Negative correlations for trees, forests, and Potts models

David Wagner
University of Waterloo
C&O Department

Combinatorial Problems raised by Statistical Mechanics
CRM February 21, 2007.

* Mason’s conjecture
* Rayleigh monotonicity
* negative correlation conditions
* a probabilistic conjecture
* the Grimmett-Winkler conjecture
* two-sums and the Potts-Rayleigh condition
* *et cetera, et cetera*…

Mason's conjecture *(1972)*

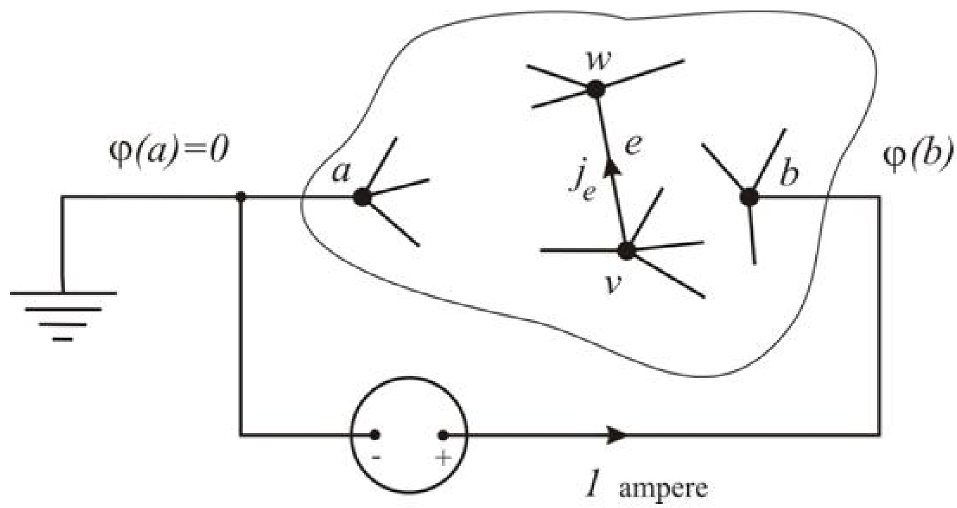
G=(V,E) a connected graph with *m* edges and *n* vertices.

I(k) = number of *k*-edge spanning forests of *G*.

Conjecture: for all $0 < k < n-1$:

$$\frac{I_k^2}{\binom{m}{k}^2} \geq \frac{I_{k-1}}{\binom{m}{k-1}} \cdot \frac{I_{k+1}}{\binom{m}{k+1}}$$

Effective conductance of a graph *(Kirchhoff, 1847)*



Effective conductance of a graph *(Kirchhoff, 1847)*

$$\mathcal{Y}_{ab}(G;\vec{y}) = \frac{1}{\varphi(b)} = \frac{T(G;\vec{y})}{T(G/ab;\vec{y})}$$

in which

$$T(H;\vec{y}) = \sum_{T} \vec{y}^{T}$$

is the sum of

$$\vec{y}^{T} = \prod_{e \in T} y_e$$

over all spanning trees T of H – that is,

the *spanning tree enumerator* of H.

Rayleigh monotonicity

for any edge *e* of *G*:

$$\frac{\partial}{\partial y_e}\mathcal{Y}_{ab}(G;\vec{y})\geq 0$$

provided that all edge-conductances *y(c)* are positive.

Rayleigh monotonicity: for distinct edges *e,f* of *G*:

$$\Delta G\{e,f\} = G_e^f G_f^e - G_{ef} G^{ef} \geq 0$$

provided that all edge-conductances are positive.

Here, *G* stands for the spanning tree enumerator of the graph *G*.

subscript -- contract the edge
superscript -- delete the edge

A more general framework….

For a finite set E $\quad \vec{y} = \{y_e : e \in E\}$

A *weight function* $\quad \omega : 2^E \to [0, \infty)$

A more general framework….

For a finite set E $\qquad \vec{y} = \{y_e : e \in E\}$

A *weight function* $\qquad \omega : 2^E \to [0, \infty)$

The associated *partition function* $\qquad Z(\omega; \vec{y}) = \sum_{S \subseteq E} \omega(S) \vec{y}^S$

A more general framework….

For a finite set E $\quad \vec{y} = \{y_e : e \in E\}$

A *weight function* $\quad \omega : 2^E \to [0, \infty)$

The associated *partition function* $\quad Z(\omega; \vec{y}) = \sum_{S \subseteq E} \omega(S) \vec{y}^S$

For distinct e, f in E,

The *Rayleigh difference* is $\quad \Delta Z\{e,f\} = Z_e^f Z_f^e - Z_{ef} Z^{ef}$

A more general framework….

For a finite set E $\qquad \vec{y} = \{y_e : e \in E\}$

A *weight function* $\qquad \omega : 2^E \to [0,\infty)$

The associated *partition function* $\qquad Z(\omega;\vec{y}) = \sum_{S \subseteq E} \omega(S)\vec{y}^S$

For distinct e, f in E,
The *Rayleigh difference* is $\qquad \Delta Z\{e,f\} = Z_e^f Z_f^e - Z_{ef} Z^{ef}$

deletion $\qquad Z^e = Z\big|_{y_e=0}$

contraction $\qquad Z_e = \frac{\partial}{\partial y_e} Z$

A probabilistic interpretation

Give each parameter *y(c)* a positive value.

Choose a subset S of E with probability $$\frac{\omega(S)\vec{y}^{S}}{Z}$$

A probabilistic interpretation

Give each parameter *y(c)* a positive value.

Choose a subset *S* of *E* with probability $\frac{\omega(S)\vec{y}^S}{Z}$

For distinct elements *e, f* of *E*,
the covariance of the events *(e in S)* and *(f in S)* is

$$Cov(e,f) = -\frac{y_e y_f}{Z^2} \Delta Z\{e,f\}$$

A probabilistic interpretation

Give each parameter *y(c)* a positive value.

Choose a subset *S* of *E* with probability $\frac{\omega(S)\vec{y}^{S}}{Z}$

For distinct elements *e, f* of *E*,
the covariance of the events *(e in S)* and *(f in S)* is

$$Cov(e,f) = -\frac{y_e y_f}{Z^2}\Delta Z\{e,f\} = y_e y_f \frac{\partial^2}{\partial y_e \partial y_f}\log Z$$

The Rayleigh condition

For any distinct *e, f* in *E,* if all *y(c)* are positive then

$$\Delta Z\{e, f\} \geq 0$$

That is, any two distinct elements of *E* are
negatively correlated
for all choices of positive weights for the parameters.

Four interesting partition functions

correspond to these weight functions
associated with a (connected) graph *G=(V,E)*

- spanning trees (B)
- spanning forests (I)
- connected spanning subgraphs (S)

$$\omega(S) = \begin{cases} 1 & \text{if } S \text{ has the correct form,} \\ 0 & \text{otherwise.} \end{cases}$$

Four interesting partition functions

correspond to these weight functions
associated with a (connected) graph *G=(V,E)*

* spanning trees (B)
* spanning forests (I)
* connected spanning subgraphs (S)

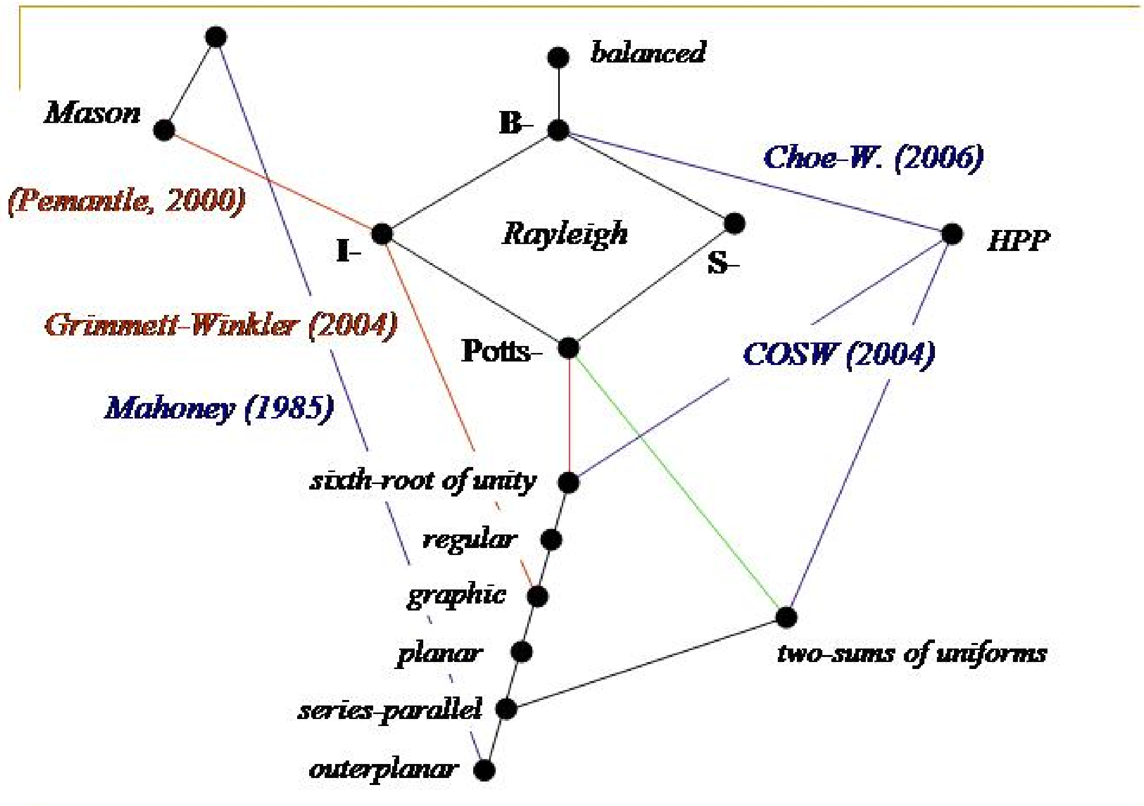
$$\omega(S) = \begin{cases} 1 & \text{if } S \text{ has the correct form,} \\ 0 & \text{otherwise.} \end{cases}$$

* Potts model

with a positive parameter *q* ….

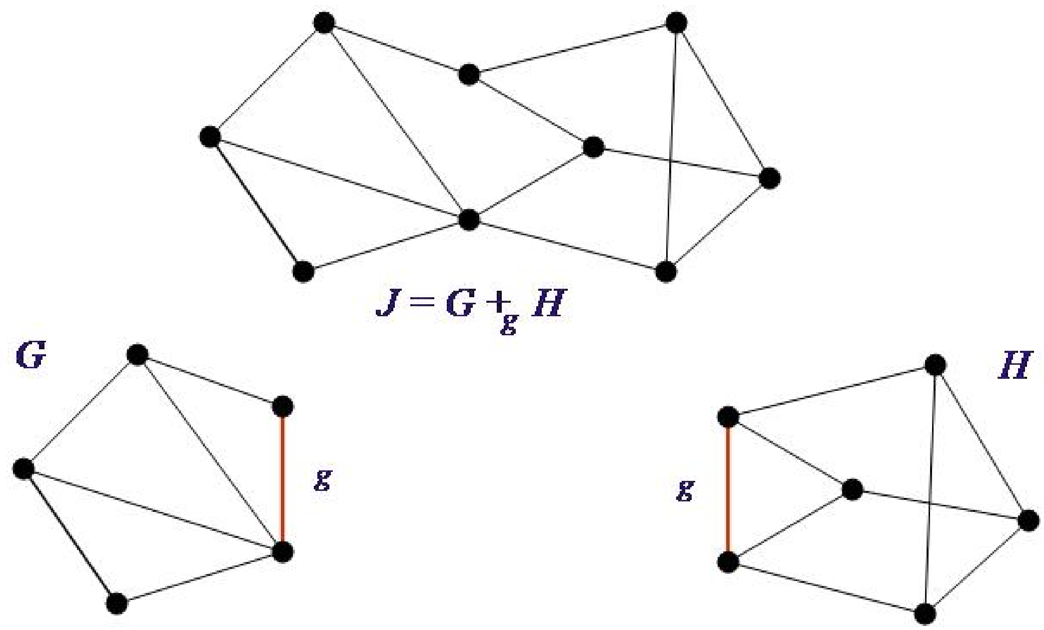
if *G=(V,E)* has *n* vertices and
(V,S) has *c(S)* components, then

$$\omega(S) = q^{c(S)-n}$$



balanced
Mason
B-
Choe-W. (2006)
(Pemantle, 2000)
Rayleigh
I-
S-
HPP
Grimmett-Winkler (2004)
Potts-
COSW (2004)
Mahoney (1985)
sixth-root of unity
regular
graphic
planar
two-sums of uniforms
series-parallel
outerplanar

The two-sum of G and H along g



Potts-model partition function of a two-sum

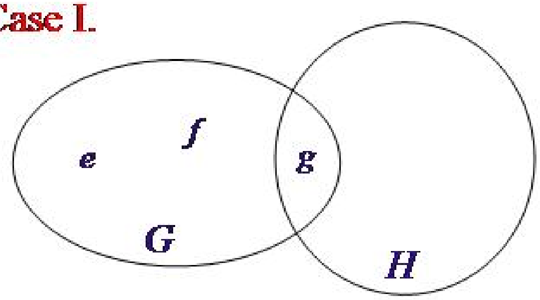
Let J be the two-sum of G and H along g.

Denote the partition function of
the Potts model of G by G, *et cetera*.

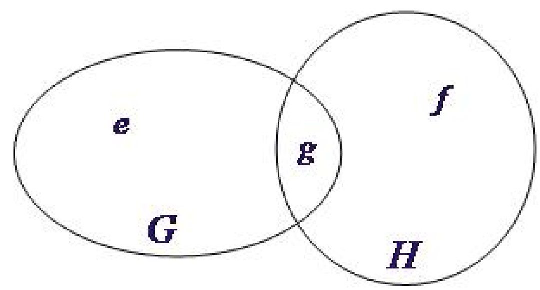
$$J = \frac{1}{1-q}\left(-qG^g H^g + G^g H_g + G_g H^g - G_g H_g\right)$$

Two cases for the Rayleigh difference $\Delta J\{e,f\} = J_e^f J_f^e - J_{ef} J^{ef}$

Case I.

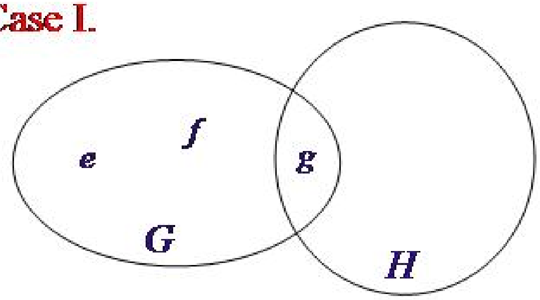


Case II.



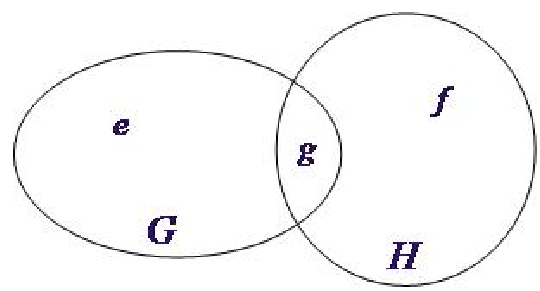
Two cases for the Rayleigh difference $\Delta J\{e,f\} = J_e^f J_f^e - J_{ef} J^{ef}$

Case I.



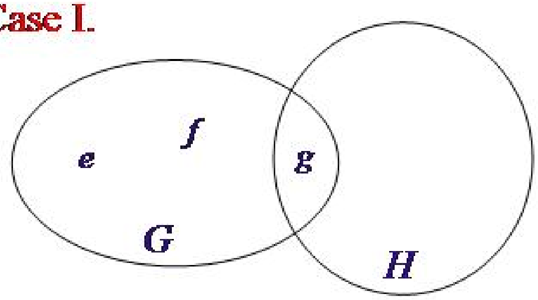
"simulate" *H* by a suitable substitution for *y(g)* in *G*

Case II.



Two cases for the Rayleigh difference $\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J^{ef}$

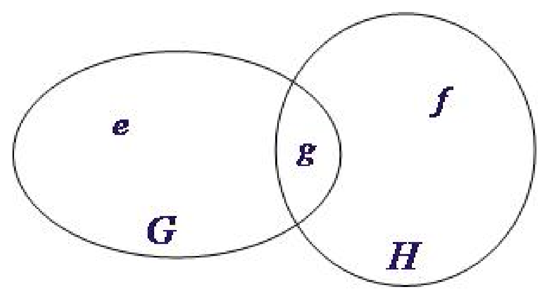
Case I.



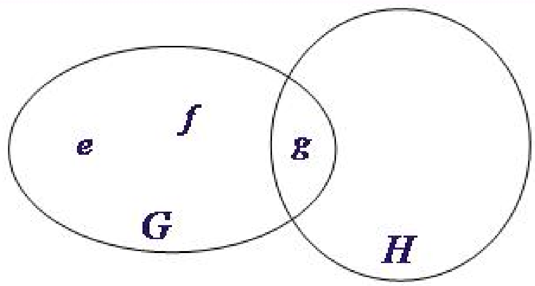
"simulate" H by a suitable substitution for $y(g)$ in G

Case II.

factor the Rayleigh difference as a product of Rayleigh differences



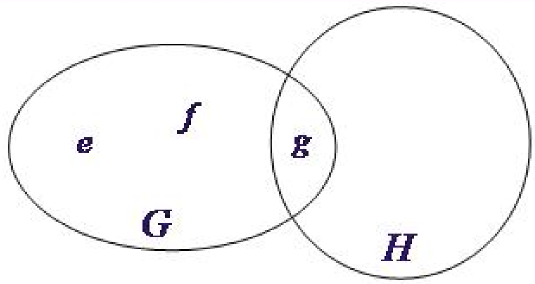
Case I.



$$G = G^g + y_g G_g$$

$$J = \frac{1}{1-q}\left(-qG^g H^g + G^g H_g + G_g H^g - G_g H_g\right)$$

Case I.

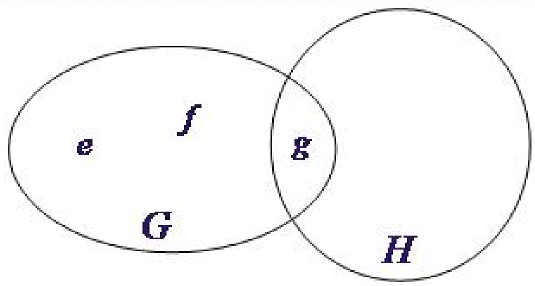


$$G = G^g + y_g G_g$$

$$J = \frac{1}{1-q}\left(-qG^g H^g + G^g H_g + G_g H^g - G_g H_g\right)$$

$$J = \frac{1}{1-q}\left[\left(-qH^g + H_g\right)G^g + \left(H^g - H_g\right)G_g\right]$$

Case I.

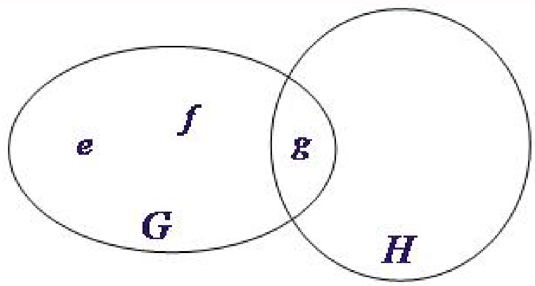


$$G = G^g + y_g G_g$$

$$J = \frac{1}{1-q}\left(-qG^g H^g + G^g H_g + G_g H^g - G_g H_g\right)$$

$$J = \frac{1}{1-q}\left[\left(-qH^g + H_g\right)G^g + \left(H^g - H_g\right)G_g\right]$$

$$J = \frac{H^g(\eta - q)}{1-q}\left[G^g + \left(\frac{1-\eta}{\eta - q}\right)G_g\right]$$ where $\eta = H_g / H^g$

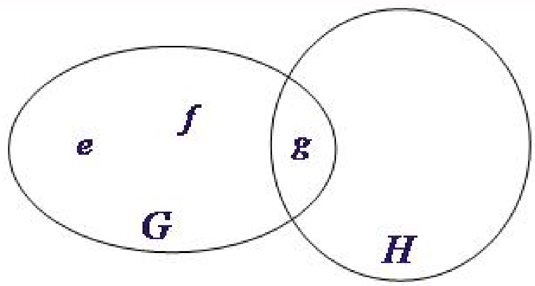


$$J = \frac{H^{g}(\eta - q)}{1-q}\left[G^{g} + \left(\frac{1-\eta}{\eta - q}\right)G_{g}\right]$$

$$\Delta J\{e,f\} = \left(\frac{H^{g}(\eta - q)}{1-q}\right)^{2}\Delta G\{e,f\}$$

where

$$y_{g} = \frac{1-\eta}{\eta - q}$$



$$J = \frac{H^g(\eta - q)}{1-q}\left[G^g + \left(\frac{1-\eta}{\eta - q}\right)G_g\right]$$

$$\Delta J\{e,f\} = \left(\frac{H^g(\eta - q)}{1-q}\right)^2 \Delta G\{e,f\}$$ where $$y_g = \frac{1-\eta}{\eta - q}$$

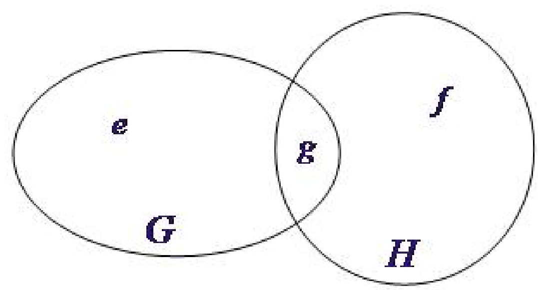
If $0 < q < 1$ then $0 < q < \eta < 1$, so that $y_g > 0$. Hence

$$\Delta J\{e,f\} \geq 0$$

$$(1-q)J = -qG^g H^g + G^g H_g + G_g H^g - G_g H_g$$

Case II.

$$\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J^{ef}$$

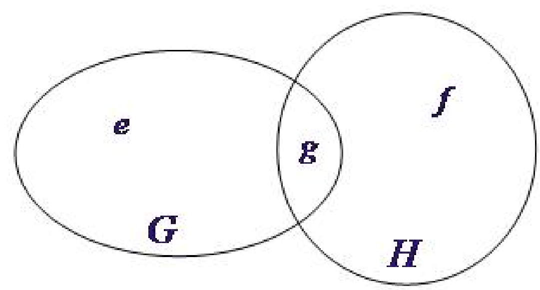


$$(1-q)J = -qG^g H^g + G^g H_g + G_g H^g - G_g H_g$$

Case II.

$$(1-q)J_e^f = -qG_e^g H^{fg} + G_e^g H_g^f + G_{eg} H^{fg} - G_{eg} H_g^f$$

$$\Delta J\{e, f\} = J_e^f J_f^e - J_{ef} J^{ef}$$



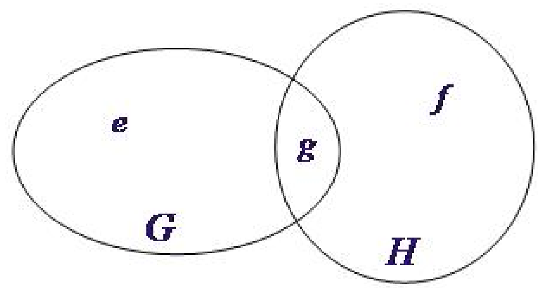
$$(1-q)J = -qG^g H^g + G^g H_g + G_g H^g - G_g H_g$$

Case II.

$$(1-q)J_e^f = -qG_e^g H^{fg} + G_e^g H_g^f + G_{eg} H^{fg} - G_{eg} H_g^f$$

$$(1-q)J_f^e = -qG^{eg} H_f^g + G^{eg} H_{fg} + G_g^e H_f^g - G_g^e H_{fg}$$

$$\Delta J\{e,f\} = J_e^f J_f^e - J_{ef} J^{ef}$$



Case II.

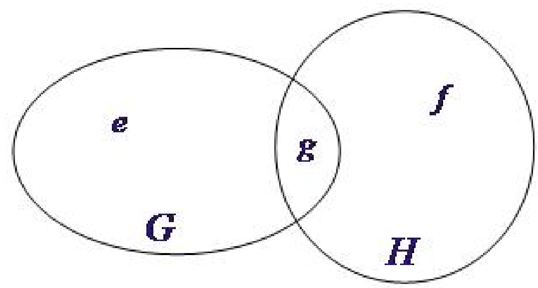
$$(1-q)J = -qG^{g}H^{g} + G^{g}H_{g} + G_{g}H^{g} - G_{g}H_{g}$$

$$(1-q)J_{e}^{f} = -qG_{e}^{g}H^{fg} + G_{e}^{g}H_{g}^{f} + G_{eg}H^{fg} - G_{eg}H_{g}^{f}$$

$$(1-q)J_{f}^{e} = -qG^{eg}H_{f}^{g} + G^{eg}H_{fg} + G_{g}^{e}H_{f}^{g} - G_{g}^{e}H_{fg}$$

$$(1-q)J_{ef} = -qG_{e}^{g}H_{f}^{g} + G_{e}^{g}H_{fg} + G_{eg}H_{f}^{g} - G_{eg}H_{fg}$$

$$\Delta J\{e,f\} = J_{e}^{f}J_{f}^{e} - J_{ef}J^{ef}$$



Case II.

$$(1-q)J = -qG^g H^g + G^g H_g + G_g H^g - G_g H_g$$

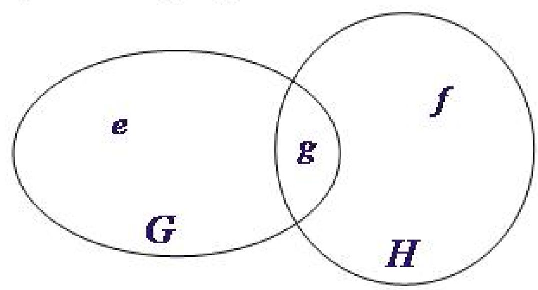
$$(1-q)J_e^f = -qG_e^g H^{fg} + G_e^g H_g^f + G_{eg} H^{fg} - G_{eg} H_g^f$$

$$(1-q)J_f^e = -qG^{eg} H_f^g + G^{eg} H_{fg} + G_g^e H_f^g - G_g^e H_{fg}$$

$$(1-q)J_{ef} = -qG_e^g H_f^g + G_e^g H_{fg} + G_{eg} H_f^g - G_{eg} H_{fg}$$

$$(1-q)J^{ef} = -qG^{eg} H^{fg} + G^{eg} H_g^f + G_g^e H^{fg} - G_g^e H_g^f$$

$$\Delta J\{e,f\} = J_e^f J_f^e - J_{ef} J^{ef}$$



$$(1-q)^2 \Delta J\{e,f\} = (1-q)^2 (J_e^f J_f^e - J_{ef} J^{ef}) =$$

$$\sum \left\{ \begin{matrix} (-qG_e^g H^{fg})(-qG^{eg} H_f^g) & (-qG_e^g H^{fg})(+G^{eg} H_{fg}) & (-qG_e^g H^{fg})(+G_g^e H_f^g) & (-qG_e^g H^{fg})(-G_g^e H_{fg}) \\ (+G_e^g H_g^f)(-qG^{eg} H_f^g) & (+G_e^g H_g^f)(+G^{eg} H_{fg}) & (+G_e^g H_g^f)(+G_g^e H_f^g) & (+G_e^g H_g^f)(-G_g^e H_{fg}) \\ (+G_{eg} H^{fg})(-qG^{eg} H_f^g) & (+G_{eg} H^{fg})(+G^{eg} H_{fg}) & (+G_{eg} H^{fg})(+G_g^e H_f^g) & (+G_{eg} H^{fg})(-G_g^e H_{fg}) \\ (-G_{eg} H_g^f)(-qG^{eg} H_f^g) & (-G_{eg} H_g^f)(+G^{eg} H_{fg}) & (-G_{eg} H_g^f)(+G_g^e H_f^g) & (-G_{eg} H_g^f)(-G_g^e H_{fg}) \end{matrix} \right\}$$

$$-\sum \left\{ \begin{matrix} (-qG_e^g H_f^g)(-qG^{eg} H^{fg}) & (-qG_e^g H_f^g)(+G^{eg} H_g^f) & (-qG_e^g H_f^g)(+G_g^e H^{fg}) & (-qG_e^g H_f^g)(-G_g^e H_g^f) \\ (+G_e^g H_{fg})(-qG^{eg} H^{fg}) & (+G_e^g H_{fg})(+G^{eg} H_g^f) & (+G_e^g H_{fg})(+G_g^e H^{fg}) & (+G_e^g H_{fg})(-G_g^e H_g^f) \\ (+G_{eg} H_f^g)(-qG^{eg} H^{fg}) & (+G_{eg} H_f^g)(+G^{eg} H_g^f) & (+G_{eg} H_f^g)(+G_g^e H^{fg}) & (+G_{eg} H_f^g)(-G_g^e H_g^f) \\ (-G_{eg} H_{fg})(-qG^{eg} H^{fg}) & (-G_{eg} H_{fg})(+G^{eg} H_g^f) & (-G_{eg} H_{fg})(+G_g^e H^{fg}) & (-G_{eg} H_{fg})(-G_g^e H_g^f) \end{matrix} \right\}$$

$$(1-q)^2 \Delta J\{e,f\} = (1-q)^2 (J_e^f J_f^e - J_{ef} J^{ef}) =$$

$$\sum \begin{Bmatrix} (-qG_e^g H^{fg})(-qG^{eg} H_f^g) & (-qG_e^g H^{fg})(+G^{eg} H_{fg}) & (-qG_e^g H^{fg})(+G_g^e H_f^g) & (-qG_e^g H^{fg})(-G_g^e H_{fg}) \\ (+G_e^g H_g^f)(-qG^{eg} H_f^g) & (+G_e^g H_g^f)(+G^{eg} H_{fg}) & (+G_e^g H_g^f)(+G_g^e H_f^g) & (+G_e^g H_g^f)(-G_g^e H_{fg}) \\ (+G_{eg} H^{fg})(-qG^{eg} H_f^g) & (+G_{eg} H^{fg})(+G^{eg} H_{fg}) & (+G_{eg} H^{fg})(+G_g^e H_f^g) & (+G_{eg} H^{fg})(-G_g^e H_{fg}) \\ (-G_{eg} H_g^f)(-qG^{eg} H_f^g) & (-G_{eg} H_g^f)(+G^{eg} H_{fg}) & (-G_{eg} H_g^f)(+G_g^e H_f^g) & (-G_{eg} H_g^f)(-G_g^e H_{fg}) \end{Bmatrix}$$

$$-\sum \begin{Bmatrix} (-qG_e^g H_f^g)(-qG^{eg} H^{fg}) & (-qG_e^g H_f^g)(+G^{eg} H_g^f) & (-qG_e^g H_f^g)(+G_g^e H^{fg}) & (-qG_e^g H_f^g)(-G_g^e H_g^f) \\ (+G_e^g H_{fg})(-qG^{eg} H^{fg}) & (+G_e^g H_{fg})(+G^{eg} H_g^f) & (+G_e^g H_{fg})(+G_g^e H^{fg}) & (+G_e^g H_{fg})(-G_g^e H_g^f) \\ (+G_{eg} H_f^g)(-qG^{eg} H^{fg}) & (+G_{eg} H_f^g)(+G^{eg} H_g^f) & (+G_{eg} H_f^g)(+G_g^e H^{fg}) & (+G_{eg} H_f^g)(-G_g^e H_g^f) \\ (-G_{eg} H_{fg})(-qG^{eg} H^{fg}) & (-G_{eg} H_{fg})(+G^{eg} H_g^f) & (-G_{eg} H_{fg})(+G_g^e H^{fg}) & (-G_{eg} H_{fg})(-G_g^e H_g^f) \end{Bmatrix}$$

$$(1-q)^2 \Delta J\{e,f\} = (1-q)^2 (J_e^f J_f^e - J_{ef} J^{ef}) =$$

$$\sum \begin{Bmatrix} (-qG_e^g H^{fg})(-qG^{eg} H_f^g) & (-qG_e^g H^{fg})(+G^{eg} H_{fg}) & (-qG_e^g H^{fg})(+G_g^e H_f^g) & (-qG_e^g H^{fg})(-G_g^e H_{fg}) \\ (+G_e^g H_g^f)(-qG^{eg} H_f^g) & (+G_e^g H_g^f)(+G^{eg} H_{fg}) & (+G_e^g H_g^f)(+G_g^e H_f^g) & (+G_e^g H_g^f)(-G_g^e H_{fg}) \\ (+G_{eg} H^{fg})(-qG^{eg} H_f^g) & (+G_{eg} H^{fg})(+G^{eg} H_{fg}) & (+G_{eg} H^{fg})(+G_g^e H_f^g) & (+G_{eg} H^{fg})(-G_g^e H_{fg}) \\ (-G_{eg} H_g^f)(-qG^{eg} H_f^g) & (-G_{eg} H_g^f)(+G^{eg} H_{fg}) & (-G_{eg} H_g^f)(+G_g^e H_f^g) & (-G_{eg} H_g^f)(-G_g^e H_{fg}) \end{Bmatrix}$$

$$-\sum \begin{Bmatrix} (-qG_e^g H_f^g)(-qG^{eg} H^{fg}) & (-qG_e^g H_f^g)(+G^{eg} H_g^f) & (-qG_e^g H_f^g)(+G_g^e H^{fg}) & (-qG_e^g H_f^g)(-G_g^e H_g^f) \\ (+G_e^g H_{fg})(-qG^{eg} H^{fg}) & (+G_e^g H_{fg})(+G^{eg} H_g^f) & (+G_e^g H_{fg})(+G_g^e H^{fg}) & (+G_e^g H_{fg})(-G_g^e H_g^f) \\ (+G_{eg} H_f^g)(-qG^{eg} H^{fg}) & (+G_{eg} H_f^g)(+G^{eg} H_g^f) & (+G_{eg} H_f^g)(+G_g^e H^{fg}) & (+G_{eg} H_f^g)(-G_g^e H_g^f) \\ (-G_{eg} H_{fg})(-qG^{eg} H^{fg}) & (-G_{eg} H_{fg})(+G^{eg} H_g^f) & (-G_{eg} H_{fg})(+G_g^e H^{fg}) & (-G_{eg} H_{fg})(-G_g^e H_g^f) \end{Bmatrix}$$

$$(1-q)^2 \Delta J\{e,f\} = (1-q)^2 (J_e^f J_f^e - J_{ef} J^{ef}) =$$

$$\sum \begin{Bmatrix} (-qG_e^g H^{fg})(-qG^{eg} H_f^g) & (-qG_e^g H^{fg})(+G^{eg} H_{fg}) & (-qG_e^g H^{fg})(+G_g^e H_f^g) & (-qG_e^g H^{fg})(-G_g^e H_{fg}) \\ (+G_e^g H_g^f)(-qG^{eg} H_f^g) & (+G_e^g H_g^f)(+G^{eg} H_{fg}) & (+G_e^g H_g^f)(+G_g^e H_f^g) & (+G_e^g H_g^f)(-G_g^e H_{fg}) \\ (+G_{eg} H^{fg})(-qG^{eg} H_f^g) & (+G_{eg} H^{fg})(+G^{eg} H_{fg}) & (+G_{eg} H^{fg})(+G_g^e H_f^g) & (+G_{eg} H^{fg})(-G_g^e H_{fg}) \\ (-G_{eg} H_g^f)(-qG^{eg} H_f^g) & (-G_{eg} H_g^f)(+G^{eg} H_{fg}) & (-G_{eg} H_g^f)(+G_g^e H_f^g) & (-G_{eg} H_g^f)(-G_g^e H_{fg}) \end{Bmatrix}$$

$$-\sum \begin{Bmatrix} (-qG_e^g H_f^g)(-qG^{eg} H^{fg}) & (-qG_e^g H_f^g)(+G^{eg} H_g^f) & (-qG_e^g H_f^g)(+G_g^e H^{fg}) & (-qG_e^g H_f^g)(-G_g^e H_g^f) \\ (+G_e^g H_{fg})(-qG^{eg} H^{fg}) & (+G_e^g H_{fg})(+G^{eg} H_g^f) & (+G_e^g H_{fg})(+G_g^e H^{fg}) & (+G_e^g H_{fg})(-G_g^e H_g^f) \\ (+G_{eg} H_f^g)(-qG^{eg} H^{fg}) & (+G_{eg} H_f^g)(+G^{eg} H_g^f) & (+G_{eg} H_f^g)(+G_g^e H^{fg}) & (+G_{eg} H_f^g)(-G_g^e H_g^f) \\ (-G_{eg} H_{fg})(-qG^{eg} H^{fg}) & (-G_{eg} H_{fg})(+G^{eg} H_g^f) & (-G_{eg} H_{fg})(+G_g^e H^{fg}) & (-G_{eg} H_{fg})(-G_g^e H_g^f) \end{Bmatrix}$$

$$
(1-q)^2 \Delta J\{e,f\} = \sum \begin{Bmatrix} (-qG_e^g H^{fg})(-G_g^e H_{fg}) \\ (+G_e^g H_g^f)(+G_g^e H_f^g) \\ (+G_{eg} H^{fg})(+G^{eg} H_{fg}) \\ (-G_{eg} H_g^f)(-qG^{eg} H_f^g) \end{Bmatrix} - \sum \begin{Bmatrix} (-qG_e^g H_f^g)(-G_g^e H_g^f) \\ (+G_e^g H_{fg})(+G_g^e H^{fg}) \\ (+G_{eg} H_f^g)(+G^{eg} H_g^f) \\ (-G_{eg} H_{fg})(-qG^{eg} H^{fg}) \end{Bmatrix}
$$

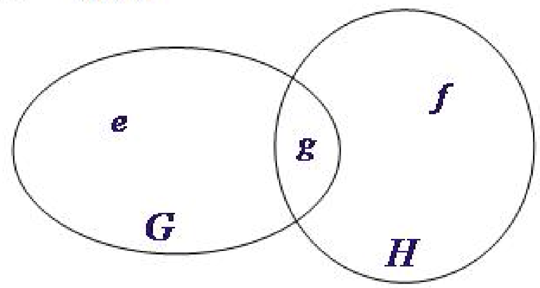
$$(1-q)^2 \Delta J\{e,f\} = \sum \left\{ \begin{matrix} (-qG_e^g H^{fg})(-G_g^e H_{fg}) \\ (+G_e^g H_g^f)(+G_g^e H_f^g) \\ (+G_{eg} H^{fg})(+G^{eg} H_{fg}) \\ (-G_{eg} H_g^f)(-qG^{eg} H_f^g) \end{matrix} \right\} - \sum \left\{ \begin{matrix} (-qG_e^g H_f^g)(-G_g^e H_g^f) \\ (+G_e^g H_{fg})(+G_g^e H^{fg}) \\ (+G_{eg} H_f^g)(+G^{eg} H_g^f) \\ (-G_{eg} H_{fg})(-qG^{eg} H^{fg}) \end{matrix} \right\}$$

$$(1-q)^2 \Delta J\{e,f\} = (1-q) \sum \left\{ \begin{matrix} -G_e^g H^{fg} G_g^e H_{fg} \\ G_e^g H_g^f G_g^e H_f^g \\ G_{eg} H^{fg} G^{eg} H_{fg} \\ -G_{eg} H_g^f G^{eg} H_f^g \end{matrix} \right\}$$

$$\Delta J\{e,f\} = \frac{1}{1-q} \cdot \Delta G\{e,g\} \cdot \Delta H\{g,f\}$$

So, if $0 < q < 1$ and all $y_c > 0$ then

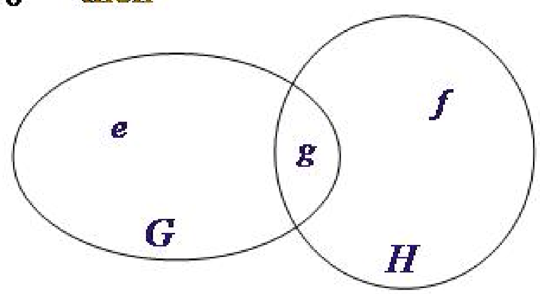
$$\Delta J\{e,f\} \geq 0$$



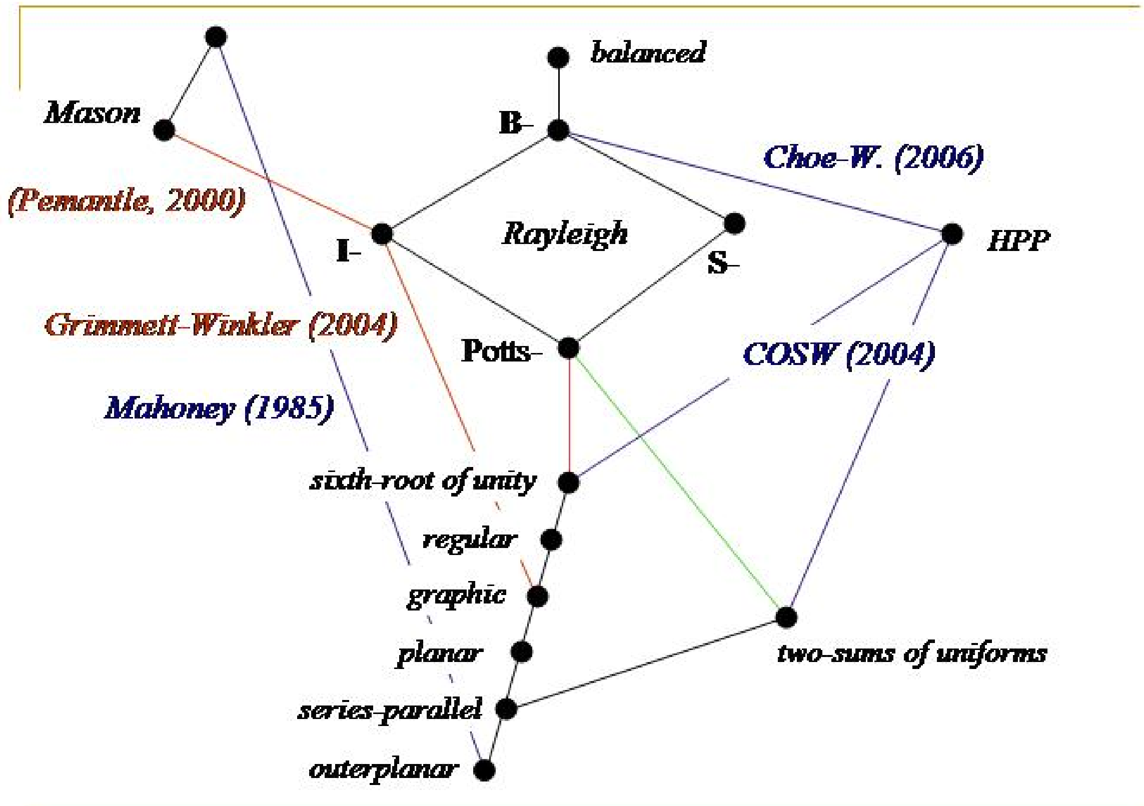
$$\Delta J\{e,f\} = \frac{1}{1-q} \cdot \Delta G\{e,g\} \cdot \Delta H\{g,f\}$$

So, if $0 < q < 1$ and all $y_e > 0$ then

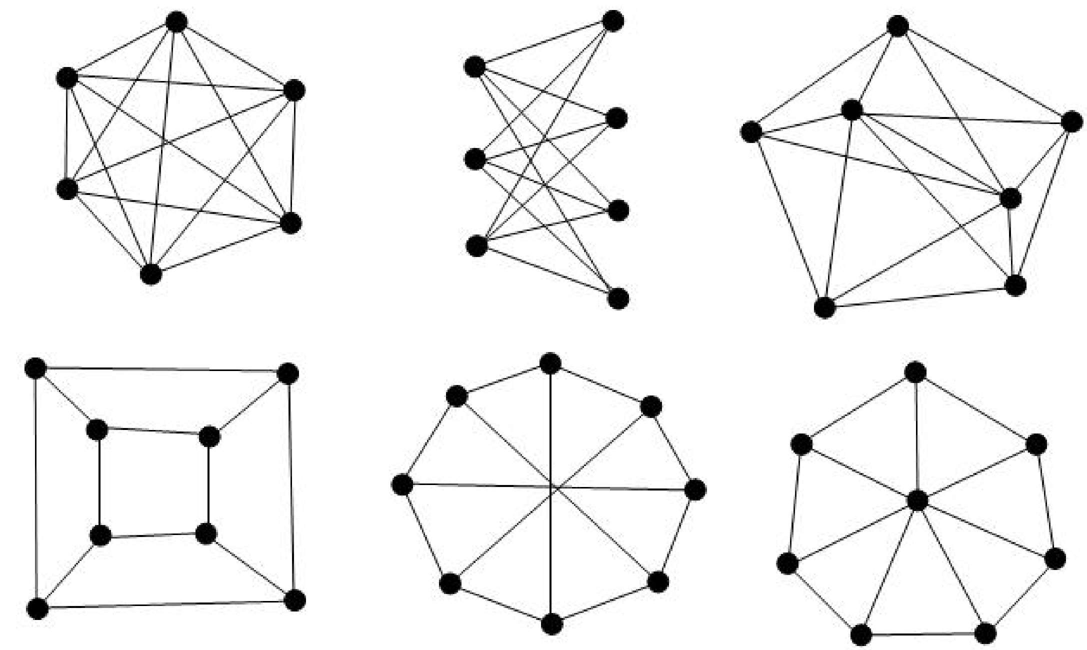
$$\Delta J\{e,f\} \geq 0$$



Thus, if G and H are both Potts-Rayleigh, then the two-sum of G and H along g is also Potts-Rayleigh.


balanced
Mason
B-
Choe-W. (2006)
(Pemantle, 2000)
I-
Rayleigh
S-
HPP
Grimmett-Winkler (2004)
Potts-
COSW (2004)
Mahoney (1985)
sixth-root of unity
regular
graphic
planar
two-sums of uniforms
series-parallel
outerplanar

Some 3-connected I-Rayleigh graphs



Negatively correlated random variables
and Mason’s conjecture
for independent sets in matroids

http://www.arxiv.org/math.CO/0602648