Random tilings

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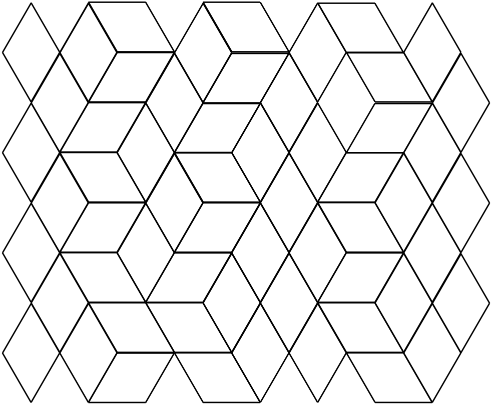
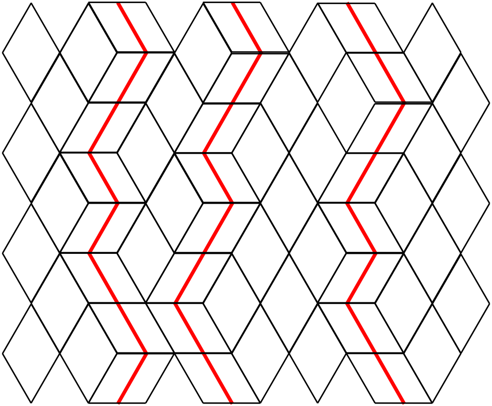
A familiar problem:

How many rhombus tilings?

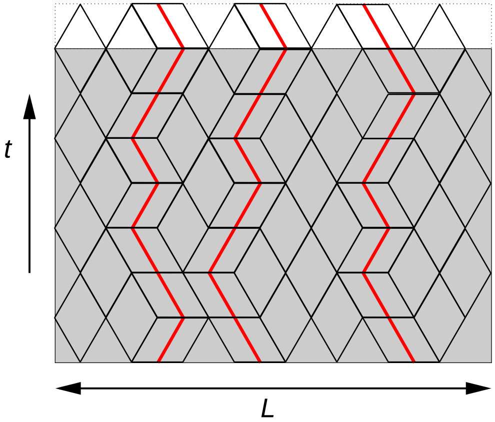


Combinatorics way: Non-intersecting lattice paths



Gessel-Viennot-Lindström. . . determinant

Stat Mech way: Transfer matrix T



Add one row at a time

Number of paths from **x** to **y**:

$$Z(x_1,\ldots,x_n;y_1,\ldots,y_n) = \langle \mathbf{x}|\mathbf{T}^t|\mathbf{y}\rangle$$

Asymptotics:

Consider periodic boundary conditions

$$Z = \sum_{\mathbf{x},\mathbf{y}} Z(\mathbf{x};\mathbf{y})\delta_{\mathbf{x},\mathbf{y}} = \mathrm{Trace}\,\mathbf{T}^t = \sum_{i=1}^{\dim \mathbf{T}} \lambda_i^t$$
$$\approx \lambda_1^t \quad (t \to \infty).$$

Need to determine eigenvalues of **T**.

One walker:

$$\mathbf{T}|x\rangle = |x-1\rangle + |x+1\rangle.$$

Ansatz for eigenvector:

$$|\Psi\rangle = \sum_{x=1}^{L} \psi(x)|x\rangle$$

Eigenvalue equation on coefficients:

$$\lambda\psi(x) = \psi(x+1) + \psi(x-1)$$

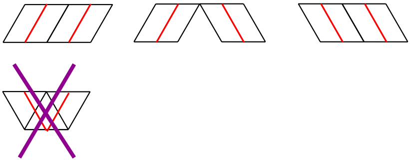
Solved by:

$$\psi(x) = z^x, \qquad \lambda = z + z^{-1}.$$

Periodic boundaries $\psi(L+1) = \psi(1)$:

$$z^L = 1.$$

Two walkers:



$$\mathbf{T}|x_1, x_2\rangle = \sum_{e_i=\pm 1} |x_1 + e_1, x_2 + e_2\rangle \quad (x_2 > x_1 + 1),$$

$$\mathbf{T}|x, x+1\rangle = |x-1, x\rangle + |x-1, x+2\rangle + |x+1, x+2\rangle.$$

Eigenvalue equation

$$\mathbf{T} \sum_{x_1, x_2} \psi(x_1, x_2)|x_1, x_2\rangle = \lambda \sum_{x_1, x_2} \psi(x_1, x_2)|x_1, x_2\rangle.$$

Ansatz:

$$\psi(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_1^{x_2}$$

Solution:

$$\lambda = \sum_{i=1,2} z_i + z_i^{-1},$$

$$\psi(x, x) = 0 \Rightarrow A_{12} + A_{21} = 0$$

Boundary condition $\psi(x, L+1) = \psi(1, x)$:

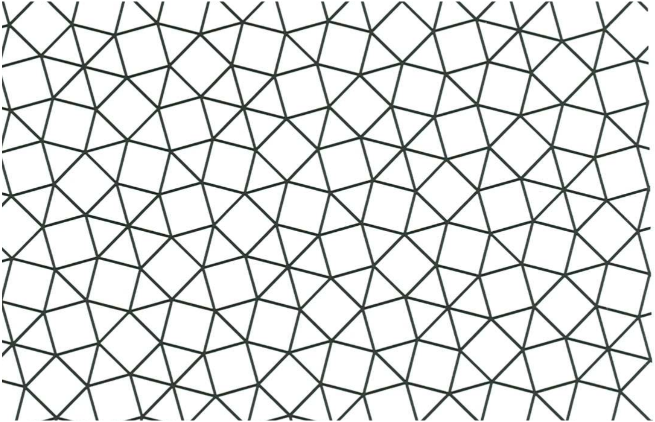
$$z_2^L = \frac{A_{21}}{A_{12}} = -1 = z_1^L.$$

General solution (n walkers):

$$\lambda = \sum_{i=1}^{n} z_i + z_i^{-1},$$
$$z_i^L = (-)^n.$$

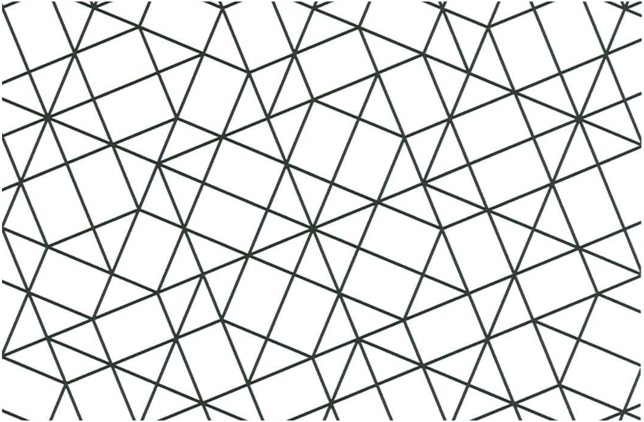
This requires nontrivial factorisation of multiparticle scattering.

Rectangle-Triangle tilings



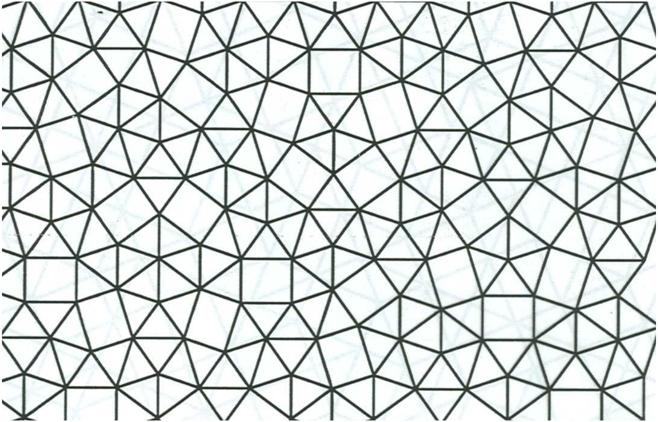
12-fold Square-Triangle tiling (angles are multiples of $\pi/6$).

Rectangle-Triangle tilings



8-fold Rectangle-Triangle tiling ($\pi/8$).

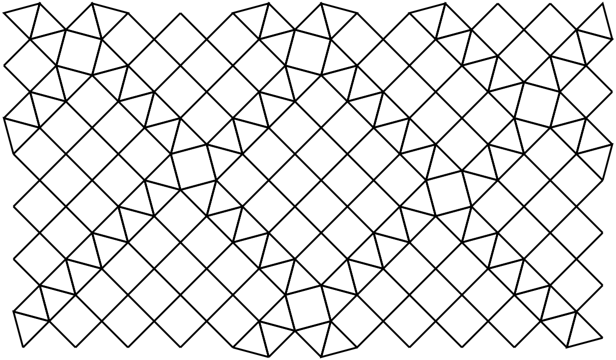
Rectangle-Triangle tilings



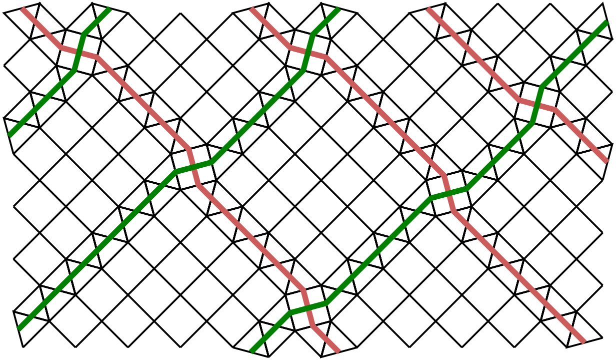
10-fold Rectangle-Triangle tiling ($\pi/10$).

Have not solved combinatorial problem: Number of tilings in a finite geometry.

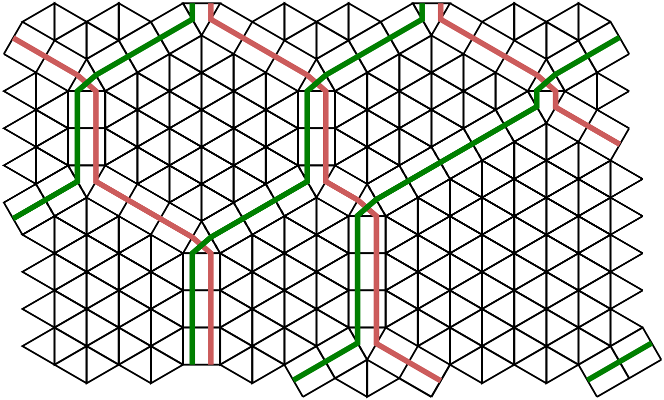
Can do Stat. Mech. problem: Asymptotics on a torus
Develop walker picture.



12-fold tiling when squares dominate

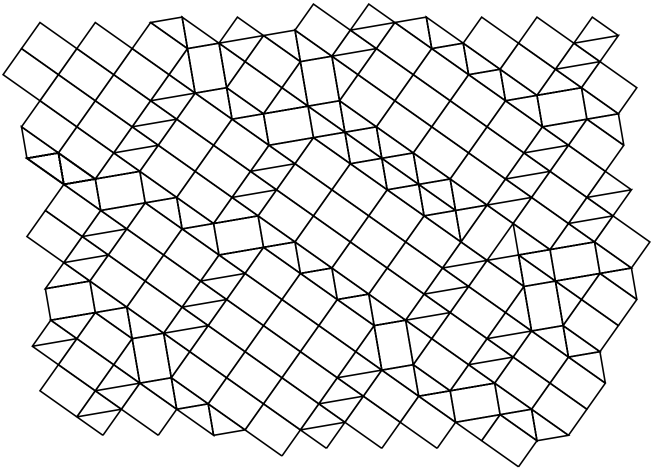


12-fold tiling when squares dominate

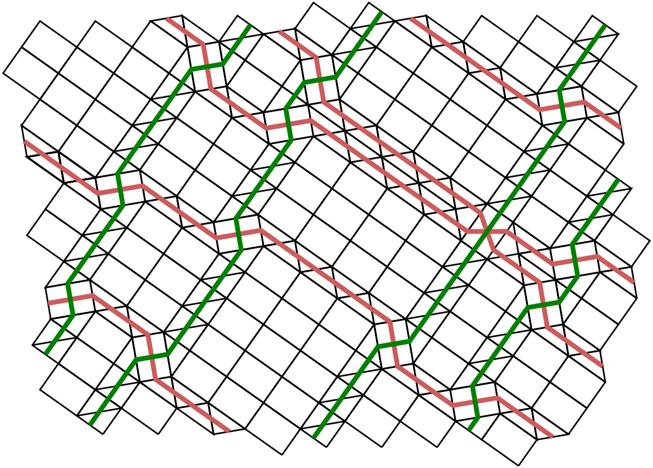


12-fold tiling when triangles dominate

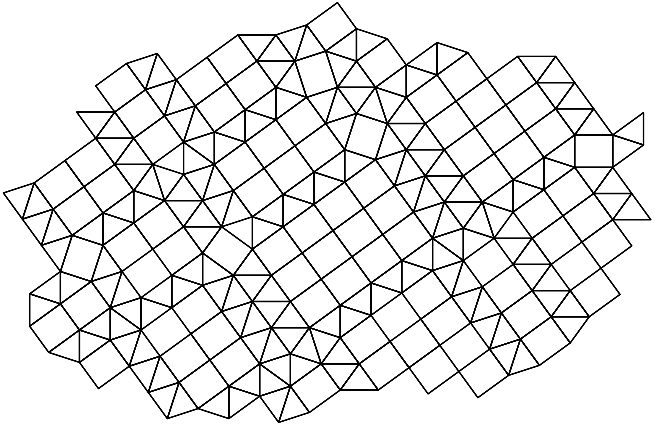
Chicken wire model
(applications to force networks in granular matter)



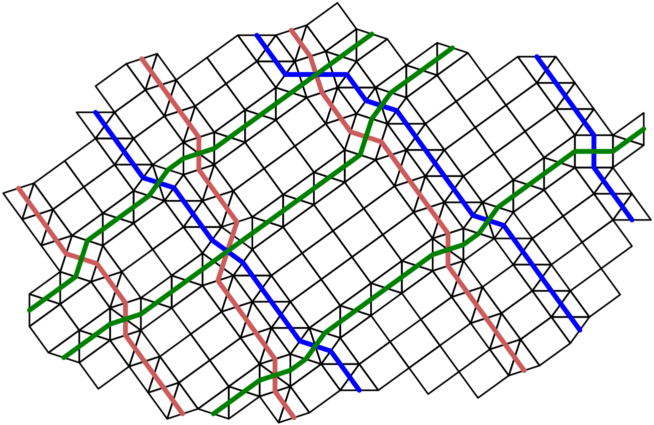
8-fold tiling when rectangles dominate



8-fold tiling when rectangles dominate



10-fold tiling when rectangles dominate



10-fold tiling when rectangles dominate

Diagonalisation of the Transfer matrix

12- and 8-fold random tiling

Ansatz for eigenvector:

$$\psi(\mathbf{x}, \mathbf{y}) = \sum_{\pi,\rho} A(\pi, \rho) \prod_{i=1}^{n} z_{\pi_i}^{x_i} \prod_{j=1}^{m} w_{\rho_j}^{y_j}$$

Program: Adjust z_i, w_j and A such that ψ satisfies the eigenvalue equation:

$$\lambda\psi(\mathbf{x}, \mathbf{y}) = (\mathbf{T}\psi)(\mathbf{x}, \mathbf{y}).$$

All walkers separated,

$$(\mathbf{T}\psi)(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x} + \mathbf{1}, \mathbf{y} - \mathbf{1})$$

which implies:

$$\lambda = \prod_{i=1}^{n} z_i \prod_{j=1}^{m} w_j^{-1}$$

Exceptions where two walkers collide imply:

$$A(\ldots, \pi_i, \rho_j, \ldots) = A(\ldots, \rho_j, \pi_i, \ldots) \left(z_{\pi_i}^{-1} w_{\rho_j}^{-1} + z_{\pi_i} w_{\rho_j} \right)$$

Trivial factors between amplitudes for like walkers.

Periodic boundary conditions give

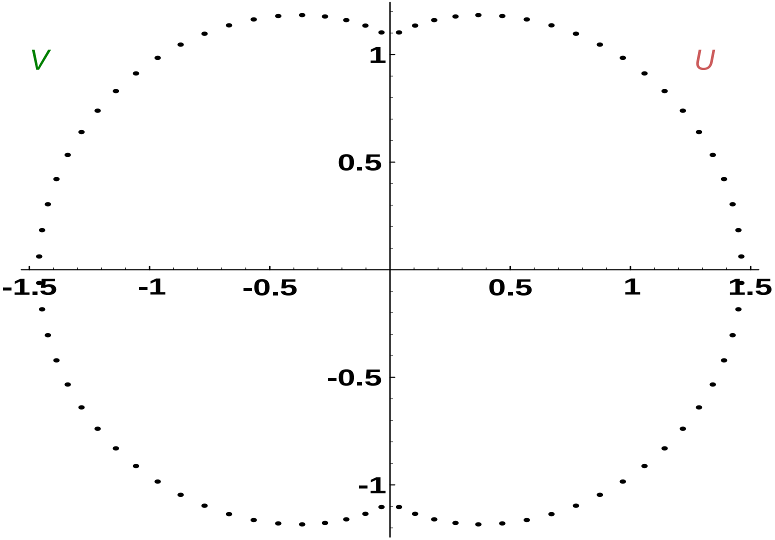
$$A(\ldots,\pi_n)z_{\pi_n}^L = A(\pi_n,\ldots)$$
$$A(\ldots,\rho_m)w_{\rho_m}^L = A(\rho_m,\ldots)$$

After change of variables:

$$\lambda = \prod_{i=1}^{n} u_i \prod_{j=1}^{m} v_j,$$

$$u_i^L = (-)^{n-1}\prod_{j=1}^{m}(u_i - v_j),$$

$$v_j^L = (-)^{m-1}\prod_{i=1}^{n}(u_i - v_j).$$



Solutions of the square-triangle equations corresponding to the largest eigenvalue.

10-fold random tiling

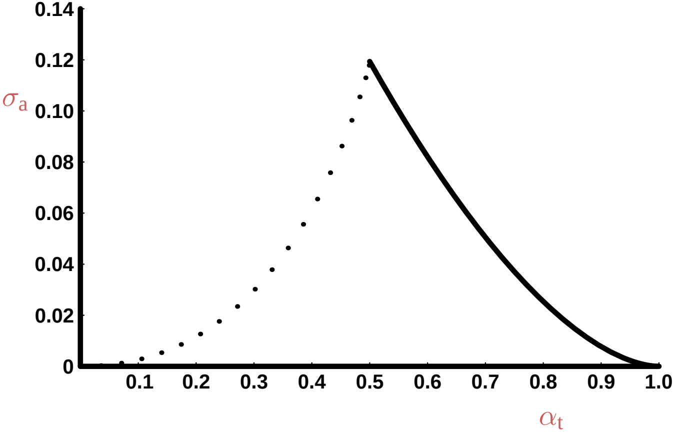
The solution of the eigenvalue problem has the following form

$$\lambda = \prod_{i=1}^{n} u_i \prod_{j=1}^{m} v_j,$$

$$u_i^L = (-)^{n-1} \prod_{j=1}^{m} (u_i - v_j) \prod_{l=1}^{n_2} (w_l + u_i - u_i^{-1}),$$

$$v_j^L = (-)^{m-1} \prod_{i=1}^{n} (u_i - v_j),$$

$$(-)^{n_2} = \prod_{i=1}^{n} (w_l + u_i - u_i^{-1}).$$



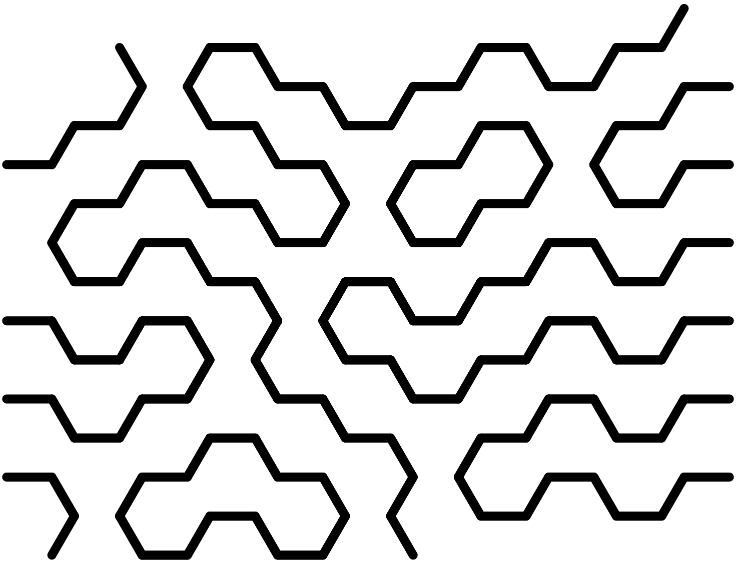
$\lim_{L\to\infty} L^{-1} \log \lambda$ (entropy) versus triangle area fraction.

Entropies per vertex:

til.	exact	numerical
8	$\log 2^2 - \sqrt{2}\log(1+\sqrt{2})$	0.139844
10	$\frac{1}{2}\log\frac{5^5}{4^4} - \sqrt{5}\log\left(\frac{1+\sqrt{5}}{2}\right)$	0.174984
12	$\log(2^2 3^3) - 2\sqrt{3}\log(2+\sqrt{3})$	0.120055

Can compute curve for 8 and 12-fold tiling.

Mapping to a loop model



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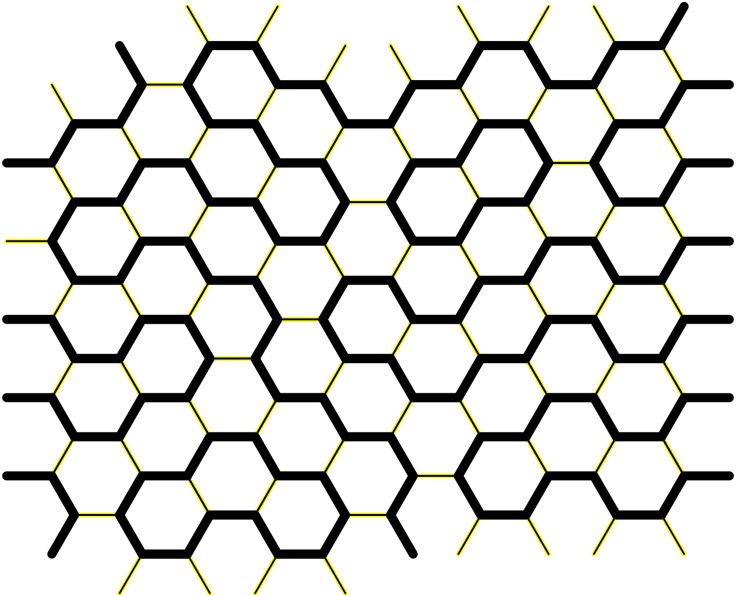

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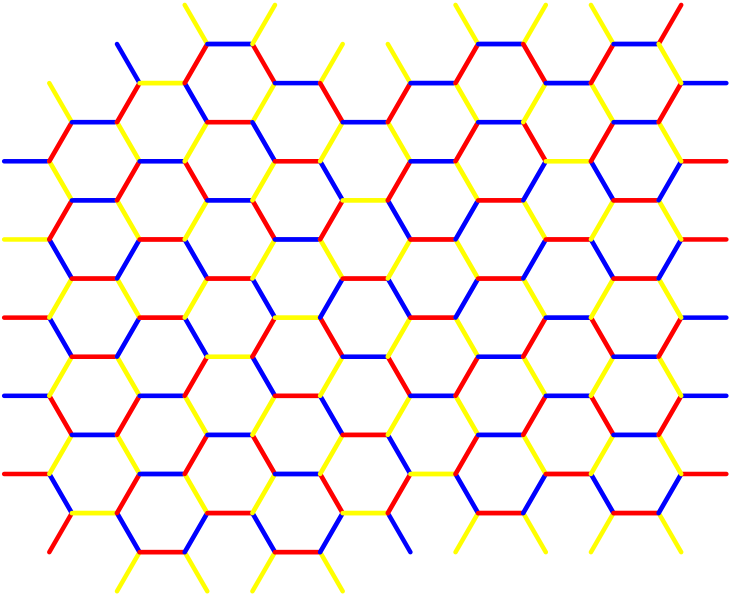

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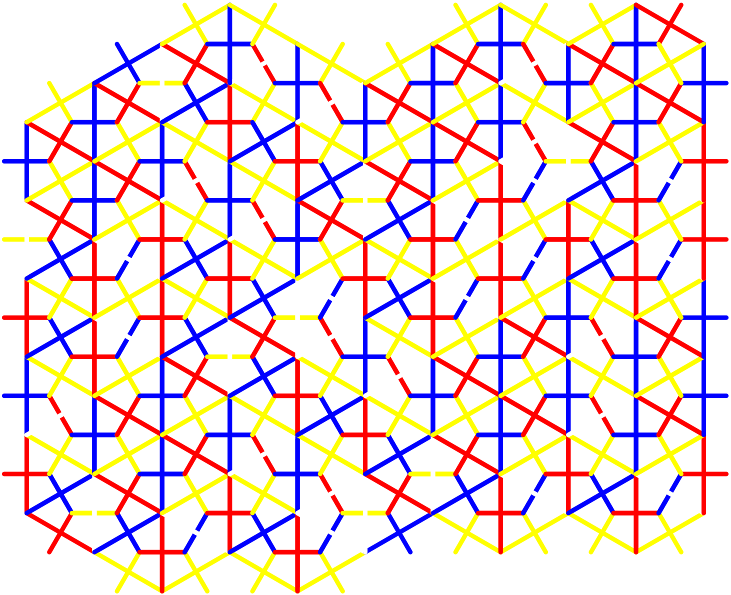

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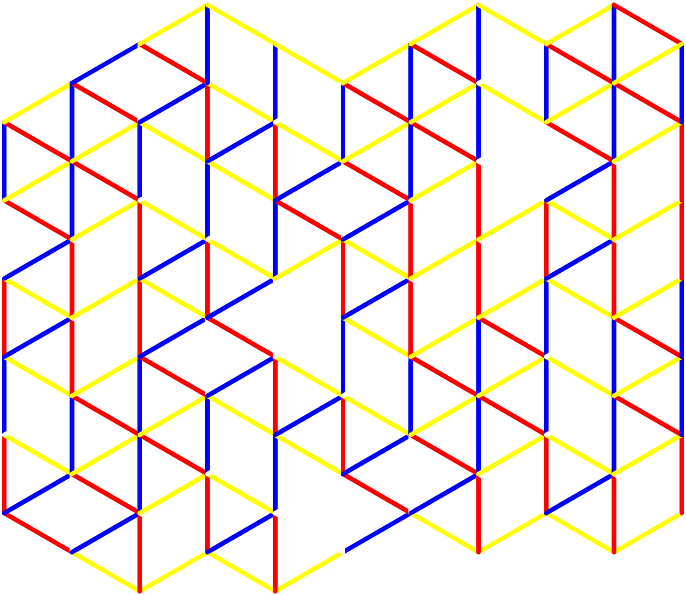

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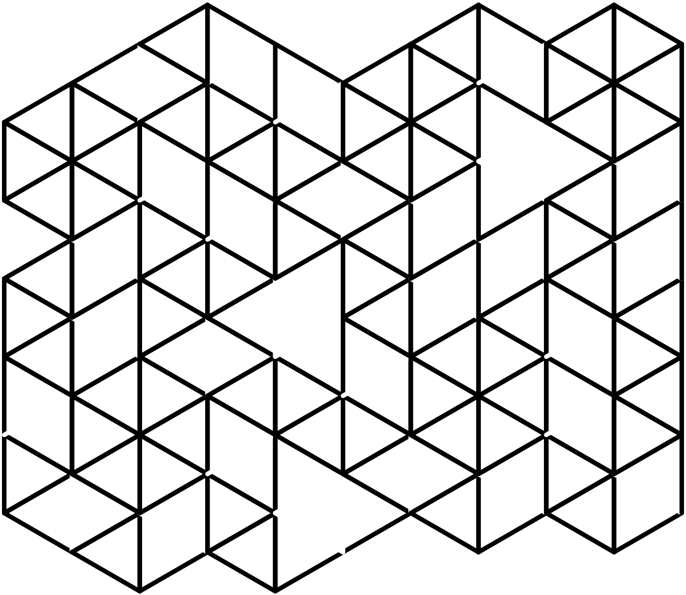

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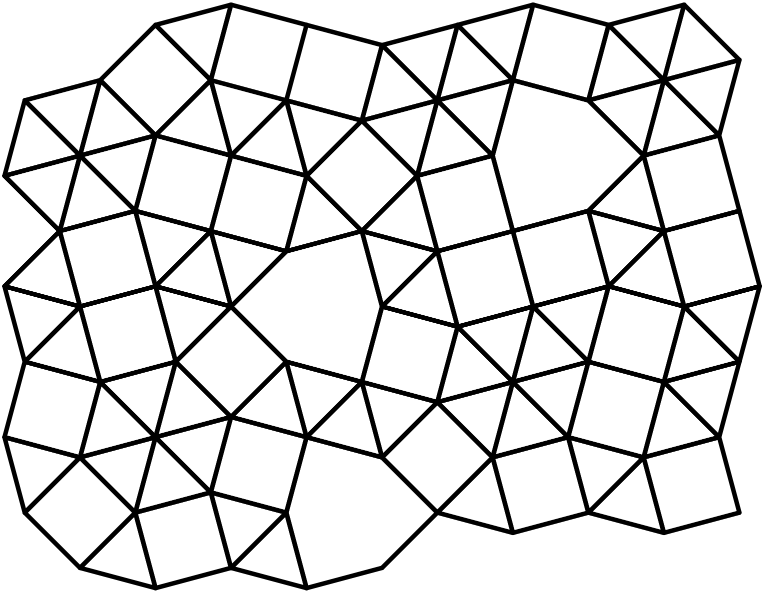

Outlook

- Tilings in finite geometry (determinant-like formulas)

- Rectangle-Triangle tilings are “nice” (correspond to solvable lattice models)

- Hidden combinatorics a la Razumov and Stroganov (talk by Di Francesco)

Thank You