

Negative correlations for trees, forests, and Potts models

David Wagner
University of Waterloo
C&O Department

Combinatorial Problems raised by Statistical Mechanics
CRM February 21, 2007.

- * Mason's conjecture
 - * Rayleigh monotonicity
 - * negative correlation conditions
 - * a probabilistic conjecture
 - * the Grimmett-Winkler conjecture
 - * two-sums and the Potts-Rayleigh condition
 - * *et cetera, et cetera...*
-

Mason's conjecture (1972)

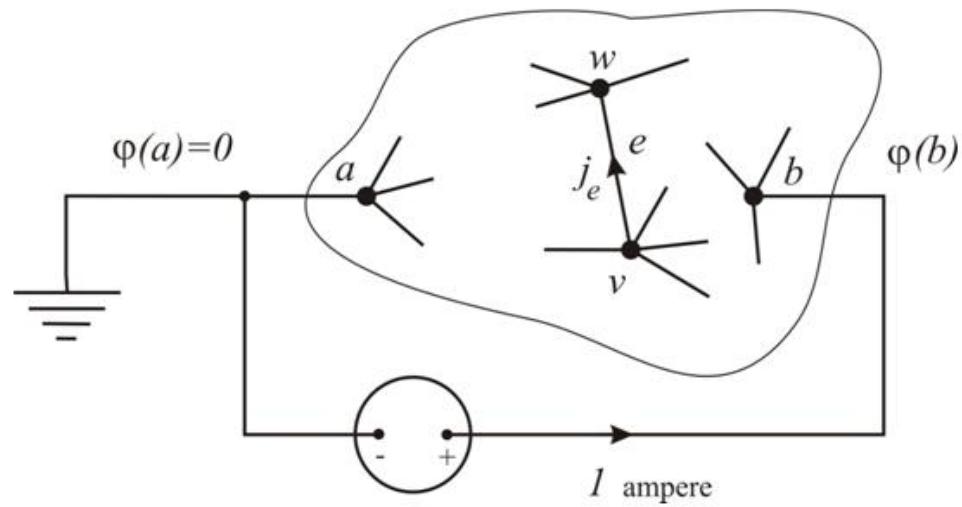
$G = (V, E)$ a connected graph with m edges and n vertices.

$I(k)$ = number of k -edge spanning forests of G .

Conjecture: for all $0 < k < n-1$:

$$\frac{I_k^2}{\binom{m}{k}^2} \geq \frac{I_{k-1}}{\binom{m}{k-1}} \cdot \frac{I_{k+1}}{\binom{m}{k+1}}$$

Effective conductance of a graph (Kirchhoff, 1847)



Effective conductance of a graph (Kirchhoff, 1847)

$$y_{ab}(G; \bar{y}) = \frac{1}{\varphi(b)} = \frac{T(G; \bar{y})}{T(G / ab; \bar{y})}$$

in which

$$T(H; \bar{y}) = \sum_T \bar{y}^T$$

is the sum of

$$\bar{y}^T = \prod_{e \in T} y_e$$

over all spanning trees T of H – that is,

the *spanning tree enumerator* of H .

Rayleigh monotonicity

for any edge e of G :

$$\frac{\partial}{\partial y_e} \mathcal{Y}_{ab}(G, \bar{y}) \geq 0$$

provided that all edge-conductances $y(c)$ are positive.

Rayleigh monotonicity: for distinct edges e, f of G :

$$\Delta G\{e, f\} = G_e^f G_f^e - G_{ef} G^{ef} \geq 0$$

provided that all edge-conductances are positive.

Here, G stands for the spanning tree enumerator of the graph G .

subscript – contract the edge
superscript – delete the edge

A more general framework....

For a finite set E

$$\vec{y} = \{y_e : e \in E\}$$

A *weight function*

$$\omega : 2^E \rightarrow [0, \infty)$$

A more general framework....

For a finite set E

$$\vec{y} = \{y_e : e \in E\}$$

A *weight function*

$$\omega : 2^E \rightarrow [0, \infty)$$

The associated *partition function*

$$Z(\omega; \vec{y}) = \sum_{S \subseteq E} \omega(S) \vec{y}^S$$

A more general framework....

For a finite set E

$$\vec{y} = \{y_e : e \in E\}$$

A weight function

$$\omega : 2^E \rightarrow [0, \infty)$$

The associated partition function

$$Z(\omega; \vec{y}) = \sum_{S \subseteq E} \omega(S) \vec{y}^S$$

For distinct e, f in E ,

The Rayleigh difference is $\Delta Z\{e, f\} = Z_e^f Z_f^e - Z_{ef} Z^{ef}$

A more general framework....

For a finite set E

$$\vec{y} = \{y_e : e \in E\}$$

A weight function

$$\omega : 2^E \rightarrow [0, \infty)$$

The associated partition function

$$Z(\omega; \vec{y}) = \sum_{S \subseteq E} \omega(S) \vec{y}^S$$

For distinct e, f in E ,

The Rayleigh difference is $\Delta Z\{e, f\} = Z_e^f Z_f^e - Z_{ef} Z^{ef}$

deletion

$$Z^e = Z|_{y_e=0}$$

contraction

$$Z_e = \frac{\partial}{\partial y_e} Z$$

A probabilistic interpretation

Give each parameter $y(c)$ a positive value.

Choose a subset S of E with probability

$$\frac{\omega(S) \bar{y}^S}{Z}$$

A probabilistic interpretation

Give each parameter $y(c)$ a positive value.

Choose a subset S of E with probability

$$\frac{\omega(S) \bar{y}^S}{Z}$$

For distinct elements e, f of E ,

the covariance of the events $(e \text{ in } S)$ and $(f \text{ in } S)$ is

$$Cov(e, f) = -\frac{y_e y_f}{Z^2} \Delta Z\{e, f\}$$

A probabilistic interpretation

Give each parameter $y(c)$ a positive value.

Choose a subset S of E with probability

$$\frac{\omega(S) \bar{y}^S}{Z}$$

For distinct elements e, f of E ,

the covariance of the events $(e \text{ in } S)$ and $(f \text{ in } S)$ is

$$Cov(e, f) = -\frac{y_e y_f}{Z^2} \Delta Z\{e, f\} = y_e y_f \frac{\partial^2}{\partial y_e \partial y_f} \log Z$$

The Rayleigh condition

For any distinct e, f in E , if all $y(c)$ are positive then

$$\Delta Z\{e, f\} \geq 0$$

That is, any two distinct elements of E are
negatively correlated
for all choices of positive weights for the parameters.

Four interesting partition functions

correspond to these weight functions
associated with a (connected) graph $G=(V,E)$

- * spanning trees (B)
- * spanning forests (I)
- * connected spanning subgraphs (S)

$$\omega(S) = \begin{cases} 1 & \text{if } S \text{ has the correct form,} \\ 0 & \text{otherwise.} \end{cases}$$

Four interesting partition functions

correspond to these weight functions
associated with a (connected) graph $G=(V,E)$

- * spanning trees (B)
- * spanning forests (I)
- * connected spanning subgraphs (S)

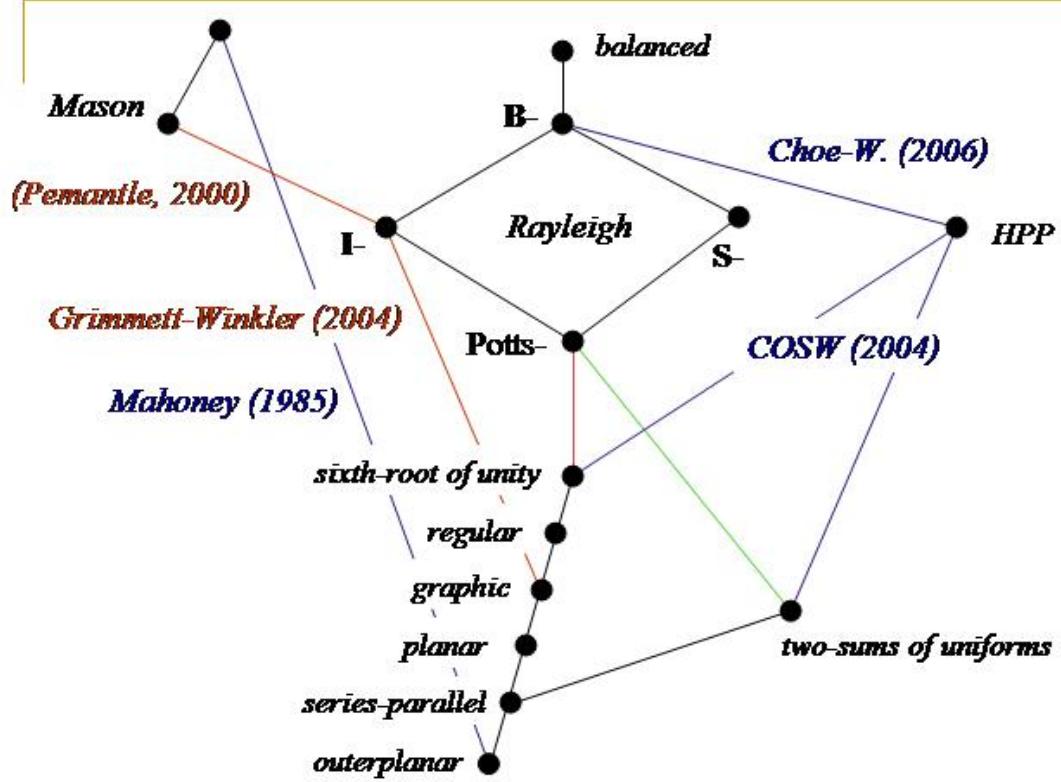
$$\omega(S) = \begin{cases} 1 & \text{if } S \text{ has the correct form,} \\ 0 & \text{otherwise.} \end{cases}$$

- * Potts model

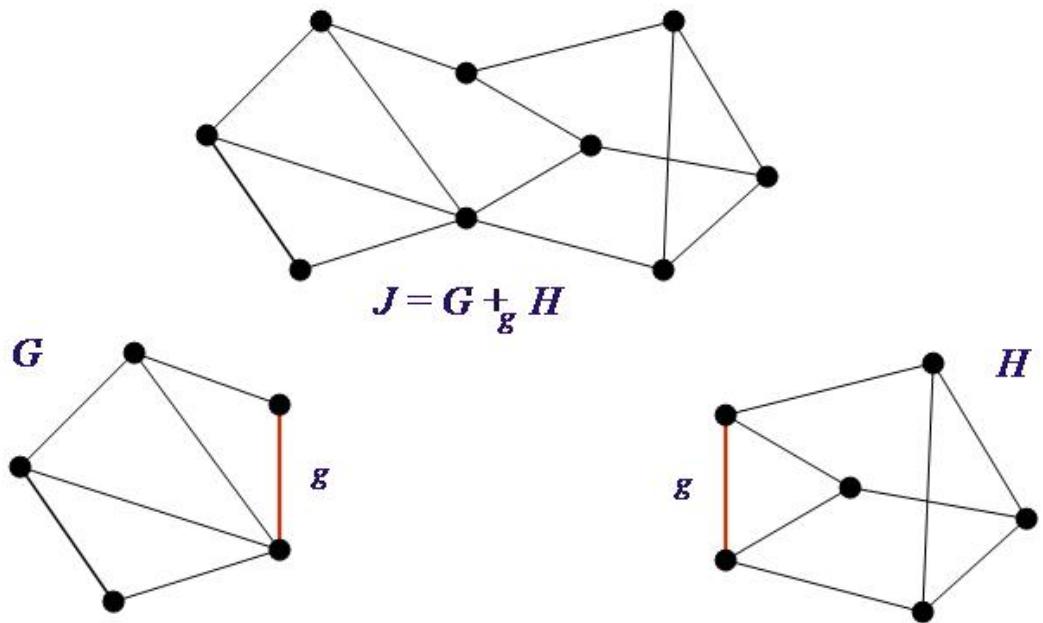
with a positive parameter q ...

if $G=(V,E)$ has n vertices and
 (V,S) has $c(S)$ components, then

$$\omega(S) = q^{c(S)-n}$$



The two-sum of G and H along g



Potts-model partition function of a two-sum

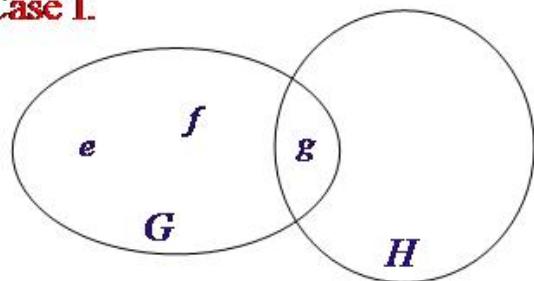
Let J be the two-sum of G and H along g .

Denote the partition function of
the Potts model of G by G , *et cetera*

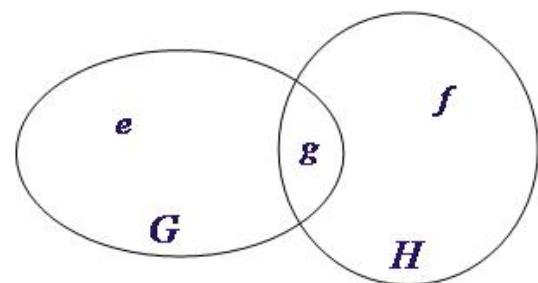
$$J = \frac{1}{1-q} (-qG^g H^g + G^g H_g + G_g H^g - G_g H_g)$$

Two cases for the Rayleigh difference $\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J_{fe}$

Case I.

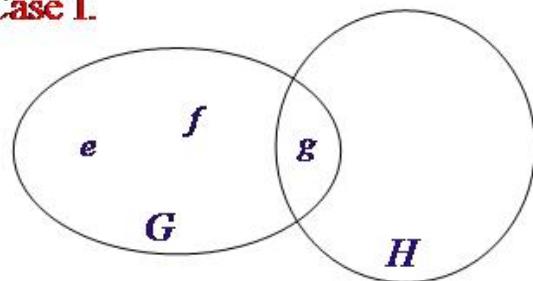


Case II.



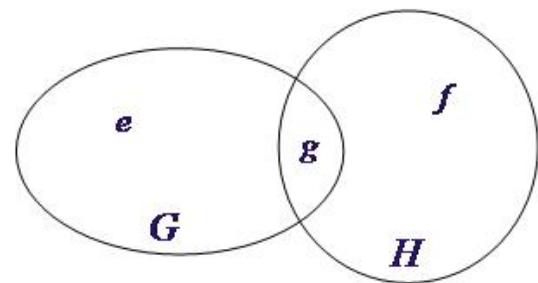
Two cases for the Rayleigh difference $\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J_{fe}$

Case I.



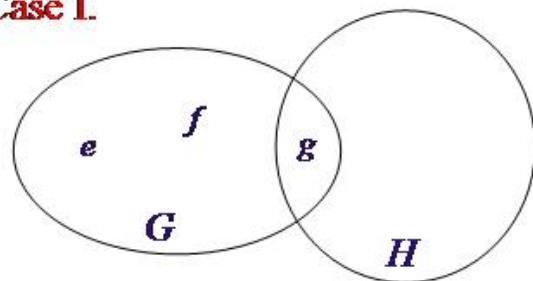
"simulate" H by a suitable substitution for $y(g)$ in G

Case II.



Two cases for the Rayleigh difference $\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J_{fe}$

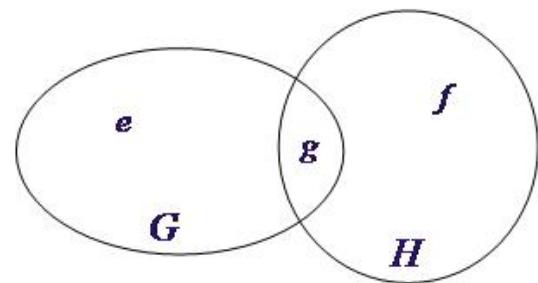
Case I.



"simulate" H by a suitable substitution for $y(g)$ in G

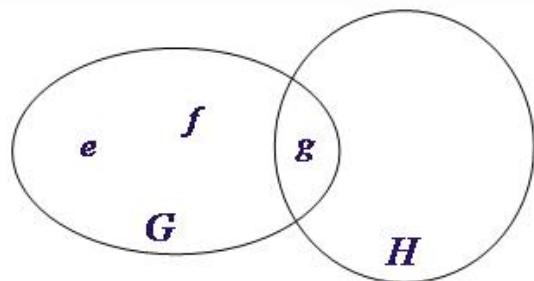
Case II.

factor the Rayleigh difference
as a product of Rayleigh
differences



Case I.

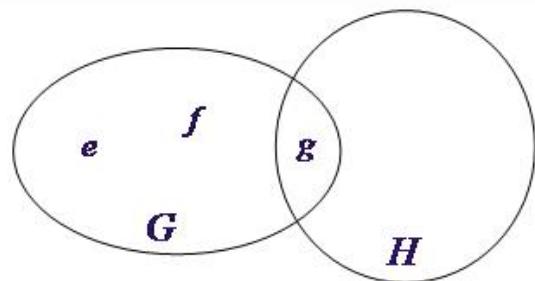
$$G = G^g + y_g G_g$$



$$J = \frac{1}{1-q} (-q G^g H^g + G^g H_g + G_g H^g - G_g H_g)$$

Case I.

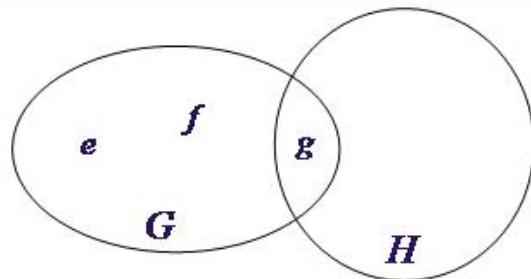
$$G = G^g + y_g G_g$$



$$J = \frac{1}{1-q} (-qG^gH^g + G^gH_g + G_gH^g - G_gH_g)$$

$$J = \frac{1}{1-q} [(-qH^g + H_g)G^g + (H^g - H_g)G_g]$$

Case I.

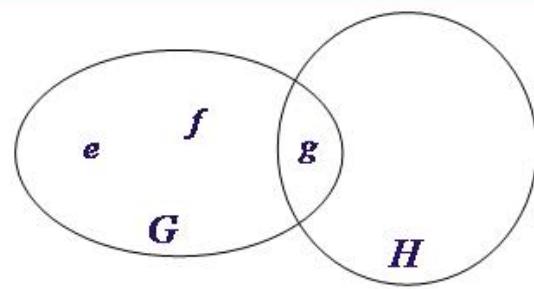


$$G = G^g + y_g G_g$$

$$J = \frac{1}{1-q} (-qG^gH^g + G^gH_g + G_gH^g - G_gH_g)$$

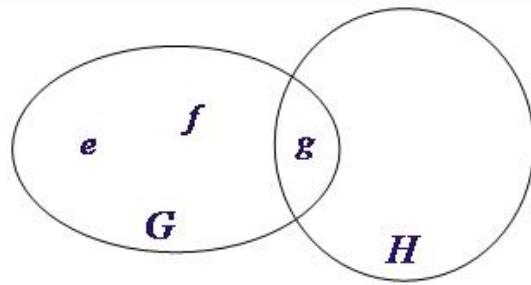
$$J = \frac{1}{1-q} [(-qH^g + H_g)G^g + (H^g - H_g)G_g]$$

$$J = \frac{H^g(\eta - q)}{1-q} \left[G^g + \left(\frac{1-\eta}{\eta - q} \right) G_g \right] \quad \text{where} \quad \eta = H_g / H^g$$



$$J = \frac{H^g(\eta - q)}{1-q} \left[G^g + \left(\frac{1-\eta}{\eta - q} \right) G_g \right]$$

$$\Delta J\{e, f\} = \left(\frac{H^g(\eta - q)}{1-q} \right)^2 \Delta G\{e, f\} \quad \text{where} \quad y_g = \frac{1-\eta}{\eta - q}$$



$$J = \frac{H^g(\eta - q)}{1-q} \left[G^g + \left(\frac{1-\eta}{\eta - q} \right) G_g \right]$$

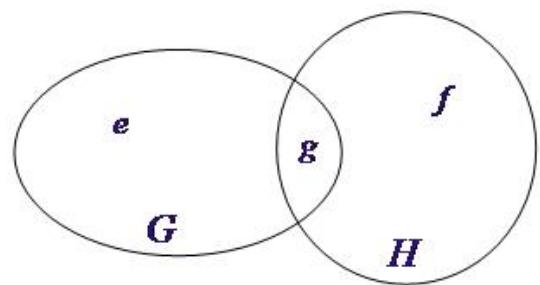
$$\Delta J\{e, f\} = \left(\frac{H^g(\eta - q)}{1-q} \right)^2 \Delta G\{e, f\} \quad \text{where} \quad y_g = \frac{1-\eta}{\eta - q}$$

If $0 < q < 1$ then $0 < q < \eta < 1$, so that $y_g > 0$. Hence

$$\Delta J\{e, f\} \geq 0$$

$$(1-q)J = -qG^g H^g + G^g H_g + G_g H^g - G_g H_g \quad \text{Case II.}$$

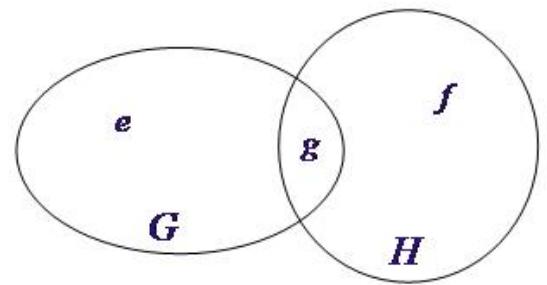
$$\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J^{ef}$$



$$(1-q)J = -qG^g H^g + G^g H_g + G_g H^g - G_g H_g \quad \text{Case II.}$$

$$(1-q)J_e^f = -qG_e^g H^{fg} + G_e^g H_g^f + G_{eg} H^{fg} - G_{eg} H_g^f$$

$$\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J^{ef}$$

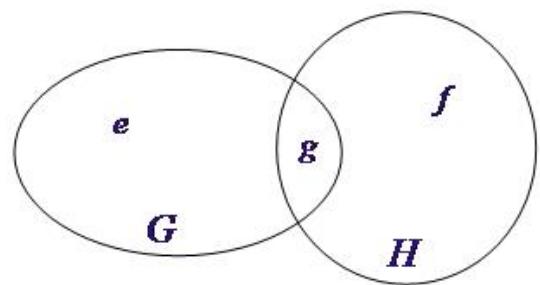


$$(1-q)J = -qG^g H^g + G^g H_g + G_g H^g - G_g H_g \quad \text{Case II.}$$

$$(1-q)J_e^f = -qG_e^g H^{fg} + G_e^g H_g^f + G_{eg} H^{fg} - G_{eg} H_g^f$$

$$(1-q)J_f^e = -qG^{eg} H_f^g + G^{eg} H_{fg} + G_g^e H_f^g - G_g^e H_{fg}$$

$$\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J^{ef}$$



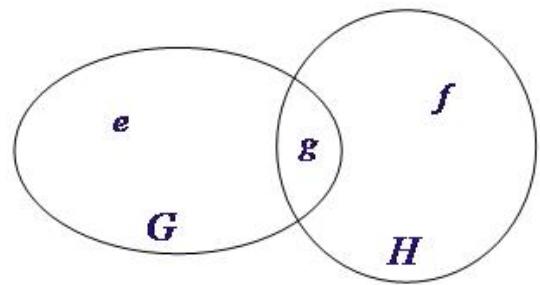
$$(1-q)J = -qG^g H^g + G^g H_g + G_g H^g - G_g H_g \quad \text{Case II.}$$

$$(1-q)J_e^f = -qG_e^g H^{fg} + G_e^g H_g^f + G_{eg} H^{fg} - G_{eg} H_g^f$$

$$(1-q)J_f^e = -qG^{eg} H_f^e + G^{eg} H_{fg} + G_g^e H_f^e - G_g^e H_{fg}$$

$$(1-q)J_{ef} = -qG_e^g H_f^e + G_e^g H_{fg} + G_{eg} H_f^e - G_{eg} H_{fg}$$

$$\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J^{ef}$$



$$(1-q)J = -qG^g H^g + G^g H_g + G_g H^g - G_g H_g \quad \text{Case II.}$$

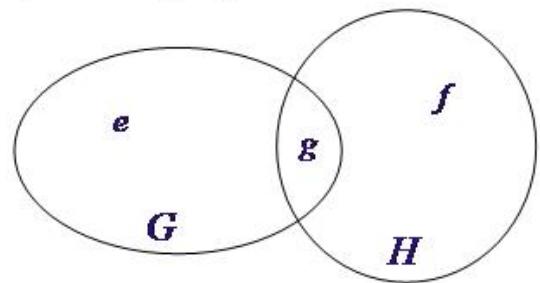
$$(1-q)J_e^f = -qG_e^g H^{fg} + G_e^g H_g^f + G_{eg} H^{fg} - G_{eg} H_g^f$$

$$(1-q)J_f^e = -qG^{eg} H_f^e + G^{eg} H_{fg} + G_g^e H_f^e - G_g^e H_{fg}$$

$$(1-q)J_{ef} = -qG_e^g H_f^e + G_e^g H_{fg} + G_{eg} H_f^e - G_{eg} H_{fg}$$

$$(1-q)J^{ef} = -qG^{eg} H^{fg} + G^{eg} H_g^f + G_g^e H^{fg} - G_g^e H_g^f$$

$$\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J^{ef}$$



$$(1-q)^2 \Delta J\{e, f\} = (1-q)^2 (J_e^f J_f^e - J_{ef} J^{fe}) =$$

$$\begin{aligned} & \sum \left\{ \begin{array}{cccc} (-qG_e^e H^k)(-qG^e H^k) & (-qG_e^e H^k)(+G^e H_g^k) & (-qG_e^e H^k)(+G_e^e H_f^k) & (-qG_e^e H^k)(-G_e^e H_k^k) \\ (+G_e^e H_g^f)(-qG^e H^k) & (+G_e^e H_g^f)(+G^e H_g^k) & (+G_e^e H_g^f)(+G_e^e H_f^k) & (+G_e^e H_g^f)(-G_e^e H_k^k) \\ (+G_{eg}^e H^k)(-qG^e H^k) & (+G_{eg}^e H^k)(+G^e H_g^k) & (+G_{eg}^e H^k)(+G_e^e H_f^k) & (+G_{eg}^e H^k)(-G_e^e H_k^k) \\ (-G_{eg}^e H_g^f)(-qG^e H^k) & (-G_{eg}^e H_g^f)(+G^e H_g^k) & (-G_{eg}^e H_g^f)(+G_e^e H_f^k) & (-G_{eg}^e H_g^f)(-G_e^e H_k^k) \end{array} \right\} \\ & - \sum \left\{ \begin{array}{cccc} (-qG_e^e H_f^e)(-qG^e H^k) & (-qG_e^e H_f^e)(+G^e H_g^k) & (-qG_e^e H_f^e)(+G_e^e H^k) & (-qG_e^e H_f^e)(-G_e^e H_g^k) \\ (+G_e^e H_k^k)(-qG^e H^k) & (+G_e^e H_k^k)(+G^e H_g^k) & (+G_e^e H_k^k)(+G_e^e H^k) & (+G_e^e H_k^k)(-G_e^e H_g^k) \\ (+G_{eg}^e H_f^e)(-qG^e H^k) & (+G_{eg}^e H_f^e)(+G^e H_g^k) & (+G_{eg}^e H_f^e)(+G_e^e H^k) & (+G_{eg}^e H_f^e)(-G_e^e H_g^k) \\ (-G_{eg}^e H_k^k)(-qG^e H^k) & (-G_{eg}^e H_k^k)(+G^e H_g^k) & (-G_{eg}^e H_k^k)(+G_e^e H^k) & (-G_{eg}^e H_k^k)(-G_e^e H_g^k) \end{array} \right\} \end{aligned}$$

$$(1-q)^2 \Delta J\{e, f\} = (1-q)^2 (J_e^f J_f^e - J_{ef} J^{fe}) =$$

$$\sum \begin{cases} (-qG_e^e H^k)(-qG^{ek} H^e_f) & (-qG_e^e H^k)(+G^{ek} H^e_g) & (-qG_e^e H^k)(+G^e H^e_f) & (-qG_e^e H^k)(-G^e H^e_g) \\ (+G_e^e H^f_g)(-qG^{ek} H^e_f) & (+G_e^e H^f_g)(+G^{ek} H^e_g) & (+G_e^e H^f_g)(+G^e H^e_f) & (+G_e^e H^f_g)(-G^e H^e_g) \\ (+G_{eg} H^k)(-qG^{ek} H^e_f) & (+G_{eg} H^k)(+G^{ek} H^e_g) & (+G_{eg} H^k)(+G^e H^e_f) & (+G_{eg} H^k)(-G^e H^e_g) \\ (-G_{eg} H^f_g)(-qG^{ek} H^e_f) & (-G_{eg} H^f_g)(+G^{ek} H^e_g) & (-G_{eg} H^f_g)(+G^e H^e_f) & (-G_{eg} H^f_g)(-G^e H^e_g) \end{cases}$$

$$- \sum \begin{cases} (-qG_e^e H^e_f)(-qG^{ek} H^k) & (-qG_e^e H^e_f)(+G^{ek} H^e_g) & (-qG_e^e H^e_f)(+G^e H^k) & (-qG_e^e H^e_f)(-G^e H^e_g) \\ (+G_e^e H^k_g)(-qG^{ek} H^k) & (+G_e^e H^k_g)(+G^{ek} H^e_g) & (+G_e^e H^k_g)(+G^e H^k) & (+G_e^e H^k_g)(-G^e H^e_g) \\ (+G_{eg} H^e_f)(-qG^{ek} H^k) & (+G_{eg} H^e_f)(+G^{ek} H^e_g) & (+G_{eg} H^e_f)(+G^e H^k) & (+G_{eg} H^e_f)(-G^e H^e_g) \\ (-G_{eg} H^k_g)(-qG^{ek} H^k) & (-G_{eg} H^k_g)(+G^{ek} H^e_g) & (-G_{eg} H^k_g)(+G^e H^k) & (-G_{eg} H^k_g)(-G^e H^e_g) \end{cases}$$

$$(1-q)^2 \Delta J\{e, f\} = (1-q)^2 (J_e^f J_f^e - J_{ef} J^{fe}) =$$

$$\sum \begin{cases} (-qG_e^e H^k_f)(-qG^{ek} H^e_f) & (-qG_e^e H^k_f)(+G^{ek} H^e_g) & (-qG_e^e H^k_f)(+G^e H^e_f) & (-qG_e^e H^k_f)(-G^e H^e_g) \\ (+G_e^e H^f_g)(-qG^{ek} H^e_f) & (+G_e^e H^f_g)(+G^{ek} H^e_g) & (+G_e^e H^f_g)(+G^e H^e_f) & (+G_e^e H^f_g)(-G^e H^e_g) \\ (+G_{eg} H^k_f)(-qG^{ek} H^e_f) & (+G_{eg} H^k_f)(+G^{ek} H^e_g) & (+G_{eg} H^k_f)(+G^e H^e_f) & (+G_{eg} H^k_f)(-G^e H^e_g) \\ (-G_{eg} H^f_g)(-qG^{ek} H^e_f) & (-G_{eg} H^f_g)(+G^{ek} H^e_g) & (-G_{eg} H^f_g)(+G^e H^e_f) & (-G_{eg} H^f_g)(-G^e H^e_g) \end{cases}$$

$$- \sum \begin{cases} (-qG_e^e H^e_f)(-qG^{ek} H^k_f) & (-qG_e^e H^e_f)(+G^{ek} H^f_g) & (-qG_e^e H^e_f)(+G^e H^k_f) & (-qG_e^e H^e_f)(-G^e H^f_g) \\ (+G_e^e H^k_g)(-qG^{ek} H^k_f) & (+G_e^e H^k_g)(+G^{ek} H^f_g) & (+G_e^e H^k_g)(+G^e H^k_f) & (+G_e^e H^k_g)(-G^e H^f_g) \\ (+G_{eg} H^e_f)(-qG^{ek} H^k_f) & (+G_{eg} H^e_f)(+G^{ek} H^f_g) & (+G_{eg} H^e_f)(+G^e H^k_f) & (+G_{eg} H^e_f)(-G^e H^f_g) \\ (-G_{eg} H^k_g)(-qG^{ek} H^k_f) & (-G_{eg} H^k_g)(+G^{ek} H^f_g) & (-G_{eg} H^k_g)(+G^e H^k_f) & (-G_{eg} H^k_g)(-G^e H^f_g) \end{cases}$$

$$(1-q)^2 \Delta J\{e, f\} = (1-q)^2 (J_e^f J_f^e - J_{ef} J^{fe}) =$$

$$\sum \left\{ \begin{array}{cccc} (-qG_e^e H^k)(-qG^{ek} H^e_f) & (-qG_e^e H^k)(+G^{ek} H^e_g) & (-qG_e^e H^k)(+G^e H^e_f) & (-qG_e^e H^k)(-G^e H^e_g) \\ (+G_e^e H^f)(-qG^{ek} H^e_f) & (+G_e^e H^f)(+G^{ek} H^e_g) & (+G_e^e H^f)(+G^e H^e_f) & (+G_e^e H^f)(-G^e H^e_g) \\ (+G_{eg} H^k)(-qG^{ek} H^e_f) & (+G_{eg} H^k)(+G^{ek} H^e_g) & (+G_{eg} H^k)(+G^e H^e_f) & (+G_{eg} H^k)(-G^e H^e_g) \\ (-G_{eg} H^f)(-qG^{ek} H^e_f) & (-G_{eg} H^f)(+G^{ek} H^e_g) & (-G_{eg} H^f)(+G^e H^e_f) & (-G_{eg} H^f)(-G^e H^e_g) \end{array} \right\}$$

$$- \sum \left\{ \begin{array}{cccc} (-qG_e^e H^e_f)(-qG^{ek} H^k) & (-qG_e^e H^e_f)(+G^{ek} H^e_g) & (-qG_e^e H^e_f)(+G^e H^k) & (-qG_e^e H^e_f)(-G^e H^e_g) \\ (+G_e^e H^k)(-qG^{ek} H^k) & (+G_e^e H^k)(+G^{ek} H^e_g) & (+G_e^e H^k)(+G^e H^k) & (+G_e^e H^k)(-G^e H^e_g) \\ (+G_{eg} H^e_f)(-qG^{ek} H^k) & (+G_{eg} H^e_f)(+G^{ek} H^e_g) & (+G_{eg} H^e_f)(+G^e H^k) & (+G_{eg} H^e_f)(-G^e H^e_g) \\ (-G_{eg} H^k)(-qG^{ek} H^k) & (-G_{eg} H^k)(+G^{ek} H^e_g) & (-G_{eg} H^k)(+G^e H^k) & (-G_{eg} H^k)(-G^e H^e_g) \end{array} \right\}$$

$$(1-q)^2 \Delta J\{e, f\} = \sum \begin{Bmatrix} (-qG_e^g H^k) \times (-G_g^e H_f^k) \\ (+G_e^g H_g^f) \times (+G_g^e H_f^k) \\ (+G_{eg}^e H^k) \times (+G_{eg}^e H_f^k) \\ (-G_{eg}^e H_g^f) \times (-qG_{eg}^e H_f^k) \end{Bmatrix} - \sum \begin{Bmatrix} (-qG_e^g H_f^g) \times (-G_g^e H_g^f) \\ (+G_e^g H_k^k) \times (+G_g^e H_k^k) \\ (+G_{eg}^e H_f^g) \times (+G_{eg}^e H_g^f) \\ (-G_{eg}^e H_k^k) \times (-qG_{eg}^e H_k^k) \end{Bmatrix}$$

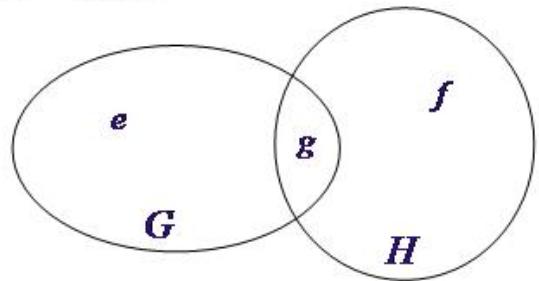
$$(1-q)^2 \Delta J\{e, f\} = \sum \begin{Bmatrix} (-qG_e^g H^k) \times (-G_g^e H_{fg}) \\ (+G_e^g H_g^f) \times (+G_g^e H_f^g) \\ (+G_{eg}^e H^k) \times (+G_{eg}^e H_{fg}) \\ (-G_{eg}^e H_g^f) \times (-qG_{eg}^e H_f^g) \end{Bmatrix} - \sum \begin{Bmatrix} (-qG_e^g H_f^g) \times (-G_g^e H_g^f) \\ (+G_e^g H_{fg}) \times (+G_g^e H^k) \\ (+G_{eg}^e H_f^g) \times (+G_{eg}^e H_g^f) \\ (-G_{eg}^e H_{fg}) \times (-qG_{eg}^e H^k) \end{Bmatrix}$$

$$(1-q)^2 \Delta J\{e, f\} = (1-q) \sum \begin{Bmatrix} -G_e^g H^{fg} G_g^e H_{fg} \\ G_e^g H_g^f G_g^e H_f^g \\ G_{eg}^e H^{fg} G_{eg}^e H_{fg} \\ -G_{eg}^e H_g^f G_{eg}^e H_f^g \end{Bmatrix}$$

$$\Delta J\{e, f\} = \frac{1}{1-q} \cdot \Delta G\{e, g\} \cdot \Delta H\{g, f\}$$

So, if $0 < q < 1$ and all $y_e > 0$ then

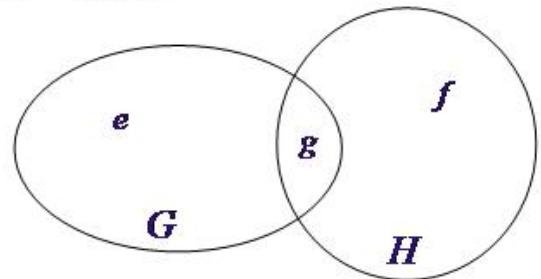
$$\Delta J\{e, f\} \geq 0$$



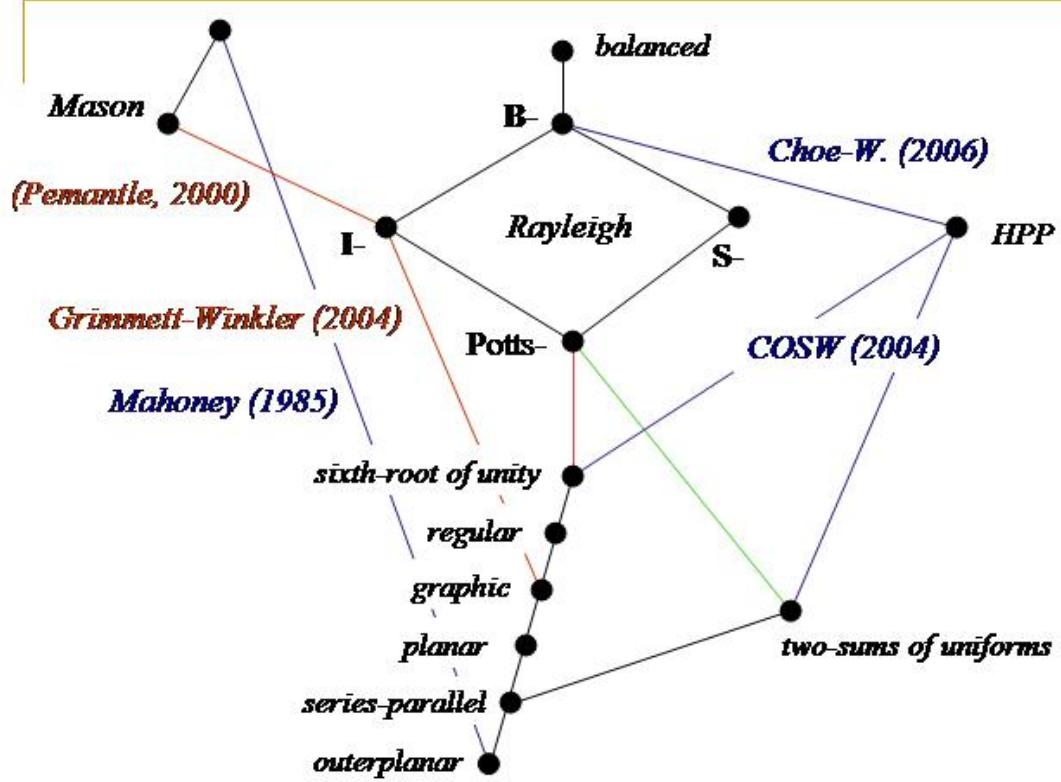
$$\Delta J\{e, f\} = \frac{1}{1-q} \cdot \Delta G\{e, g\} \cdot \Delta H\{g, f\}$$

So, if $0 < q < 1$ and all $y_e > 0$ then

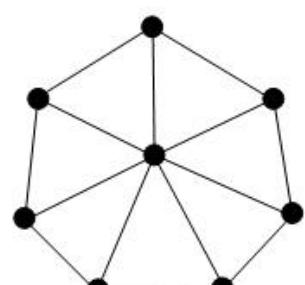
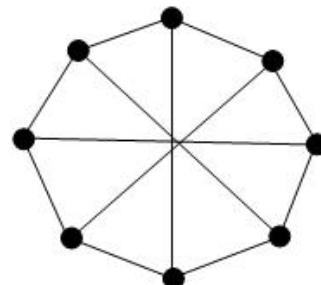
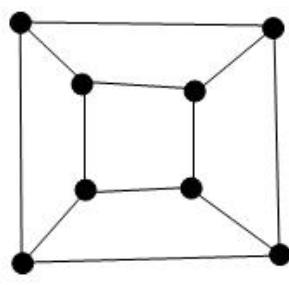
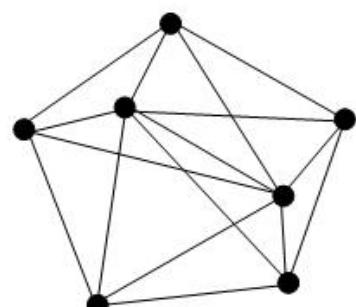
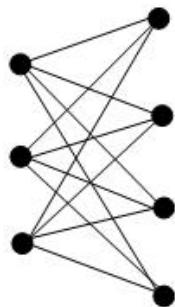
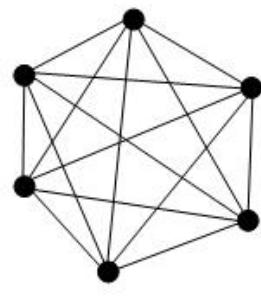
$$\Delta J\{e, f\} \geq 0$$



Thus, if G and H are both Potts-Rayleigh, then the two-sum of G and H along g is also Potts-Rayleigh.



Some 3-connected I-Rayleigh graphs



**Negatively correlated random variables
and Mason's conjecture
for independent sets in matroids**

<http://www.arxiv.org/math.CO/0602648>
