^{-\frac{d}{2}}.$$

where

$e(c)$ is the number of edges of c

and

$\gamma(c)$ is the graph complexity of c , that is, the number of spanning subtrees of c .

The hard-core continuum gas in one dimension

Consider N hard particles of diameter 1 on a line segment, of the form $[-D, D]$.

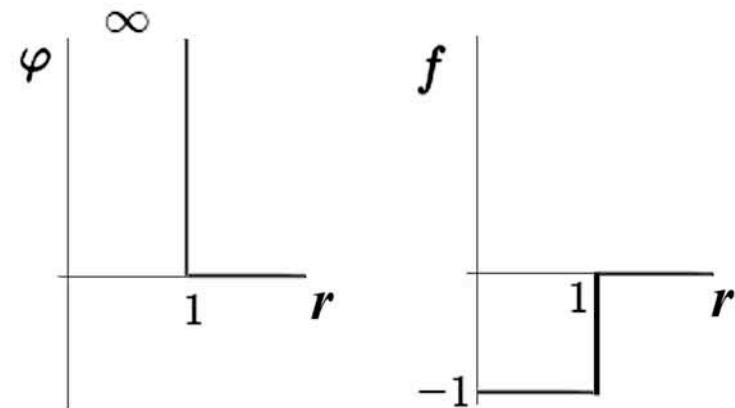


The *hard-core* constraint translates into the interaction potential

$$\varphi(r) = \infty, \text{ if } r < 1, \text{ and } \varphi(r) = 0, \text{ if } r \geq 1$$

and the Mayer function

$$f(r) = -\chi(r < 1).$$



$$f_{ij} = -\chi(|x_i - x_j| < 1) \iff f_{ij} = \chi(|x_i - x_j| \geq 1)$$

Second Mayer weight, in this case :

$$w(c) = (-1)^{e(c)} \int_{\mathbb{R}^{N-1}} \prod_{\{i,j\} \in c; x_N=0} \chi(|x_i - x_j| < 1) dx_1 \dots dx_{N-1}$$

Global formulas

$$\frac{P}{kT} = \mathcal{C}_w(z)$$

It is known (see *) that for the hard-core gas, the pressure of the system is given by

$$\frac{P}{kT} = L(z),$$

where $L(z)$ denotes the *Lambert function*, defined by the functional equation

$$L(z) \exp(L(z)) = z.$$

- * D. C. Brydges and J. Z. Imbrie, “Dimensional Reduction Formulas for Branched Polymer Correlation Functions.” Journal of Statistical Physics 110 (2003), 503–518. arXiv:math-ph/0203055 v2.
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Here, we give a combinatorial proof of this result 

Proposition *In the thermodynamic limit $D \rightarrow \infty$, the pressure of the continuous unidimensional hard-core gas model is given by*

$$\frac{P}{kT} = L(z). \quad (\text{classical Lambert function})$$

Proof. Note that the Lambert function satisfies $L(z) = -T(-z)$, where $T(z)$ is the exponential generating function of labelled rooted trees. Let us consider the particles on a segment of the form $[0, 2D]$. Then, since the $N!$ possible relative positions of the x_i give rise to integrals of equal value, the grand-canonical partition function can be written as

$$\begin{aligned} Z_{gr}(D, z) &= \sum_{N \geq 0} \frac{z^N}{N!} \int_{[0, 2D]^N} \prod_{i < j} \chi(|x_i - x_j| \geq 1) dx_1 dx_2 \cdots dx_N \\ &= \sum_{N \geq 0} z^N \int_{0 \leq x_1} dx_1 \int_{x_1+1 \leq x_2} dx_2 \cdots \int_{x_{N-1}+1 \leq x_N \leq 2D} dx_N. \end{aligned}$$

Now the integral is the volume of the rectangular simplex

$$0 \leq x_1 \leq x_2 - 1 \leq x_3 - 2 \leq \cdots \leq x_N - N + 1 \leq 2D - N + 1,$$

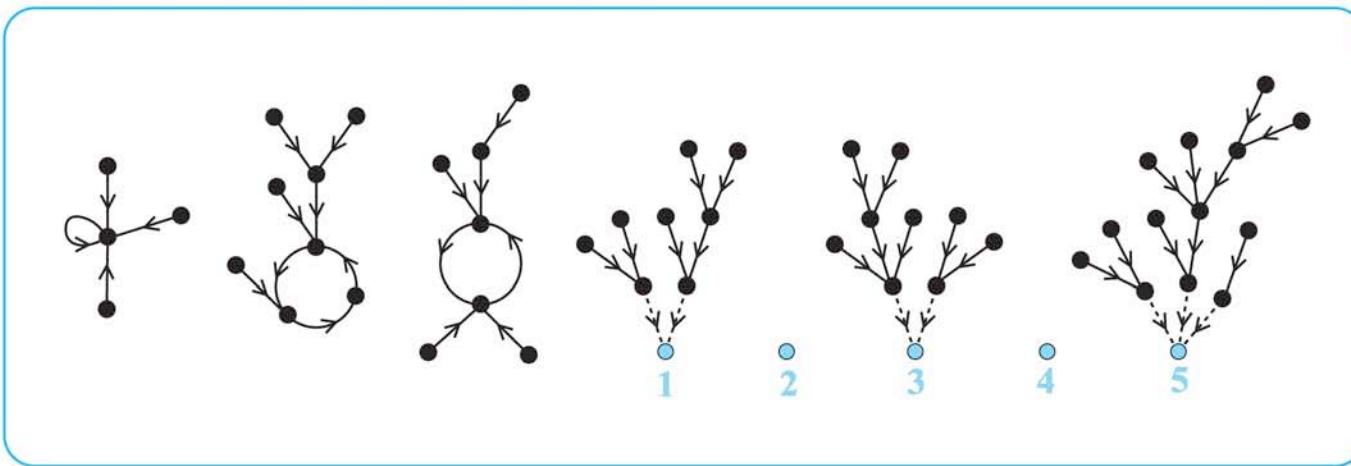
and has value $(2D - N + 1)^N / N!$.



It follows that

$$Z_{gr}(-D, -z) = \sum_{N \geq 0} (N + 2D - 1)^N z^N / N!,$$

which is the exponential generating function of structures which consist of functions of the form $f : U \rightarrow U + V$, where U is a finite set (of varying size N), V is a fixed set of size $|V| = 2D - 1$ and $+$ denotes the disjoint union.



Let f_n be the number of connected endofunctions on an n -element set and $\sum_{n \geq 0} f_n z^n / n!$ be their generating function. Then we have

$$\begin{aligned} Z_{gr}(-D, -z) &= \exp(F(z) + (2D - 1)T(z)) \\ \frac{P}{kT} &= \lim_{D \rightarrow \infty} \frac{1}{2D} \log Z_{gr}(D, z) = \lim_{D \rightarrow \infty} \frac{1}{2D} (F(-z) - (2D + 1)T(-z)) \\ &= -T(-z) = L(z). \end{aligned}$$

Corollary Let N be an integer ≥ 1 ; the total weight $|\mathcal{C}[N]|_w$ of the set of all connected graphs over the set $[N] = \{1, 2, \dots, N\}$ of vertices is given by

$$\sum_{c \in \mathcal{C}[N]} w(c) = (-N)^{N-1}.$$

Proof. This follows immediately from the fact that

$$\mathcal{C}_w(z) = \frac{P}{kT} = -T(-z)$$

by extracting coefficients. (A more direct combinatorial proof will be given by Bernardi in the next talk)

Proposition The fact that $\mathcal{C}_w(z) = L(z)$ is equivalent to the equation

$$\mathcal{B}_w(z) = z \log(1 - z).$$

Proof. It is clear from $\mathcal{C}_w^\bullet(z) = z \exp(\mathcal{B}'_w(\mathcal{C}_w^\bullet(z)))$ that any one of the functions $\mathcal{C}_w(z)$ and $\mathcal{B}_w(z)$ determines uniquely the other. Hence it suffices to prove that

$$L^\bullet(z) = z \exp(B'(L^\bullet(z))),$$

where the function $B(z)$ is defined by

$$B(z) = z \log(1 - z),$$

in order to establish the Proposition.

Proposition *Let N be an integer ≥ 2 ; the total weight of the set of all 2-connected graphs with N vertices is given by*

$$|\mathcal{B}[N]|_w = \sum_{c \in \mathcal{B}[N]} w(c) = -N(N-2)!$$

Proof. This follows immediately from

$$\mathcal{B}_w(z) = z \log(1-z)$$

by extracting coefficients.

Corollary *For the hard-core gas, the virial equation has the form*

$$\frac{P}{kT} = \frac{\overline{N}}{V} + \left(\frac{\overline{N}}{V}\right)^2 + \left(\frac{\overline{N}}{V}\right)^3 + \dots \quad (d=1)$$

Proof. $\gamma_n(T) = -\frac{(n-1)}{n!} |\mathcal{B}[n]|_w = -\frac{(n-1)}{n!} \times -n(n-2)! = 1.$

Individual weights $w(c)$

Question Can we compute the individual weights $w(c)$ of given connected graphs c and interpret them in terms of other graph invariants?

The Ehrhart polynomial

While trying to answer these questions, we have made the following observation.
Except for the sign, the weight

$$w(c) = (-1)^{e(c)} \int_{\mathbb{R}^{N-1}} \prod_{\{i,j\} \in c; x_N=0} \chi(|x_i - x_j| < 1) dx_1 \dots dx_{N-1}$$

can be seen as the volume of a convex polytope $\mathcal{P}(c)$ in \mathbb{R}^N bounded by the constraints $|x_i - x_j| \leq 1$, for $\{i, j\} \in c$, with $x_N = 0$.

We can compute this volume using Ehrhart polynomials



Theorem (Ehrhart) Let \mathcal{P} be a convex polytope of dimension d in \mathbb{R}^m , with vertices having integer coordinates. Let $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ denote the n -fold expansion of \mathcal{P} , and $I(\mathcal{P}, n)$, the number of points with integer coordinates which lie inside $n\mathcal{P}$. Then $I(\mathcal{P}, n)$ is a polynomial function of n of degree d whose leading coefficient is the volume $\text{Vol}(\mathcal{P})$ of \mathcal{P} .

In order to apply Ehrhart's theorem, we proved the following:

Proposition Let c be a connected graph with its N vertices labelled $\{1, 2, \dots, N\}$, and define the convex polytope $\mathcal{P}(c) \subset \mathbb{R}^N$ by

$$\mathcal{P}(c) = \{X \in \mathbb{R}^N \mid x_N = 0 \text{ and } |x_i - x_j| \leq 1 \ \forall \{i, j\} \in c\},$$

where $X = (x_1, \dots, x_N)$. Then the vertices of $\mathcal{P}(c)$ have integer coordinates.

It follows that the volume of $\mathcal{P}(c)$ and the weight $w(c) = (-1)^{e(c)} \text{Vol}(\mathcal{P}(c))$ can be obtained by computing the Ehrhart polynomial $I(\mathcal{P}(c), n)$.

We have carried out the computation of $w(c)$ for all 2-connected graph c having $N \leq 6$ vertices.

Particular families of weight values

Proposition *For the complete graph K_N , we have*

$$w(K_N) = (-1)^{\binom{N}{2}} N.$$

Proposition *For the (unoriented) cycle C_N with N vertices, we have*

$$\begin{aligned} w(C_N) &= (-1)^N \frac{2^N}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^N dt \\ &= \frac{(-1)^N}{(N-1)!} \sum_{i=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^i \binom{N}{i} (N-2i)^{N-1} \\ &\sim (-2)^N \left(\frac{3}{2\pi N} \right)^{\frac{1}{2}} \left(1 - \frac{3}{20N} - \frac{13}{1120N^2} + \dots \right). \end{aligned}$$

Proof of

$$w(C_N) = (-1)^N \frac{2^N}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^N dt$$

Write $\chi(x) := \chi(|x| \leq 1)$ for simplicity. Then

$$\begin{aligned} w(C_N) &= (-1)^N \int_{\mathbb{R}^{N-1}} \prod_{\{i,j\} \in C_N ; x_N=0} \chi(x_i - x_j) dx_1 \dots dx_{N-1} \\ &= (-1)^N \int_{\mathbb{R}^{N-1}} \chi(t_1 + t_2 + \dots + t_{N-1}) \chi(t_1) \dots \chi(t_{N-1}) dt_1 \dots dt_{N-1} \end{aligned}$$

$t_i = x_i - x_{i+1}, \text{ for } 1 \leq i \leq N-1.$

Let U_1, U_2, \dots, U_{N-1} be independent identically distributed uniform random variables on the interval $[-1, 1]$, with common density function $u(x) = \frac{1}{2}\chi(x)$ and let $S = U_1 + \dots + U_{N-1}$.

Then,

$$w(C_N) = (-1)^N 2^{N-1} \text{Prob}(-1 \leq S \leq 1).$$

The density function $s(x)$ of S is given by the $(N-1)$ -fold convolution product

$$s = u^{*(N-1)} = u * \dots * u \quad (N-1 \text{ factors}),$$

$$w(C_N) = (-1)^N 2^{N-1} \int_{-1}^1 s(\xi) d\xi = (-2)^N \int_{-\infty}^{\infty} u^{*(N-1)}(\xi) u(0 - \xi) d\xi = (-2)^N u^{*N}(0).$$

Taking Fourier transform, we get

$$\widehat{u^{*N}}(t) = (\hat{u}(t))^N = \left(\frac{\sin t}{t} \right)^N.$$

Taking inverse Fourier transform, we find

$$u^{*N}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{u^{*N}}(t) e^{itx} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^N e^{itx} dt.$$

Finally : let $x = 0$.

Computations using graph homomorphisms

As observed by Bodo Lass **(*)**, it is possible to evaluate the volume of the polytope $\mathcal{P}(c)$ by decomposing it into a certain number $\nu(c)$ of subpolytopes which are all simplexes of volume $1/(N - 1)!$.

Each subpolytope is obtained by fixing the integral parts and the relative positions of the fractional parts of the coordinates x_1, \dots, x_N of points $X \in \mathcal{P}(c)$.

The number of such configurations will then yield $\nu(c)$ and we will have

$$\text{Vol}(\mathcal{P}(c)) = \nu(c)/(N - 1)!.$$

(*) Personal communication, 2005.

In order to make this correspondence more precise, we consider the “fractional representation” of real numbers

$$\mathbb{R} \rightarrow ([0, 1] \times \mathbb{Z}) : x \mapsto (\xi_x, h_x),$$

where

$$h_x = \lfloor x \rfloor \quad \text{integral part of } x,$$

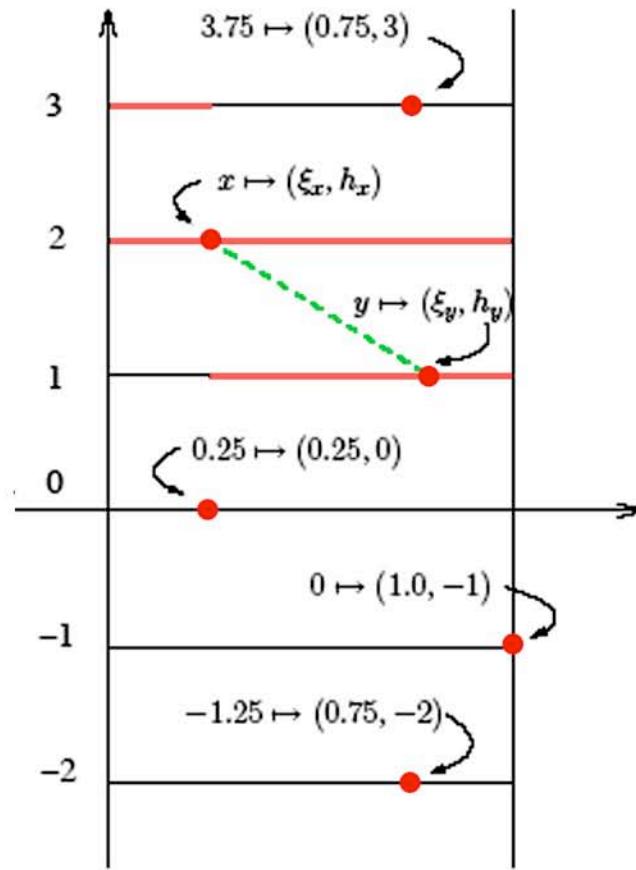
$$\xi_x = x - h_x \quad \text{fractional part of } x,$$

so that

$$x = \xi_x + h_x.$$

Exception : $0 \mapsto (1.0, -1)$,

as if 0 was infinitesimally negative.



$|x - y| < 1$ translates into

“ $\xi_x \neq \xi_y$ and assuming $\xi_x < \xi_y$, then $h_x = h_y$ or $h_x = h_y + 1$ ”

(null or negative slope from x to y)

Now consider a connected graph c with vertex set $V = [N] = \{1, 2, \dots, N\}$, let $X = (x_1, \dots, x_N)$ be a point in the polytope $\mathcal{P}(c)$.

$x_i \mapsto (\xi_i, h_i)$ Recall that
 $x_N = 0$ so that $\xi_N = 1.0$ and $h_N = -1$, with our convention.

We can assume that all the fractional parts ξ_i are distinct.

We form a subpolytope of $\mathcal{P}(c)$ by keeping the “heights” h_1, h_2, \dots, h_N fixed and as well as the relative positions (total order) of the fractional parts $\xi_1, \xi_2, \dots, \xi_N$.

Such polytopes are denoted $\mathcal{P}(h, \beta)$, where

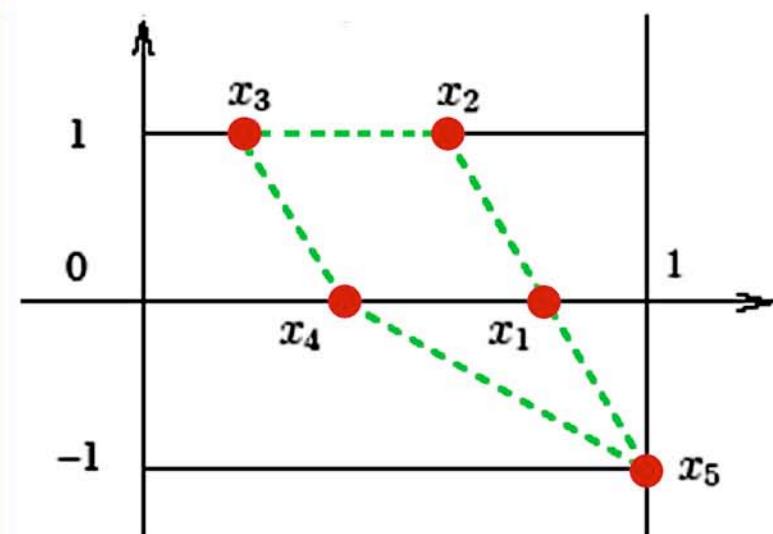
$h : V \rightarrow \mathbb{Z}$ height function $i \mapsto h_i$

$\beta : V \rightarrow [N]$ $\beta(i)$ gives the rank of ξ_i

and are graphically characterized by their centroid $X_{h, \beta}$ obtained by setting

$$\xi_i = \beta(i)/N \qquad \beta(N) = N$$

and drawing a dotted line segment between x_i and x_j for each edge $\{i, j\}$ of c .



Fractional representation of a simplicial subpolytope of $\mathcal{P}(C_5)$

Proposition Let c be a connected graph with vertex set $V = [N]$ and consider a function $h : V \rightarrow \mathbb{Z}$ and a bijection $\beta : V \rightarrow [N]$. Then the pair (h, β) determines a valid subpolytope $\mathcal{P}(h, \beta)$ of $\mathcal{P}(c)$ if and only if the following condition is satisfied:

$$\text{for any edge } \{i, j\} \text{ of } c, \quad \beta(i) < \beta(j) \text{ implies } h_i = h_j \text{ or } h_i = h_j + 1. \quad (*)$$

Proposition Let c be a connected graph and let (h, β) be such that condition $(*)$ is satisfied. Then the volume of the associated subpolytope $\mathcal{P}(h, \beta)$ is equal to

$$1/(N - 1)!.$$

Proposition Let c be a connected graph and let $\nu(c)$ be the number of pairs (h, β) such that the condition $(*)$ is satisfied. Then the volume of the polytope $\mathcal{P}(c)$ is given by

$$\text{Vol}(\mathcal{P}(c)) = \nu(c)/(N - 1)!.$$

Corollary $w(c) = (-1)^{e(c)} \text{Vol}(\mathcal{P}(c)) = (-1)^{e(c)} \nu(c)/(N - 1)!$,

$$w(K_N) = (-1)^{\binom{N}{2}} N, \quad w(K_N \setminus e) = (-1)^{\binom{N}{2}-1} \left(N + \frac{2}{(N - 1)} \right), \quad \text{etc}$$

Proof of

$$w(K_N) = (-1)^{\binom{N}{2}} N$$

Indeed, since all edges are present in the complete graph, there are only N possible height sequences, of the form $(0, \dots, 0, -1, \dots, -1)$, from $(0, \dots, 0, -1)$ to $(-1, \dots, -1)$, and any of the $(N - 1)!$ permutations β for which $\beta(N) = N$ gives rise to a legal configuration (h, β) .

Hence

$$\nu(K_N) = N(N - 1)!$$

and the result follows since

$$w(c) = (-1)^{e(c)} \text{Vol}(\mathcal{P}(c)) = (-1)^{e(c)} \nu(c)/(N - 1)!$$

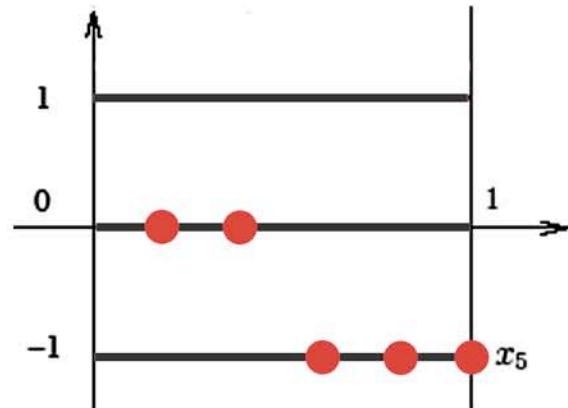


Table for 2-connected graphs

Key:

number	degree sequence of c		
graph c	Ehrhart Pol. in base n^i	nb of labellings	nb of spanning subtrees
	Ehrhart Pol. in base $\binom{n}{i}$	Polytope's volume	volume $\times (n - 1)!$

Typical entry:

8.3	$(4,3,3,2,2,2)$			
	$\frac{37}{3} n^5 + \frac{185}{6} n^4 + \frac{104}{3} n^3 + \frac{127}{6} n^2 + 7 n + 1$ $\frac{1\binom{n}{0} + 106\binom{n}{1} + 1052\binom{n}{2} + 3168\binom{n}{3} + 3700\binom{n}{4} + 1480\binom{n}{5}}{\binom{n}{6}}$	360	32	$\frac{37}{3}$

Table for 2-connected graphs of size at most 6

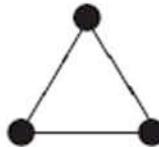
Key:

number	degree sequence of c		
graph c	Ehrhart Pol. in base n^i	nb of labellings	nb of spanning subtrees
	Ehrhart Pol. in base $\binom{n}{i}$	polytope's volume	volume $\times (n - 1)!$

With 2 vertices:

2	(1,1)		
	$\begin{array}{c c c} 2n + 1 & 1 & 1 \\ \hline 1\binom{n}{0} + 2\binom{n}{1} & 2 & 2 \end{array}$		

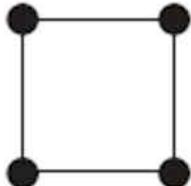
With 3 vertices:

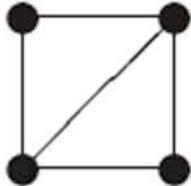
3	(2,2,2)		
	$\begin{array}{c c c} 3n^2 + 3n + 1 & 1 & 3 \\ \hline 1\binom{n}{0} + 6\binom{n}{1} + 6\binom{n}{2} & 3 & 6 \end{array}$		

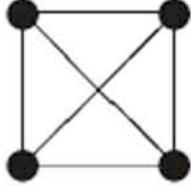
Key:

number	degree sequence of c		
graph c	Ehrhart Pol. in base n^i	nb of labellings	nb of spanning subtrees
	Ehrhart Pol. in base $\binom{n}{i}$	polytope's volume	volume $\times (n - 1)!$

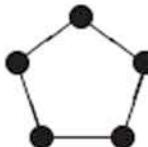
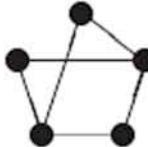
With 4 vertices:

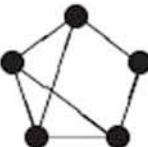
4.1	(2,2,2,2)		
	$\frac{16}{3} n^3 + 8 n^2 + \frac{14}{3} n + 1$	3	4
	$1\binom{n}{0} + 18\binom{n}{1} + 48\binom{n}{2} + 32\binom{n}{3}$	$\frac{16}{3}$	32

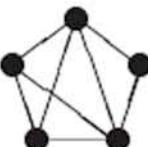
4.2	(3,3,2,2)		
	$\frac{14}{3} n^3 + 7 n^2 + \frac{13}{3} n + 1$	6	8
	$1\binom{n}{0} + 16\binom{n}{1} + 42\binom{n}{2} + 28\binom{n}{3}$	$\frac{14}{3}$	28

4.3	(3,3,3,3)		
	$4 n^3 + 6 n^2 + 4 n + 1$	1	16
	$1\binom{n}{0} + 14\binom{n}{1} + 36\binom{n}{2} + 24\binom{n}{3}$	4	24

With 5 vertices:

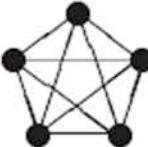
5.1	$(2,2,2,2,2)$
	$\frac{115}{12} n^4 + \frac{115}{6} n^3 + \frac{185}{12} n^2 + \frac{35}{6} n + 1$ $\frac{1(n) + 50(n)_1 + 280(n)_2 + 460(n)_3 + 230(n)_4}{12} \quad \left \begin{array}{c c} \frac{115}{12} & 230 \\ \hline & \end{array} \right.$
5.2	$(3,3,2,2,2)$
	$\frac{49}{6} n^4 + \frac{49}{3} n^3 + \frac{83}{6} n^2 + \frac{17}{3} n + 1$ $\frac{1(n) + 44(n)_1 + 240(n)_2 + 392(n)_3 + 196(n)_4}{60} \quad \left \begin{array}{c c} \frac{49}{6} & 196 \\ \hline & \end{array} \right.$
5.3	$(3,3,2,2,2)$
	$8 n^4 + 16 n^3 + 14 n^2 + 6 n + 1$ $\frac{1(n) + 44(n)_1 + 236(n)_2 + 384(n)_3 + 192(n)_4}{10} \quad \left \begin{array}{c c} 8 & 192 \\ \hline & \end{array} \right.$
5.4	$(4,4,2,2,2)$
	$\frac{15}{2} n^4 + 15 n^3 + \frac{27}{2} n^2 + 6 n + 1$ $\frac{1(n) + 42(n)_1 + 222(n)_2 + 360(n)_3 + 180(n)_4}{10} \quad \left \begin{array}{c c} \frac{15}{2} & 180 \\ \hline & \end{array} \right.$
5.5	$(4,3,3,2,2)$
	$\frac{29}{4} n^4 + \frac{29}{2} n^3 + \frac{51}{4} n^2 + \frac{11}{2} n + 1$ $\frac{1(n) + 40(n)_1 + 214(n)_2 + 348(n)_3 + 174(n)_4}{60} \quad \left \begin{array}{c c} \frac{29}{4} & 174 \\ \hline & \end{array} \right.$

5.6	$(3,3,3,3,2)$
	$\frac{\frac{41}{6}n^4 + \frac{41}{3}n^3 + \frac{73}{6}n^2 + \frac{16}{3}n + 1}{1\binom{n}{0} + 38\binom{n}{1} + 202\binom{n}{2} + 328\binom{n}{3} + 164\binom{n}{4}} \Bigg \begin{array}{c c} 30 & 24 \\ \frac{41}{6} & 164 \end{array}$

5.7	$(4,4,3,3,2)$
	$\frac{\frac{19}{3}n^4 + \frac{38}{3}n^3 + \frac{35}{3}n^2 + \frac{16}{3}n + 1}{1\binom{n}{0} + 36\binom{n}{1} + 188\binom{n}{2} + 304\binom{n}{3} + 152\binom{n}{4}} \Bigg \begin{array}{c c} 30 & 40 \\ \frac{19}{3} & 152 \end{array}$

5.8	$(4,3,3,3,3)$
	$\frac{6n^4 + 12n^3 + 11n^2 + 5n + 1}{1\binom{n}{0} + 34\binom{n}{1} + 178\binom{n}{2} + 288\binom{n}{3} + 144\binom{n}{4}} \Bigg \begin{array}{c c} 15 & 45 \\ 6 & 144 \end{array}$

5.9	$(4,4,4,3,3)$
	$\frac{\frac{11}{2}n^4 + 11n^3 + \frac{21}{2}n^2 + 5n + 1}{1\binom{n}{0} + 32\binom{n}{1} + 164\binom{n}{2} + 264\binom{n}{3} + 132\binom{n}{4}} \Bigg \begin{array}{c c} 10 & 75 \\ \frac{11}{2} & 132 \end{array}$

5.10	$(4,4,4,4,4)$
	$\frac{5n^4 + 10n^3 + 10n^2 + 5n + 1}{1\binom{n}{0} + 30\binom{n}{1} + 150\binom{n}{2} + 240\binom{n}{3} + 120\binom{n}{4}} \Bigg \begin{array}{c c} 1 & 125 \\ 5 & 120 \end{array}$

With 6 vertices, ordered according to the number of edges:

6.1	$(2,2,2,2,2,2)$
	$\frac{88}{5} n^5 + 44 n^4 + 46 n^3 + 25 n^2 + \frac{37}{5} n + 1$ $\frac{1(n) + 140(n)_1 + 1470(n)_2 + 4500(n)_3 + 5280(n)_4 + 2112(n)_5}{\frac{88}{5}}$ 60 6

7.1	$(3,3,2,2,2,2)$
	$\frac{72}{5} n^5 + 36 n^4 + \frac{118}{3} n^3 + 23 n^2 + \frac{109}{15} n + 1$ $\frac{1(n) + 120(n)_1 + 1218(n)_2 + 3692(n)_3 + 4320(n)_4 + 1728(n)_5}{\frac{72}{5}}$ 180 15

7.2	$(3,3,2,2,2,2)$
	$\frac{439}{30} n^5 + \frac{439}{12} n^4 + \frac{118}{3} n^3 + \frac{269}{12} n^2 + \frac{211}{30} n + 1$ $\frac{1(n) + 120(n)_1 + 1232(n)_2 + 3748(n)_3 + 4390(n)_4 + 1756(n)_5}{\frac{439}{30}}$ 360 14

7.3	$(3,3,2,2,2,2)$
	$\frac{419}{30} n^5 + \frac{419}{12} n^4 + 38 n^3 + \frac{265}{12} n^2 + \frac{211}{30} n + 1$ $\frac{1(n) + 116(n)_1 + 1180(n)_2 + 3580(n)_3 + 4190(n)_4 + 1676(n)_5}{\frac{419}{30}}$ 180 16

8.1	$(4,4,2,2,2,2)$
	$\frac{64}{5} n^5 + 32 n^4 + \frac{112}{3} n^3 + 24 n^2 + \frac{118}{15} n + 1$ $\frac{1(n) + 114(n)_1 + 1104(n)_2 + 3296(n)_3 + 3840(n)_4 + 1536(n)_5}{\frac{64}{5}}$ 15 32

8.2	$(4,4,2,2,2,2)$		
	$\frac{194}{15} n^5 + \frac{97}{3} n^4 + 36 n^3 + \frac{65}{3} n^2 + \frac{106}{15} n + 1$ $1\binom{n}{0} + 110\binom{n}{1} + 1100\binom{n}{2} + 3320\binom{n}{3} + 3880\binom{n}{4} + 1552\binom{n}{5}$	180	28

8.3	$(4,3,3,2,2,2)$		
	$\frac{37}{3} n^5 + \frac{185}{6} n^4 + \frac{104}{3} n^3 + \frac{127}{6} n^2 + 7 n + 1$ $1\binom{n}{0} + 106\binom{n}{1} + 1052\binom{n}{2} + 3168\binom{n}{3} + 3700\binom{n}{4} + 1480\binom{n}{5}$	360	32

8.4	$(4,3,3,2,2,2,2)$		
	$\frac{127}{10} n^5 + \frac{127}{4} n^4 + \frac{106}{3} n^3 + \frac{85}{4} n^2 + \frac{209}{30} n + 1$ $1\binom{n}{0} + 108\binom{n}{1} + 1080\binom{n}{2} + 3260\binom{n}{3} + 3810\binom{n}{4} + 1524\binom{n}{5}$	720	29

8.5	$(4,3,3,2,2,2)$		
	$\frac{188}{15} n^5 + \frac{94}{3} n^4 + \frac{104}{3} n^3 + \frac{62}{3} n^2 + \frac{34}{5} n + 1$ $1\binom{n}{0} + 106\binom{n}{1} + 1064\binom{n}{2} + 3216\binom{n}{3} + 3760\binom{n}{4} + 1504\binom{n}{5}$	360	30

8.6	$(3,3,3,3,2,2)$		
	$\frac{188}{15} n^5 + \frac{94}{3} n^4 + \frac{104}{3} n^3 + \frac{62}{3} n^2 + \frac{34}{5} n + 1$ $1\binom{n}{0} + 106\binom{n}{1} + 1064\binom{n}{2} + 3216\binom{n}{3} + 3760\binom{n}{4} + 1504\binom{n}{5}$	180	30

8.7	$(3,3,3,3,2,2)$		
	$\frac{361}{30} n^5 + \frac{361}{12} n^4 + \frac{100}{3} n^3 + \frac{239}{12} n^2 + \frac{199}{30} n + 1$ $1\binom{n}{0} + 102\binom{n}{1} + 1022\binom{n}{2} + 3088\binom{n}{3} + 3610\binom{n}{4} + 1444\binom{n}{5}$	180	32

8.8	$(3,3,3,3,2,2)$		
	$\frac{176}{15} n^5 + \frac{88}{3} n^4 + \frac{100}{3} n^3 + \frac{62}{3} n^2 + \frac{104}{15} n + 1$ $1\binom{n}{0} + 102\binom{n}{1} + 1004\binom{n}{2} + 3016\binom{n}{3} + 3520\binom{n}{4} + 1408\binom{n}{5}$	90	36

8.9	$(3,3,3,3,2,2)$		
	$\frac{117}{10} n^5 + \frac{117}{4} n^4 + \frac{98}{3} n^3 + \frac{79}{4} n^2 + \frac{199}{30} n + 1$ $1\binom{n}{0} + 100\binom{n}{1} + 996\binom{n}{2} + 3004\binom{n}{3} + 3510\binom{n}{4} + 1404\binom{n}{5}$	360	35

9.1	$(5,5,2,2,2,2)$		
	$\frac{62}{5} n^5 + 31 n^4 + \frac{110}{3} n^3 + 24 n^2 + \frac{119}{15} n + 1$ $1\binom{n}{0} + 112\binom{n}{1} + 1074\binom{n}{2} + 3196\binom{n}{3} + 3720\binom{n}{4} + 1488\binom{n}{5}$	15	48

9.2	$(5,4,3,2,2,2)$		
	$\frac{349}{30} n^5 + \frac{349}{12} n^4 + \frac{100}{3} n^3 + \frac{251}{12} n^2 + \frac{211}{30} n + 1$ $1\binom{n}{0} + 102\binom{n}{1} + 998\binom{n}{2} + 2992\binom{n}{3} + 3490\binom{n}{4} + 1396\binom{n}{5}$	360	52

9.3	$(5,3,3,3,2,2)$		
	$\frac{169}{15} n^5 + \frac{169}{6} n^4 + 32 n^3 + \frac{119}{6} n^2 + \frac{101}{15} n + 1$ $1\binom{n}{0} + 98\binom{n}{1} + 964\binom{n}{2} + 2896\binom{n}{3} + 3380\binom{n}{4} + 1352\binom{n}{5}$	360	55

9.4	$(4,4,4,2,2,2)$		
	$\frac{113}{10} n^5 + \frac{113}{4} n^4 + 32 n^3 + \frac{79}{4} n^2 + \frac{67}{10} n + 1$ $1\binom{n}{0} + 98\binom{n}{1} + 966\binom{n}{2} + 2904\binom{n}{3} + 3390\binom{n}{4} + 1356\binom{n}{5}$	120	54

9.5	$(4,4,3,3,2,2)$		
	$11 n^5 + \frac{55}{2} n^4 + \frac{94}{3} n^3 + \frac{39}{2} n^2 + \frac{20}{3} n + 1$ $1\binom{n}{0} + 96\binom{n}{1} + 942\binom{n}{2} + 2828\binom{n}{3} + 3300\binom{n}{4} + 1320\binom{n}{5}$	180	56

9.6	$(4,4,3,3,2,2)$		
	$\frac{169}{15} n^5 + \frac{169}{6} n^4 + 32 n^3 + \frac{119}{6} n^2 + \frac{101}{15} n + 1$ $1\binom{n}{0} + 98\binom{n}{1} + 964\binom{n}{2} + 2896\binom{n}{3} + 3380\binom{n}{4} + 1352\binom{n}{5}$	360	55

9.7	$(4,4,3,3,2,2)$		
	$\frac{163}{15} n^5 + \frac{163}{6} n^4 + \frac{94}{3} n^3 + \frac{119}{6} n^2 + \frac{34}{5} n + 1$ $1\binom{n}{0} + 96\binom{n}{1} + 934\binom{n}{2} + 2796\binom{n}{3} + 3260\binom{n}{4} + 1304\binom{n}{5}$	180	60

9.8	$(4,4,3,3,2,2)$		
	$\frac{161}{15} n^5 + \frac{161}{6} n^4 + \frac{94}{3} n^3 + \frac{121}{6} n^2 + \frac{104}{15} n + 1$ $1\binom{n}{0} + 96\binom{n}{1} + 926\binom{n}{2} + 2764\binom{n}{3} + 3220\binom{n}{4} + 1288\binom{n}{5}$	$\left \begin{array}{c cc} 90 & 64 \\ \frac{161}{15} & 1288 \end{array} \right.$	

9.9	$(4,4,3,3,2,2)$		
	$\frac{161}{15} n^5 + \frac{161}{6} n^4 + \frac{92}{3} n^3 + \frac{115}{6} n^2 + \frac{33}{5} n + 1$ $1\binom{n}{0} + 94\binom{n}{1} + 920\binom{n}{2} + 2760\binom{n}{3} + 3220\binom{n}{4} + 1288\binom{n}{5}$	$\left \begin{array}{c cc} 720 & 61 \\ \frac{161}{15} & 1288 \end{array} \right.$	

9.10	$(4,3,3,3,3,2)$		
	$\frac{51}{5} n^5 + \frac{51}{2} n^4 + \frac{88}{3} n^3 + \frac{37}{2} n^2 + \frac{97}{15} n + 1$ $1\binom{n}{0} + 90\binom{n}{1} + 876\binom{n}{2} + 2624\binom{n}{3} + 3060\binom{n}{4} + 1224\binom{n}{5}$	$\left \begin{array}{c cc} 360 & 69 \\ \frac{51}{5} & 1224 \end{array} \right.$	

9.11	$(4,3,3,3,3,2)$		
	$\frac{103}{10} n^5 + \frac{103}{4} n^4 + \frac{88}{3} n^3 + \frac{73}{4} n^2 + \frac{191}{30} n + 1$ $1\binom{n}{0} + 90\binom{n}{1} + 882\binom{n}{2} + 2648\binom{n}{3} + 3090\binom{n}{4} + 1236\binom{n}{5}$	$\left \begin{array}{c cc} 360 & 66 \\ \frac{103}{10} & 1236 \end{array} \right.$	

9.12	$(4,3,3,3,3,2)$		
	$\frac{158}{15} n^5 + \frac{79}{3} n^4 + 30 n^3 + \frac{56}{3} n^2 + \frac{97}{15} n + 1$ $1\binom{n}{0} + 92\binom{n}{1} + 902\binom{n}{2} + 2708\binom{n}{3} + 3160\binom{n}{4} + 1264\binom{n}{5}$	$\left \begin{array}{c cc} 360 & 64 \\ \frac{158}{15} & 1264 \end{array} \right.$	

9.13	$(3,3,3,3,3,3)$
	$\frac{48}{5} n^5 + 24 n^4 + 28 n^3 + 18 n^2 + \frac{32}{5} n + 1$ $1\binom{n}{0} + 86\binom{n}{1} + 828\binom{n}{2} + 2472\binom{n}{3} + 2880\binom{n}{4} + 1152\binom{n}{5}$

9.14	$(3,3,3,3,3,3)$
	$\frac{49}{5} n^5 + \frac{49}{2} n^4 + 28 n^3 + \frac{35}{2} n^2 + \frac{31}{5} n + 1$ $1\binom{n}{0} + 86\binom{n}{1} + 840\binom{n}{2} + 2520\binom{n}{3} + 2940\binom{n}{4} + 1176\binom{n}{5}$

10.1	$(5,5,3,3,2,2)$
	$\frac{31}{3} n^5 + \frac{155}{6} n^4 + \frac{92}{3} n^3 + \frac{121}{6} n^2 + 7 n + 1$ $1\binom{n}{0} + 94\binom{n}{1} + 896\binom{n}{2} + 2664\binom{n}{3} + 3100\binom{n}{4} + 1240\binom{n}{5}$

10.2	$(5,4,4,3,2,2)$
	$\frac{301}{30} n^5 + \frac{301}{12} n^4 + \frac{88}{3} n^3 + \frac{227}{12} n^2 + \frac{199}{30} n + 1$ $1\binom{n}{0} + 90\binom{n}{1} + 866\binom{n}{2} + 2584\binom{n}{3} + 3010\binom{n}{4} + 1204\binom{n}{5}$

10.3	$(5,4,3,3,3,2)$
	$\frac{287}{30} n^5 + \frac{287}{12} n^4 + 28 n^3 + \frac{217}{12} n^2 + \frac{193}{30} n + 1$ $1\binom{n}{0} + 86\binom{n}{1} + 826\binom{n}{2} + 2464\binom{n}{3} + 2870\binom{n}{4} + 1148\binom{n}{5}$

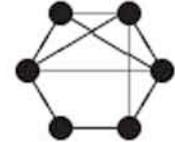
10.4	$(5,4,3,3,3,2)$
	$\frac{59}{6} n^5 + \frac{295}{12} n^4 + \frac{86}{3} n^3 + \frac{221}{12} n^2 + 13/2 n + 1$ $\frac{1(n) + 88(n)_1 + 848(n)_2 + 2532(n)_3 + 2950(n)_4 + 1180(n)_5}{\frac{59}{6}} \quad \quad \begin{matrix} 360 \\ \frac{59}{6} \end{matrix} \quad \begin{matrix} 104 \\ 1180 \end{matrix}$

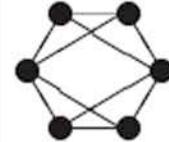
10.5	$(5,3,3,3,3,3)$
	$\frac{55}{6} n^5 + \frac{275}{12} n^4 + \frac{80}{3} n^3 + \frac{205}{12} n^2 + \frac{37}{6} n + 1$ $\frac{1(n) + 82(n)_1 + 790(n)_2 + 2360(n)_3 + 2750(n)_4 + 1100(n)_5}{\frac{55}{6}} \quad \quad \begin{matrix} 72 \\ \frac{55}{6} \end{matrix} \quad \begin{matrix} 121 \\ 1100 \end{matrix}$

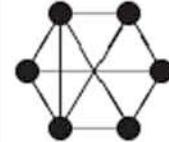
10.6	$(4,4,4,4,2,2)$
	$10 n^5 + 25 n^4 + \frac{88}{3} n^3 + 19 n^2 + \frac{20}{3} n + 1$ $\frac{1(n) + 90(n)_1 + 864(n)_2 + 2576(n)_3 + 3000(n)_4 + 1200(n)_5}{10} \quad \quad \begin{matrix} 90 \\ 10 \end{matrix} \quad \begin{matrix} 100 \\ 1200 \end{matrix}$

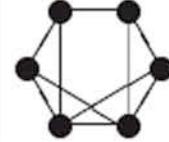
10.7	$(4,4,4,3,3,2)$
	$\frac{47}{5} n^5 + \frac{47}{2} n^4 + \frac{82}{3} n^3 + \frac{35}{2} n^2 + \frac{94}{15} n + 1$ $\frac{1(n) + 84(n)_1 + 810(n)_2 + 2420(n)_3 + 2820(n)_4 + 1128(n)_5}{\frac{47}{5}} \quad \quad \begin{matrix} 360 \\ \frac{47}{5} \end{matrix} \quad \begin{matrix} 114 \\ 1128 \end{matrix}$

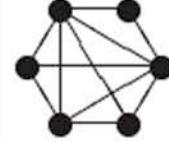
10.8	$(4,4,4,3,3,2)$
	$\frac{139}{15} n^5 + \frac{139}{6} n^4 + \frac{82}{3} n^3 + \frac{107}{6} n^2 + \frac{32}{5} n + 1$ $\frac{1(n) + 84(n)_1 + 802(n)_2 + 2388(n)_3 + 2780(n)_4 + 1112(n)_5}{\frac{139}{15}} \quad \quad \begin{matrix} 180 \\ \frac{139}{15} \end{matrix} \quad \begin{matrix} 120 \\ 1112 \end{matrix}$

10.9	$(4,4,4,3,3,2)$
	$\frac{\frac{28}{3}n^5 + \frac{70}{3}n^4 + \frac{82}{3}n^3 + \frac{53}{3}n^2 + \frac{19}{3}n + 1}{1\binom{n}{0} + 84\binom{n}{1} + 806\binom{n}{2} + 2404\binom{n}{3} + 2800\binom{n}{4} + 1120\binom{n}{5}} \quad \begin{array}{c c} 360 & 115 \\ \hline \frac{28}{3} & 1120 \end{array}$

10.10	$(4,4,3,3,3,3)$
	$\frac{\frac{136}{15}n^5 + \frac{68}{3}n^4 + \frac{80}{3}n^3 + \frac{52}{3}n^2 + \frac{94}{15}n + 1}{1\binom{n}{0} + 82\binom{n}{1} + 784\binom{n}{2} + 2336\binom{n}{3} + 2720\binom{n}{4} + 1088\binom{n}{5}} \quad \begin{array}{c c} 45 & 128 \\ \hline \frac{136}{15} & 1088 \end{array}$

10.11	$(4,4,3,3,3,3)$
	$\frac{\frac{44}{5}n^5 + 22n^4 + 26n^3 + 17n^2 + \frac{31}{5}n + 1}{1\binom{n}{0} + 80\binom{n}{1} + 762\binom{n}{2} + 2268\binom{n}{3} + 2640\binom{n}{4} + 1056\binom{n}{5}} \quad \begin{array}{c c} 60 & 135 \\ \hline \frac{44}{5} & 1056 \end{array}$

10.12	$(4,4,3,3,3,3)$
	$\frac{\frac{89}{10}n^5 + \frac{89}{4}n^4 + 26n^3 + \frac{67}{4}n^2 + \frac{61}{10}n + 1}{1\binom{n}{0} + 80\binom{n}{1} + 768\binom{n}{2} + 2292\binom{n}{3} + 2670\binom{n}{4} + 1068\binom{n}{5}} \quad \begin{array}{c c} 360 & 130 \\ \hline \frac{89}{10} & 1068 \end{array}$

11.1	$(5,5,4,3,3,2)$
	$\frac{\frac{133}{15}n^5 + \frac{133}{6}n^4 + \frac{80}{3}n^3 + \frac{107}{6}n^2 + \frac{97}{15}n + 1}{1\binom{n}{0} + 82\binom{n}{1} + 772\binom{n}{2} + 2288\binom{n}{3} + 2660\binom{n}{4} + 1064\binom{n}{5}} \quad \begin{array}{c c} 180 & 180 \\ \hline \frac{133}{15} & 1064 \end{array}$

11.2	$(5,5,3,3,3,3)$
	$\frac{26}{3} n^5 + \frac{65}{3} n^4 + 26 n^3 + \frac{52}{3} n^2 + \frac{19}{3} n + 1$ $\frac{1(n) + 80(n)_1 + 754(n)_2 + 2236(n)_3 + 2600(n)_4 + 1040(n)_5}{\frac{26}{3}} \mid \begin{array}{c c} 45 & 192 \\ \hline \frac{26}{3} & 1040 \end{array}$
11.3	$(5,4,4,4,3,2)$
	$\frac{87}{10} n^5 + \frac{87}{4} n^4 + 26 n^3 + \frac{69}{4} n^2 + \frac{63}{10} n + 1$ $\frac{1(n) + 80(n)_1 + 756(n)_2 + 2244(n)_3 + 2610(n)_4 + 1044(n)_5}{\frac{87}{10}} \mid \begin{array}{c c} 360 & 185 \\ \hline \frac{87}{10} & 1044 \end{array}$
11.4	$(5,4,4,3,3,3)$
	$\frac{124}{15} n^5 + \frac{62}{3} n^4 + \frac{74}{3} n^3 + \frac{49}{3} n^2 + \frac{91}{15} n + 1$ $\frac{1(n) + 76(n)_1 + 718(n)_2 + 2132(n)_3 + 2480(n)_4 + 992(n)_5}{\frac{124}{15}} \mid \begin{array}{c c} 360 & 209 \\ \hline \frac{124}{15} & 992 \end{array}$
11.5	$(5,4,4,3,3,3)$
	$\frac{41}{5} n^5 + \frac{41}{2} n^4 + \frac{74}{3} n^3 + \frac{33}{2} n^2 + \frac{92}{15} n + 1$ $\frac{1(n) + 76(n)_1 + 714(n)_2 + 2116(n)_3 + 2460(n)_4 + 984(n)_5}{\frac{41}{5}} \mid \begin{array}{c c} 60 & 216 \\ \hline \frac{41}{5} & 984 \end{array}$
11.6	$(4,4,4,4,4,2)$
	$\frac{42}{5} n^5 + 21 n^4 + \frac{76}{3} n^3 + 17 n^2 + \frac{94}{15} n + 1$ $\frac{1(n) + 78(n)_1 + 732(n)_2 + 2168(n)_3 + 2520(n)_4 + 1008(n)_5}{\frac{42}{5}} \mid \begin{array}{c c} 60 & 200 \\ \hline \frac{42}{5} & 1008 \end{array}$

11.7	$(4,4,4,4,3,3)$
	$\frac{241}{30} n^5 + \frac{241}{12} n^4 + 24 n^3 + \frac{191}{12} n^2 + \frac{179}{30} n + 1$ $\frac{1(n)}{1} + 74\binom{n}{1} + 698\binom{n}{2} + 2072\binom{n}{3} + 2410\binom{n}{4} + 964\binom{n}{5}$

11.8	$(4,4,4,4,3,3)$
	$8 n^5 + 20 n^4 + 24 n^3 + 16 n^2 + 6 n + 1$ $\frac{1(n)}{1} + 74\binom{n}{1} + 696\binom{n}{2} + 2064\binom{n}{3} + 2400\binom{n}{4} + 960\binom{n}{5}$

12.1	$(5,5,5,3,3,3)$
	$\frac{39}{5} n^5 + \frac{39}{2} n^4 + 24 n^3 + \frac{33}{2} n^2 + \frac{31}{5} n + 1$ $\frac{1(n)}{1} + 74\binom{n}{1} + 684\binom{n}{2} + 2016\binom{n}{3} + 2340\binom{n}{4} + 936\binom{n}{5}$

12.2	$(5,5,4,4,4,2)$
	$8 n^5 + 20 n^4 + \frac{74}{3} n^3 + 17 n^2 + \frac{19}{3} n + 1$ $\frac{1(n)}{1} + 76\binom{n}{1} + 702\binom{n}{2} + 2068\binom{n}{3} + 2400\binom{n}{4} + 960\binom{n}{5}$

12.3	$(5,5,4,4,3,3)$
	$\frac{229}{30} n^5 + \frac{229}{12} n^4 + \frac{70}{3} n^3 + \frac{191}{12} n^2 + \frac{181}{30} n + 1$ $\frac{1(n)}{1} + 72\binom{n}{1} + 668\binom{n}{2} + 1972\binom{n}{3} + 2290\binom{n}{4} + 916\binom{n}{5}$

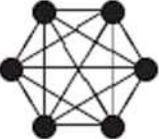
11.7	$(4,4,4,4,3,3)$
	$\frac{241}{30} n^5 + \frac{241}{12} n^4 + 24 n^3 + \frac{191}{12} n^2 + \frac{179}{30} n + 1$ $\frac{1(n)}{1} + 74\binom{n}{1} + 698\binom{n}{2} + 2072\binom{n}{3} + 2410\binom{n}{4} + 964\binom{n}{5}$

11.8	$(4,4,4,4,3,3)$
	$8 n^5 + 20 n^4 + 24 n^3 + 16 n^2 + 6 n + 1$ $\frac{1(n)}{1} + 74\binom{n}{1} + 696\binom{n}{2} + 2064\binom{n}{3} + 2400\binom{n}{4} + 960\binom{n}{5}$

12.1	$(5,5,5,3,3,3)$
	$\frac{39}{5} n^5 + \frac{39}{2} n^4 + 24 n^3 + \frac{33}{2} n^2 + \frac{31}{5} n + 1$ $\frac{1(n)}{1} + 74\binom{n}{1} + 684\binom{n}{2} + 2016\binom{n}{3} + 2340\binom{n}{4} + 936\binom{n}{5}$

12.2	$(5,5,4,4,4,2)$
	$8 n^5 + 20 n^4 + \frac{74}{3} n^3 + 17 n^2 + \frac{19}{3} n + 1$ $\frac{1(n)}{1} + 76\binom{n}{1} + 702\binom{n}{2} + 2068\binom{n}{3} + 2400\binom{n}{4} + 960\binom{n}{5}$

12.3	$(5,5,4,4,3,3)$
	$\frac{229}{30} n^5 + \frac{229}{12} n^4 + \frac{70}{3} n^3 + \frac{191}{12} n^2 + \frac{181}{30} n + 1$ $\frac{1(n)}{1} + 72\binom{n}{1} + 668\binom{n}{2} + 1972\binom{n}{3} + 2290\binom{n}{4} + 916\binom{n}{5}$

15.1	$(5,5,5,5,5,5)$
	$\begin{array}{c} 6n^5 + 15n^4 + 20n^3 + 15n^2 + 6n + 1 \\ \hline 1\binom{n}{0} + 62\binom{n}{1} + 540\binom{n}{2} + 1560\binom{n}{3} + 1800\binom{n}{4} + 720\binom{n}{5} \end{array}$

PROBLEM

Fill the **?** by “nice” combinatorial interpretations involving graphical invariants for arbitrary n .

number	degree sequence of c	
graph c	Ehrhart Pol. in base n^i ?	polytope’s volume ?
	Ehrhart Pol. in base $\binom{n}{i}$?	

END

Thank You !!