Negative hard-squares

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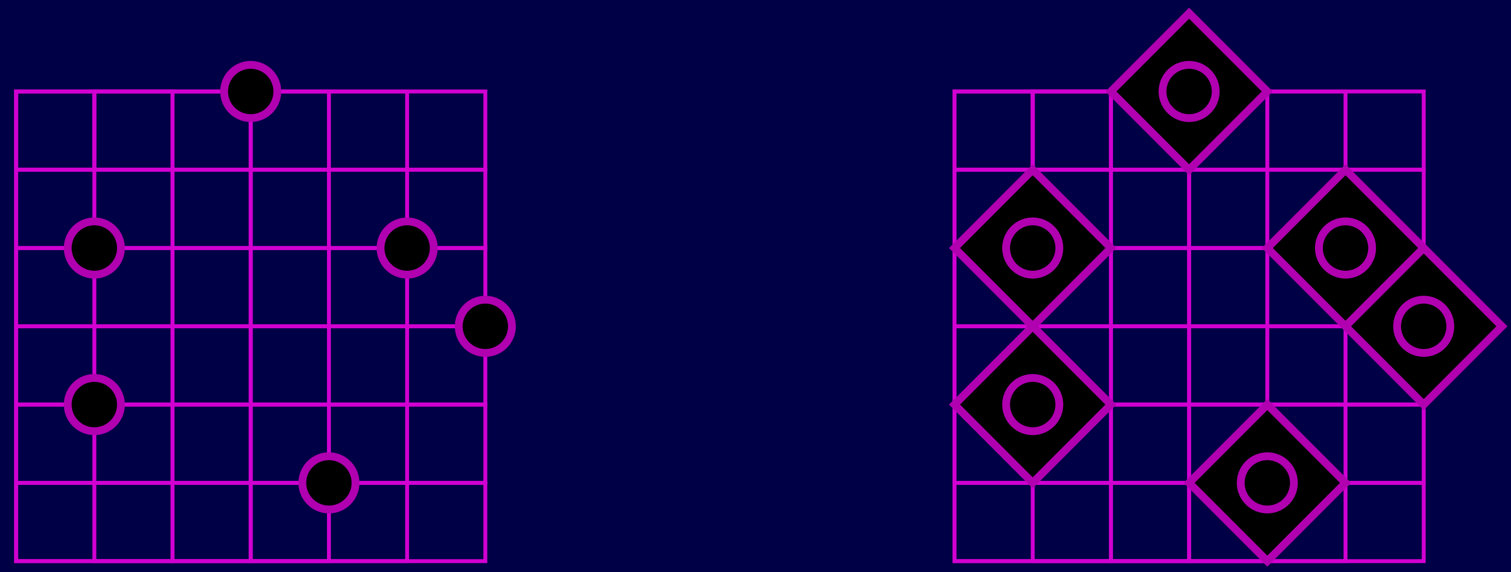
The hard-square model



An independent set of the 7×7 grid, or a hard-square configuration.

Question: find

$$Z_N(u) = \sum_I u^{|I|},$$

where the sum runs over all independent sets of the $N \times N$ grid.

The partition function of the hard-square model at activity u.

Hard squares are indeed hard

Question: find

$$Z_N(u) = \sum_I u^{|I|},$$

where the sum runs over all independent sets of the $N \times N$ grid.

Unsolved!

- even for $u = 1$,
- even in the thermodynamic limit:

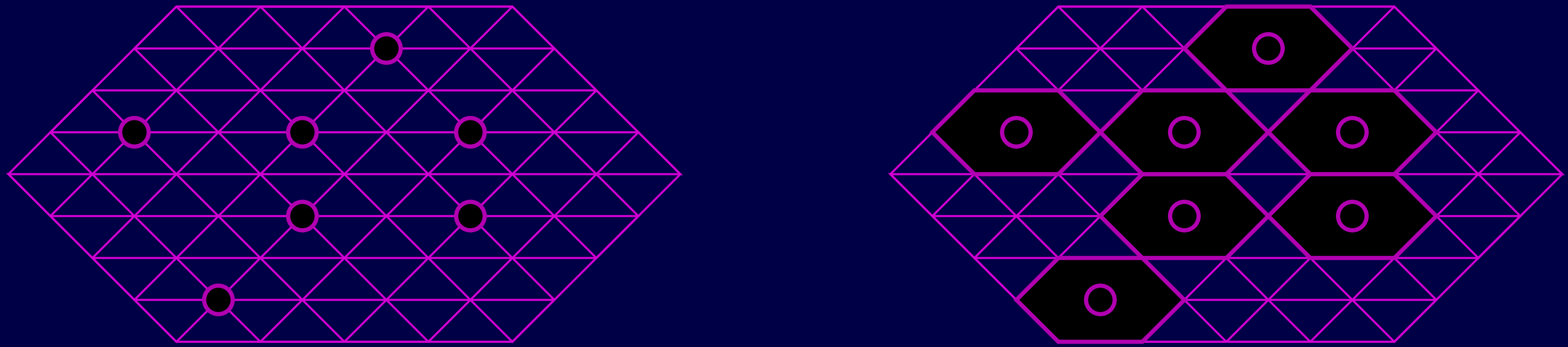
$$\lim_N Z_N(u)^{1/N^2} = ?$$

- even for $u = 1$ and in the thermodynamic limit:

$$\lim_N Z_N(1)^{1/N^2} \sim 1.503048...$$

is the (mysterious) hard-square constant.

Hard hexagons are slightly less hard



Question: find

$$Z_N(u) = \sum_I u^{|I|},$$

where the sum runs over all independent sets of the $N \times N \times N$ hexagon.

Solved by Baxter in the thermodynamic limit:

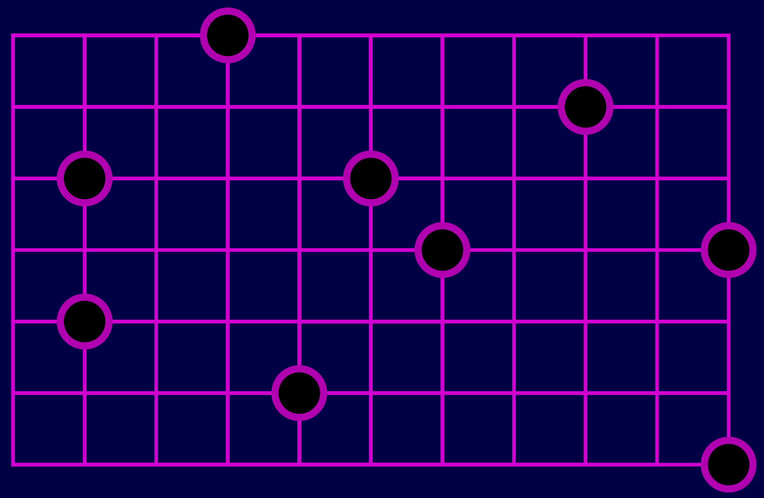
$$\lim_N Z_N(u)^{1/N^2} = \ldots$$

[Baxter, 1980]

Some conjectures of

Fendley, Schoutens and van Eerten (2004)

Hard squares at activity -1



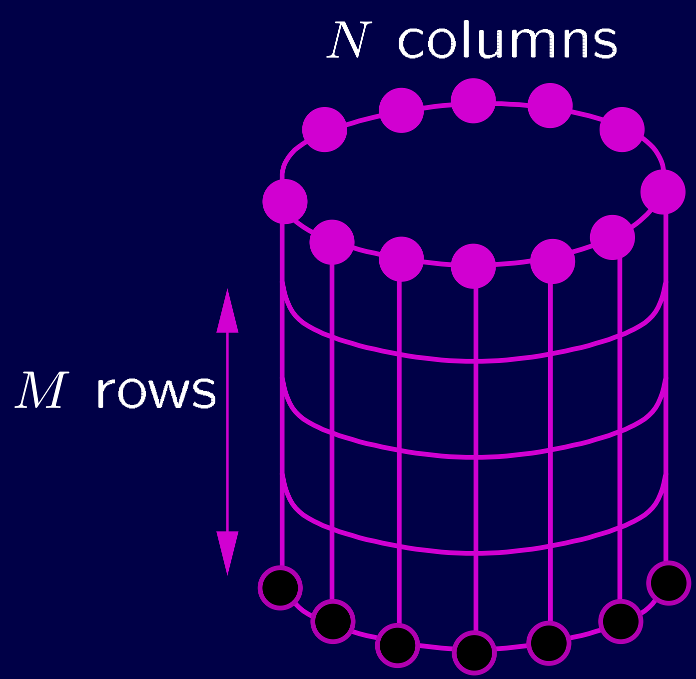
Question: find

$$Z_{M,N} := \sum_I (-1)^{|I|},$$

where the sum runs over all independent sets of the $M \times N$ grid.

The alternating number of independent sets.

Independent sets on a torus



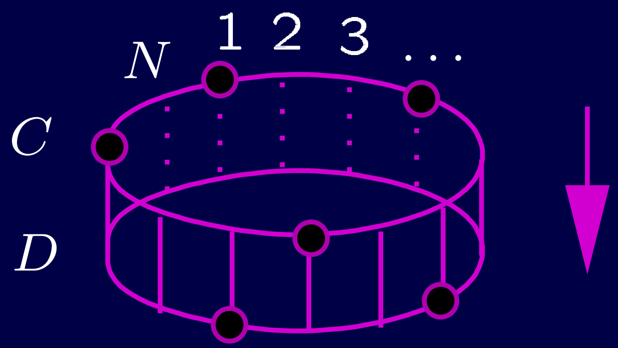
Conjecture [Fendley et al. 04]

For a torus with M rows and N columns, with M and N coprime,

$$Z_t(M,N) := \sum_I (-1)^{|I|} = 1.$$

$\Longrightarrow$ Not true for a rectangle (with open boundary conditions), nor for a cylinder.

Transfer matrices



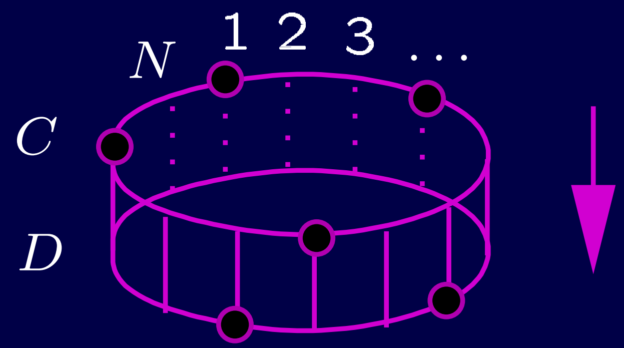
For two independent sets C and D on the N-point circle:

$$\mathbb{T}_N(C,D) = \begin{cases} (-1)^{|D|} & \text{if } C \cap D = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Example: For $N = 4$, the independent sets on the 4-point circle are $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{2,4\}$, and

$$\mathbb{T}_4 = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

Transfer matrices



For two independent sets C and D on the N-point circle:

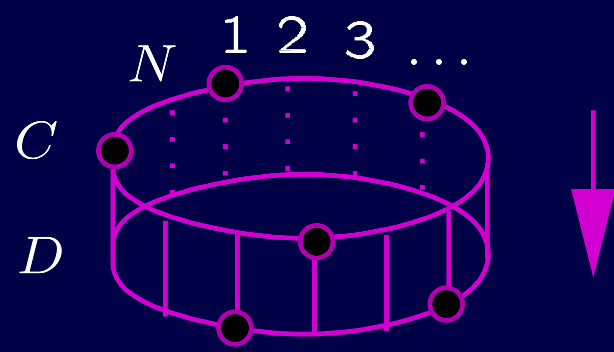
$$\mathbb{T}_N(C,D) = \begin{cases} (-1)^{|D|} & \text{if } C \cap D = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then:

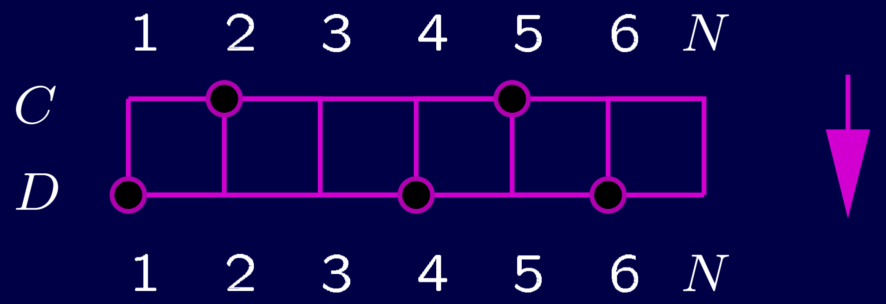
- $(\mathbb{T}_N)^M(C,C)$ is the alternating number of independent sets of the $M \times N$ torus, with boundary condition C on row 1
- $\mathrm{tr}\,(\mathbb{T}_N)^M$ is the alternating number $Z_t(M,N)$ of independent sets of the $M \times N$ torus

Conjectures on transfer matrices [Fendley et al. 04]

- For every N, all the eigenvalues of the matrix $\mathbb{T}_N$ are roots of unity



- Same conjecture for the transfer matrix $\mathbb{O}_N$ between two N-point segments



Ex.: The characteristic polynomial of the matrix $\mathbb{O}_N$

$$P_1(x) = \frac{x^3+1}{x+1}, \qquad P_4(x) = (x-1)(x^3+1)(x^4-1),$$

$$P_2(x) = (x-1)(x^2+1), \qquad P_5(x) = \frac{(x^{10}-1)(x^4+1)}{x+1},$$

$$P_3(x) = (x-1)(x^4+1), \qquad P_6(x) = (x-1)(x^2+1)(x^4-1)(x^{14}-1),$$

$$P_7(x) = \frac{(x^3+1)(x^4+1)(x^{12}-1)(x^{18}-1)}{(x+1)(x^2+1)},$$

$$P_8(x) = (x+1)(x^4+1)^2(x^8+1)(x^{16}-1)(x^{22}-1).$$

Some bad news

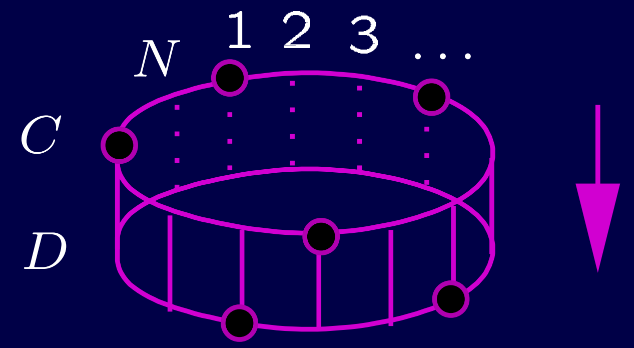
I am not going to prove these conjectures...

... but

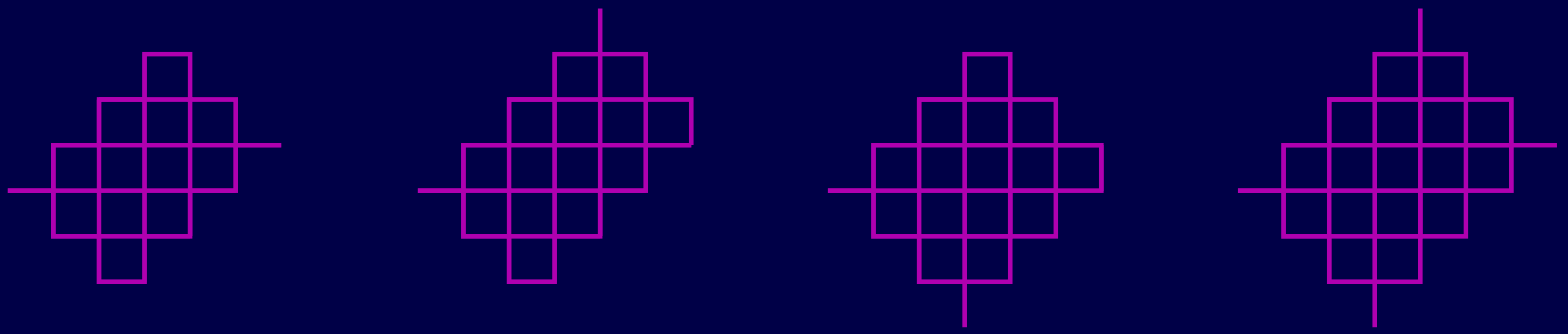
Some good news

(1) Jacob Jonsson has proved some of them:

- $Z_t(M, N) = 1$ if M and N are coprime
- All eigenvalues of the transfer matrix from circle to circle are roots of unity



(2) Similar results hold in greater generality for other shapes



$\mathcal{R}(6,8)$ $\mathcal{R}(6,9)$ $\mathcal{R}(7,8)$ $\mathcal{R}(7,9)$

One typical result

Independent sets of tilted rectangles

Let $\mathcal{R}(M,N)$ be the subgraph of $\mathbb{Z}^2$ induced by the points (x,y) satisfying

$$y \le x \le y+M-1 \quad \text{and} \quad -y \le x \le -y+N-1.$$

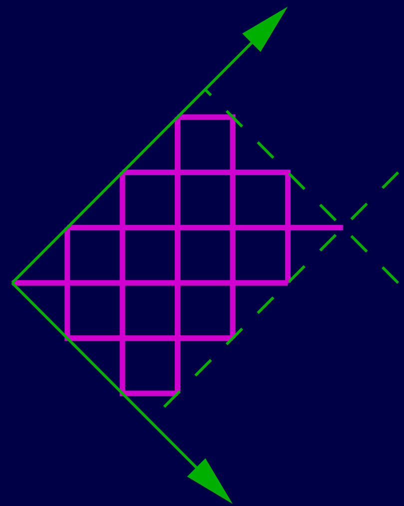
The alternating number of independent sets on $\mathcal{R}(M,N)$ is

$$Z_{\mathcal{R}}(M,N) = \sum_I (-1)^{|I|}.$$

Theorem [MBM-Linusson-Nevo 06]

- If $M \equiv_3 1$ or $N \equiv_3 1$, then $Z_{\mathcal{R}}(M,N) = 0$.
- Otherwise $Z_{\mathcal{R}}(M,N) = (-1)^{mn}$,

with $m = \lceil M/3 \rceil$ and $n = \lceil N/3 \rceil$.



$\mathcal{R}(6,8)$

Proof: Involutions on independent sets

Aim: find an involution σ such that

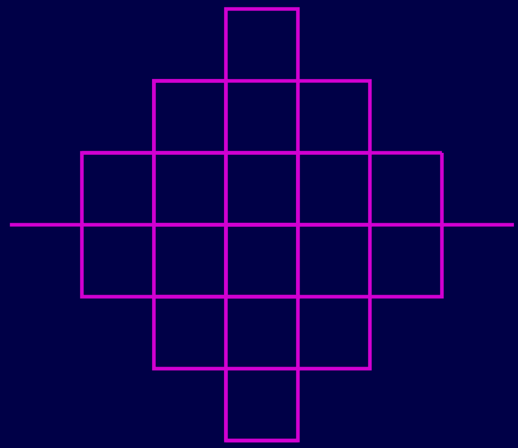
- there are few fixed points (ideally, 0 or 1)
- if $I \neq \sigma(I)$, then I and $\sigma(I)$ differ at only one vertex. The sets I and $\sigma(I)$ are said to be matched together.

$\Rightarrow \sigma$ is sign-reversing on the matched sets.

$$Z = \sum_{I} (-1)^{|I|} = \sum_{I \text{ unmatched}} (-1)^{|I|}$$

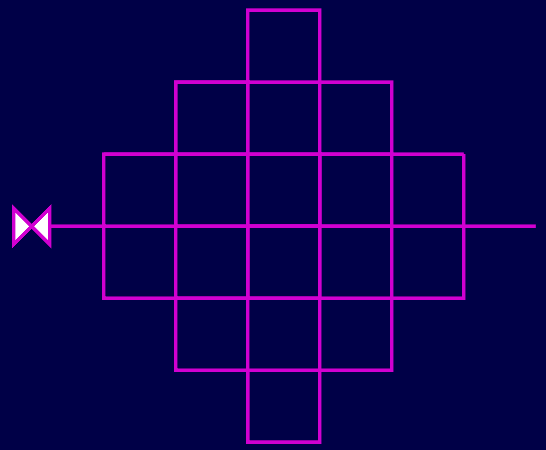
How to match independent sets

$\mathcal{R}(8,8)$



How to match independent sets

$\mathcal{R}(8,8)$



How to match independent sets

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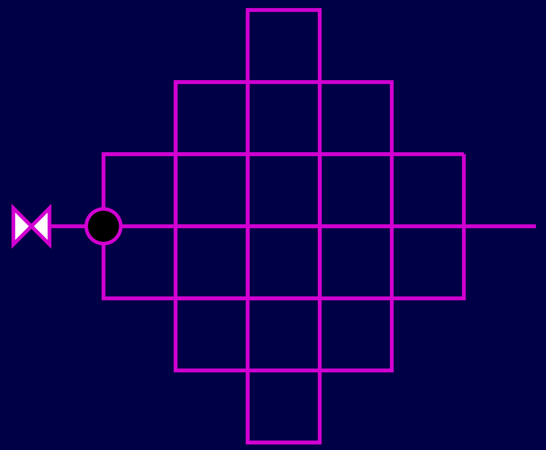

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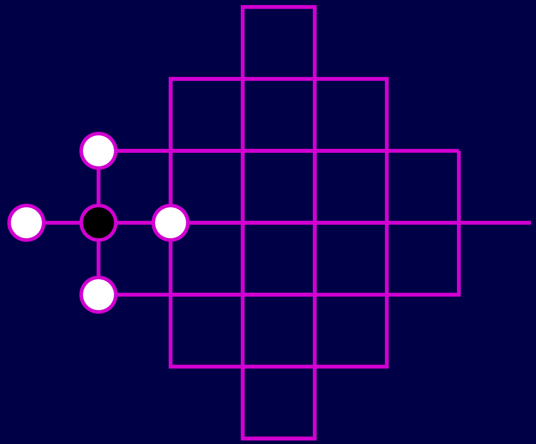

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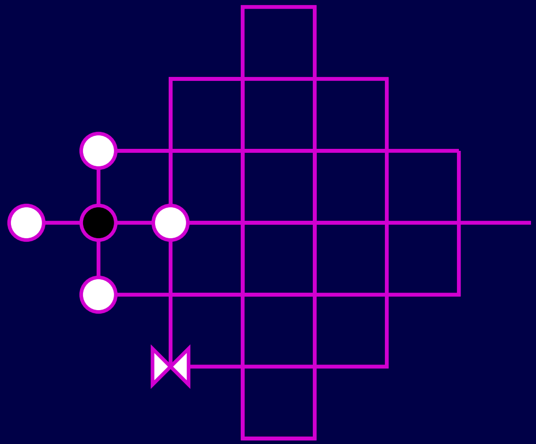

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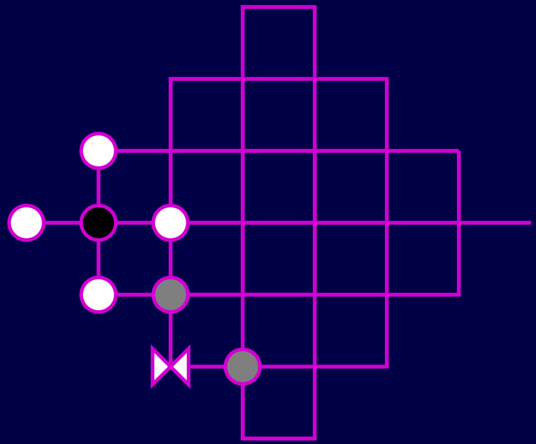

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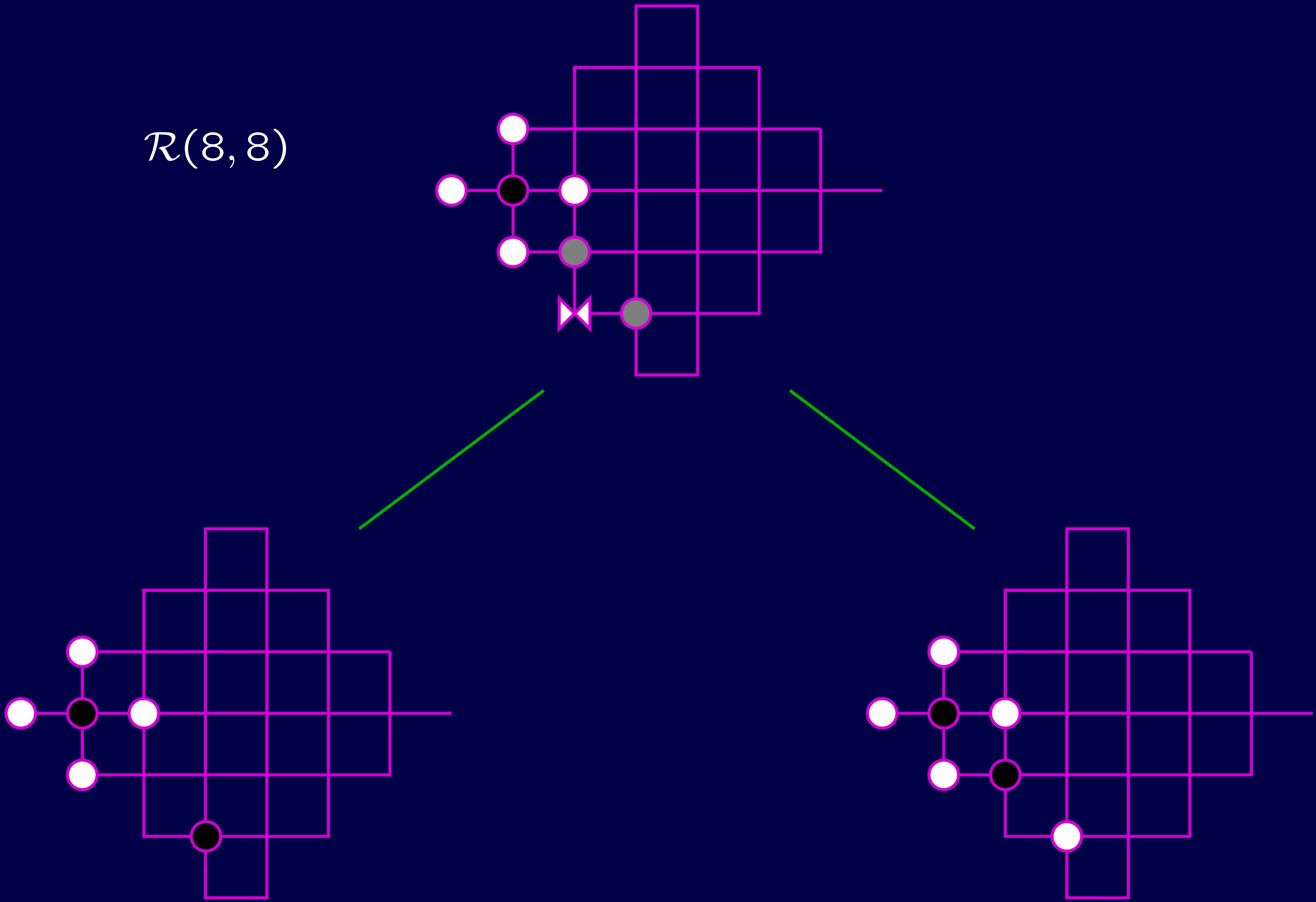

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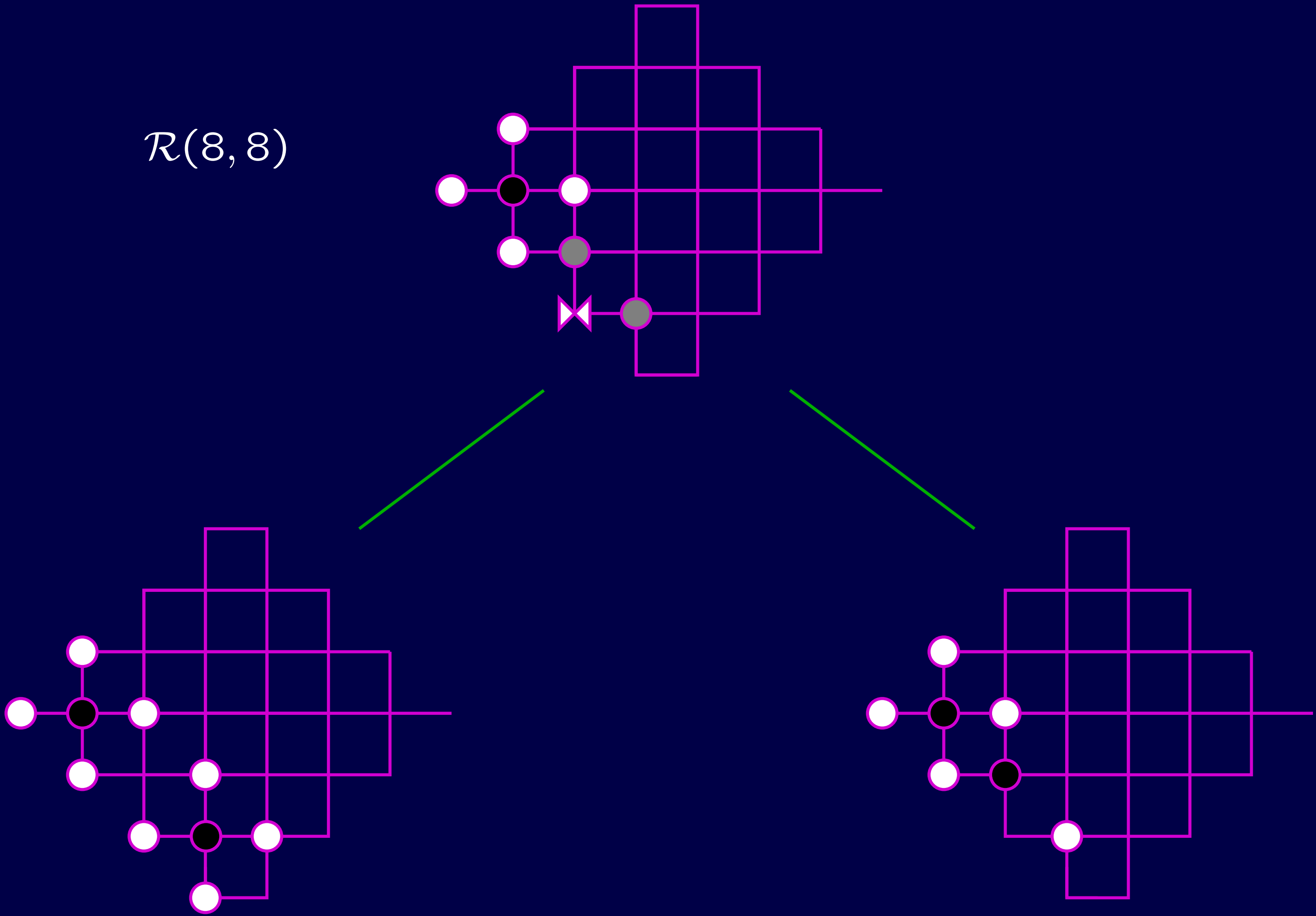

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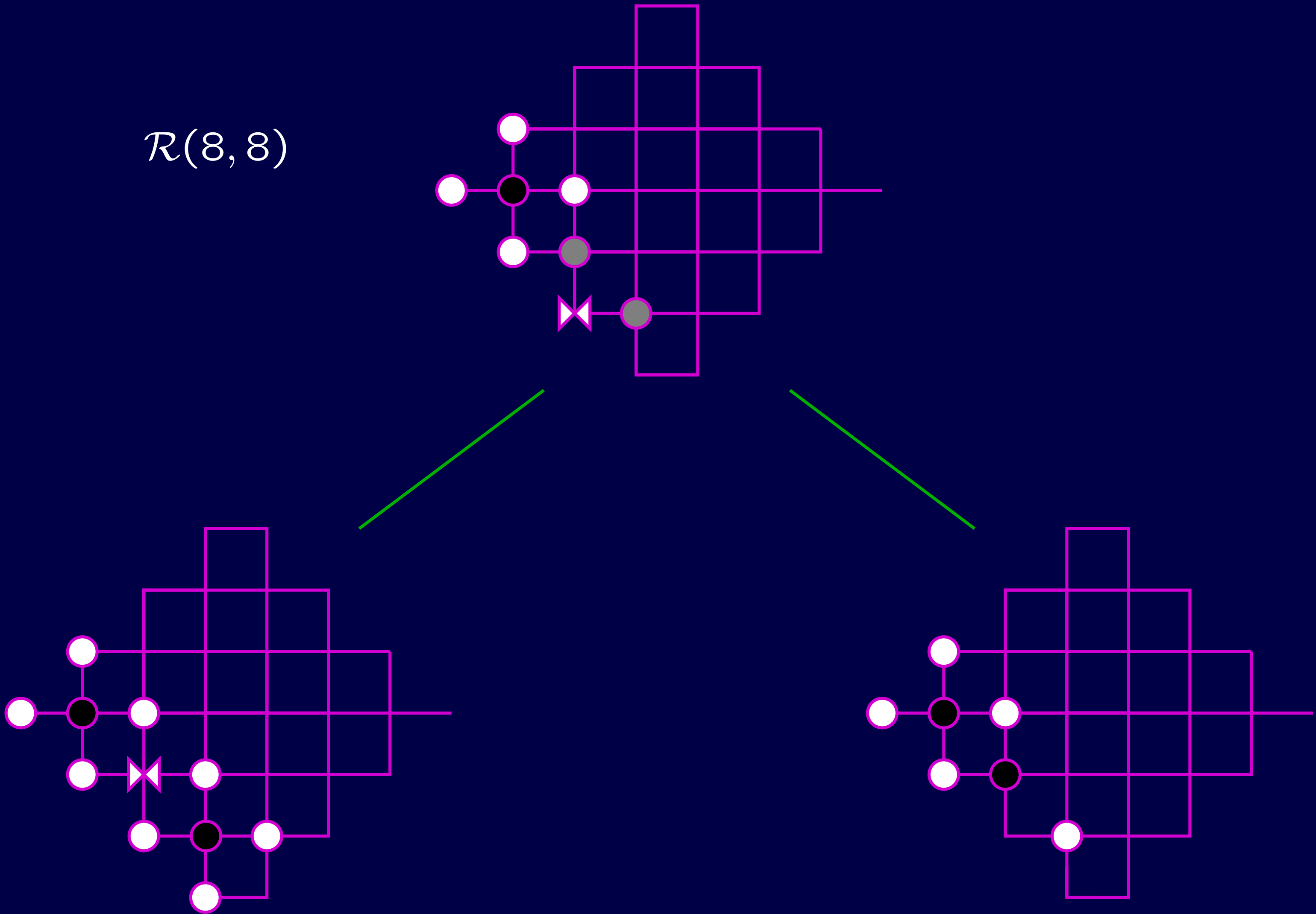
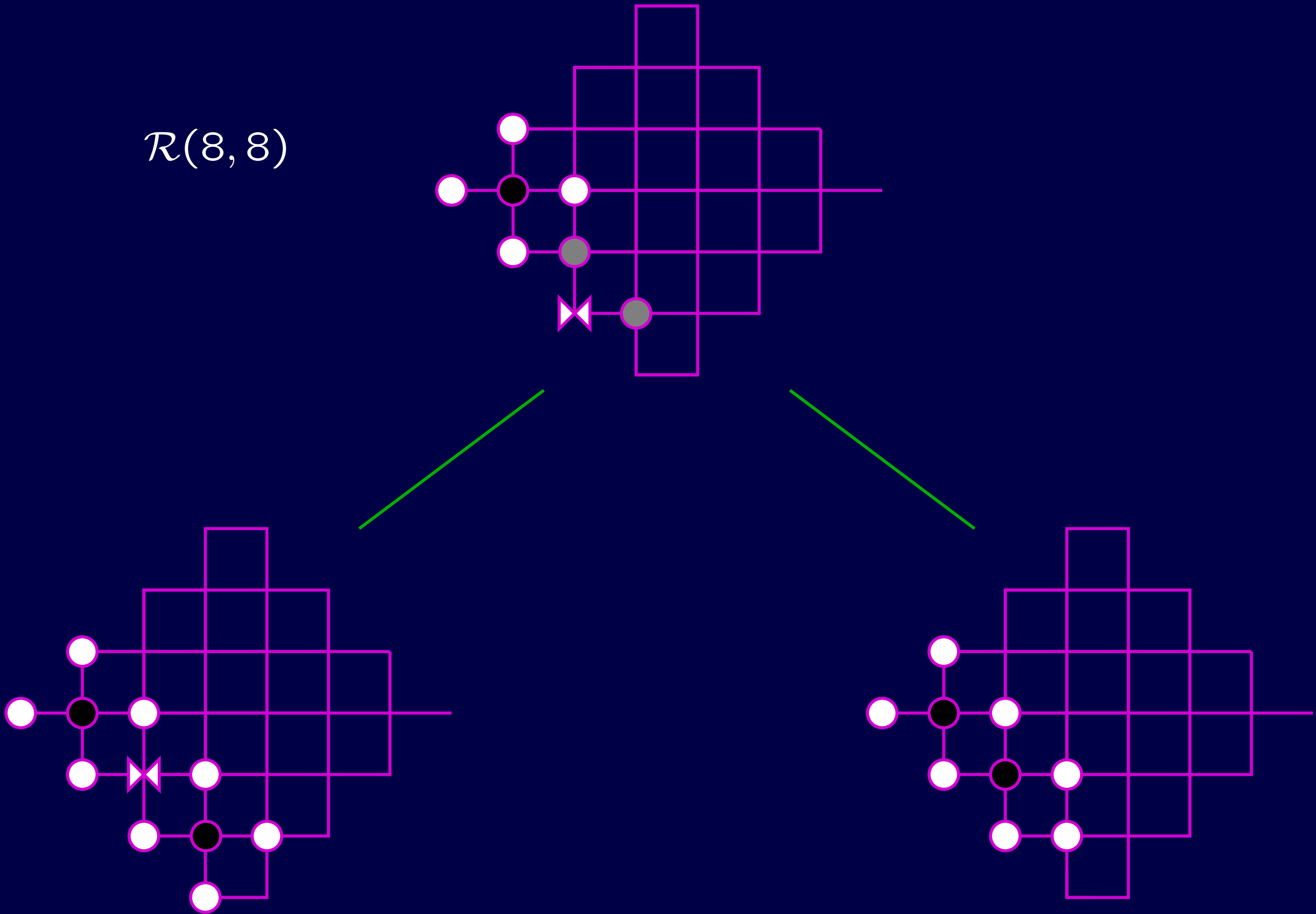


Perfect matching

How to match independent sets

$\mathcal{R}(8,8)$



Perfect matching

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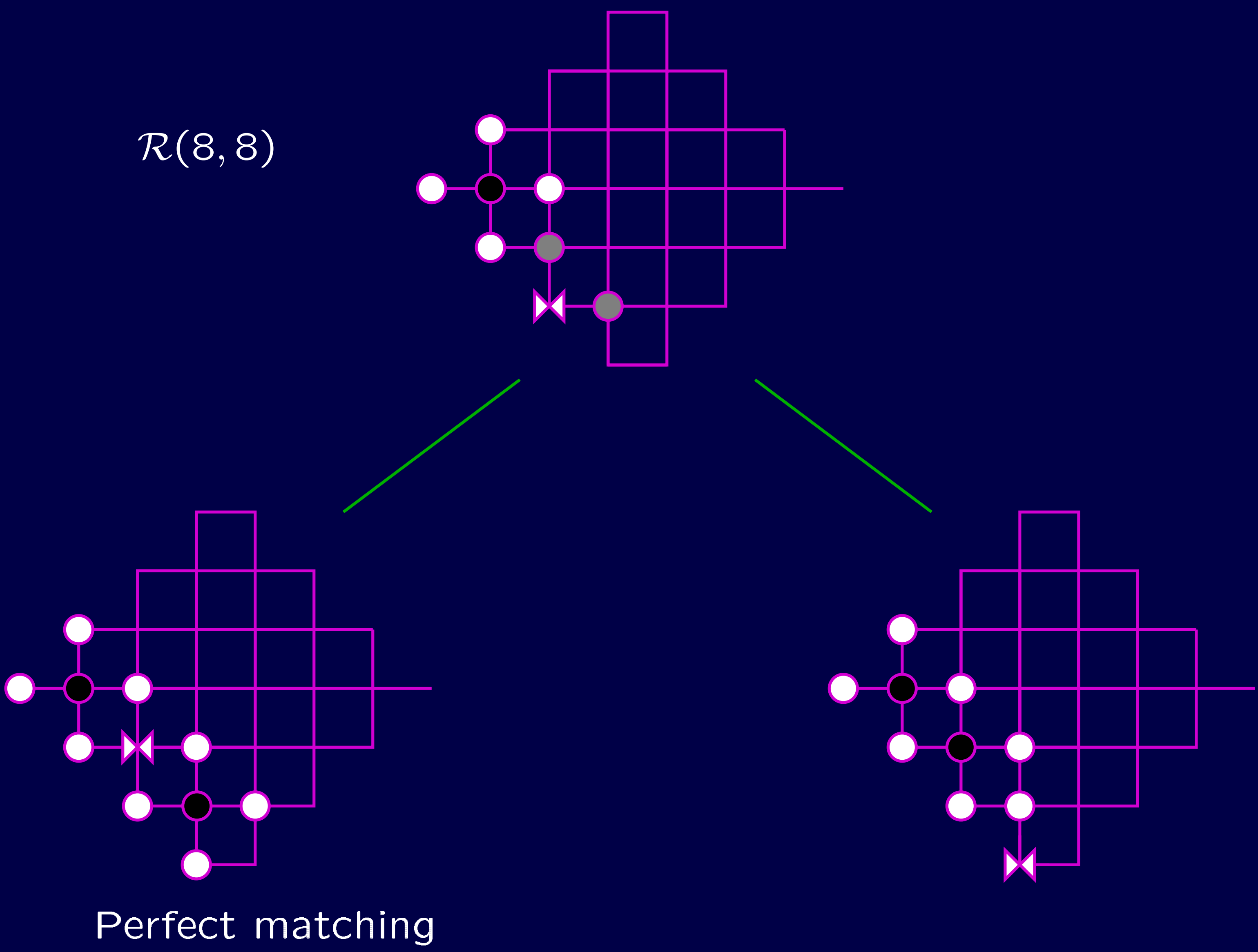

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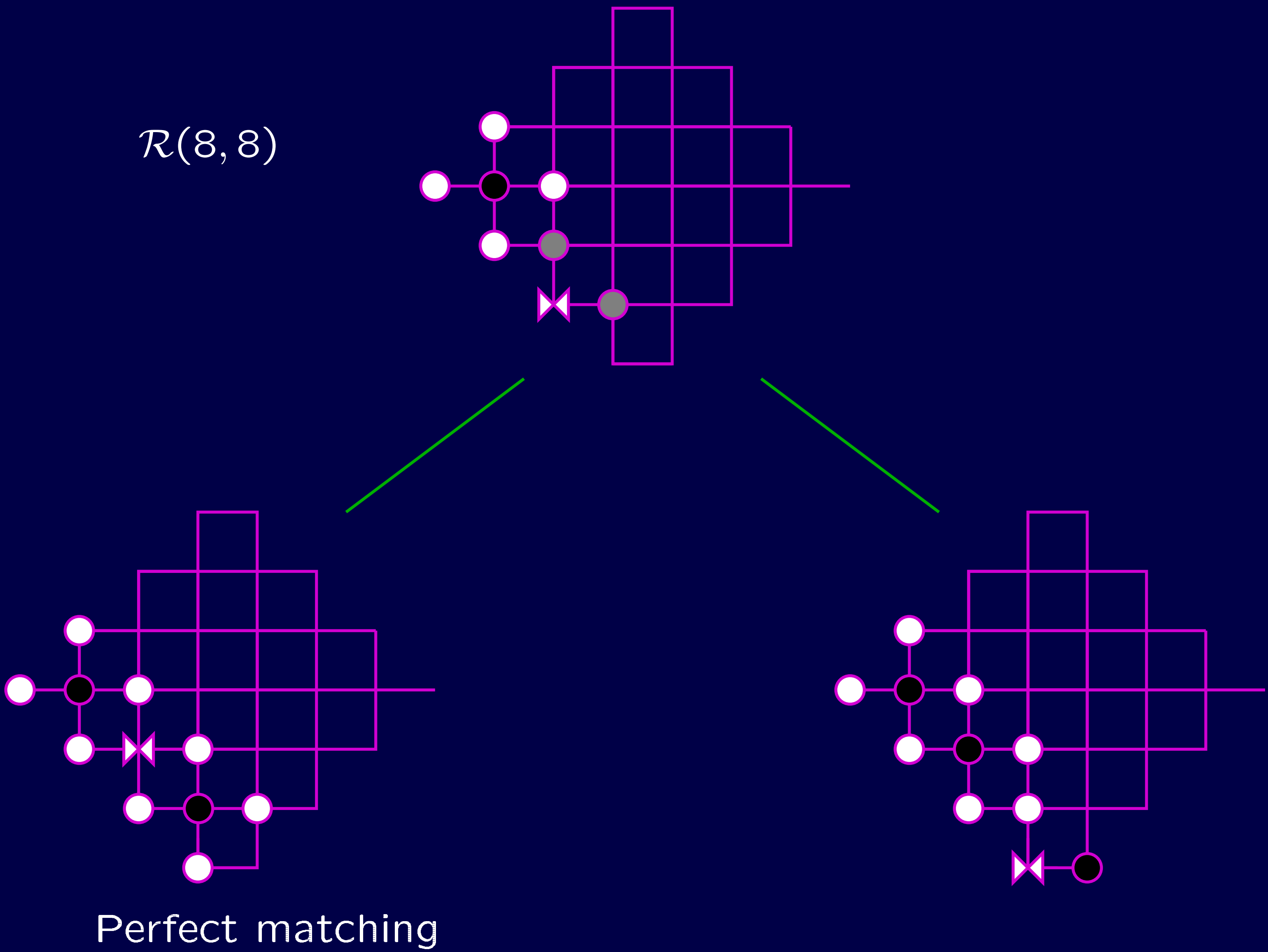
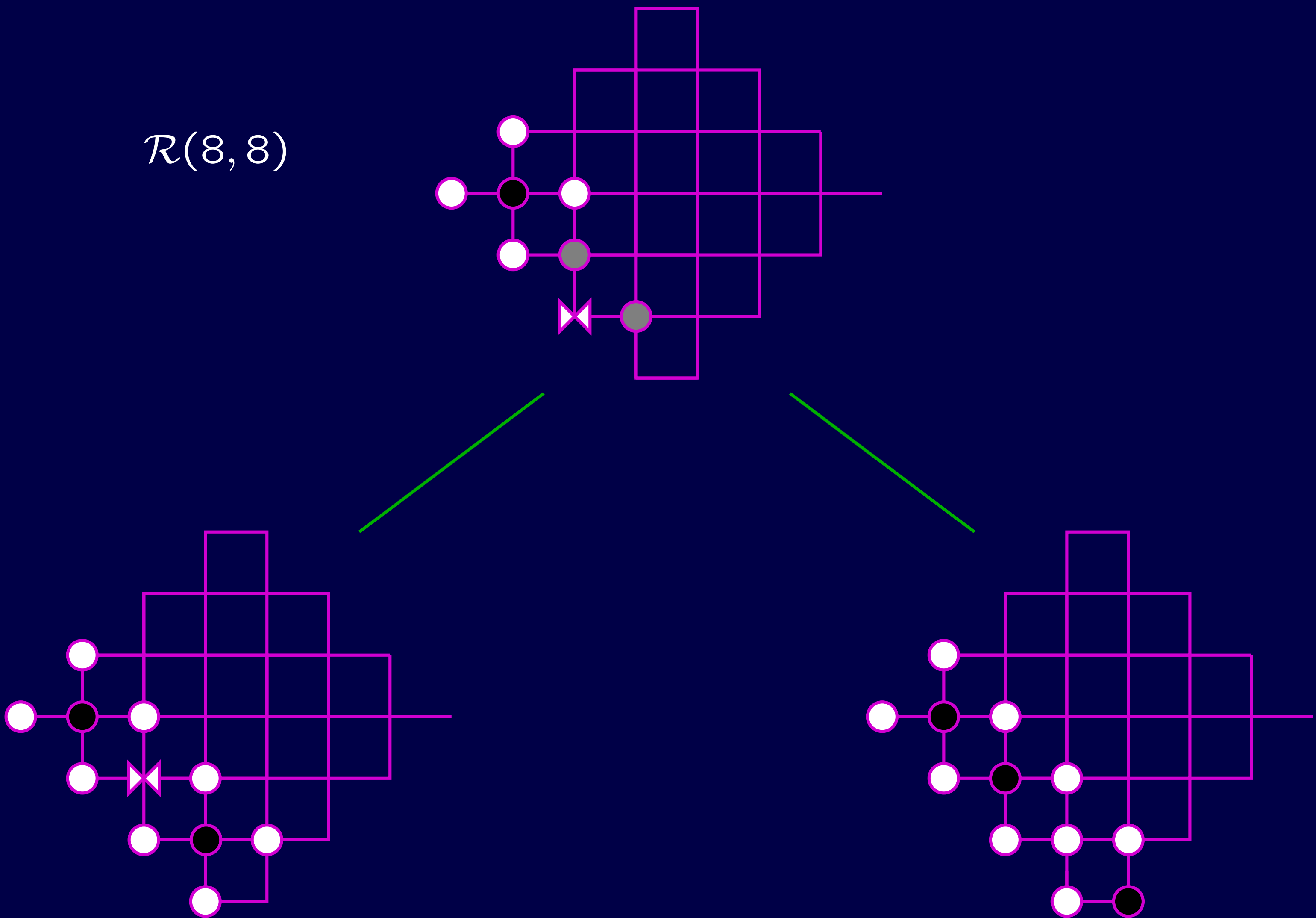


How to match independent sets



Perfect matching

$\mathcal{R}(8,5) \quad \Rightarrow$ Recursion

Independent sets of tilted rectangles

Let $\mathcal{R}(M,N)$ be the subgraph of $\mathbb{Z}^2$ induced by the points (x,y) satisfying

$$y \le x \le y + M - 1 \quad \text{and} \quad -y \le x \le -y + N - 1.$$

The alternating number of independent sets on $\mathcal{R}(M,N)$ is

$$Z_{\mathcal{R}}(M,N) = \sum_I (-1)^{|I|}.$$

Theorem [BM-L-N 06]

- If $M \equiv_3 1$ or $N \equiv_3 1$, then $Z_{\mathcal{R}}(M,N) = 0$.
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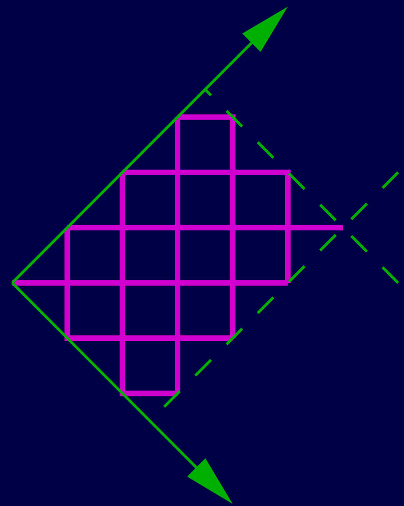
with $m = \lceil M/3 \rceil$ and $n = \lceil N/3 \rceil$.

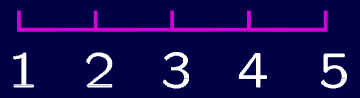


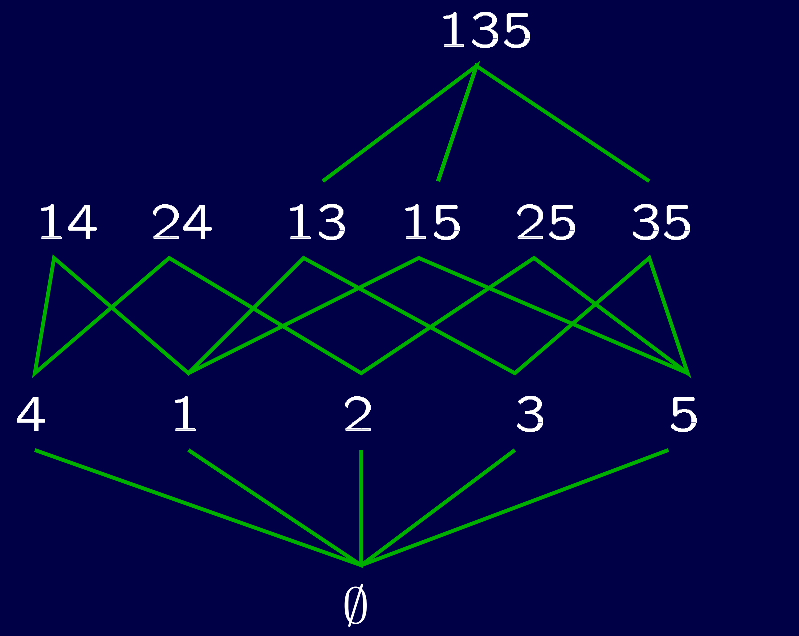
$\mathcal{R}(6,8)$

Bonus: The topology of the independence complex

The topology of the independence complex

- Graph G

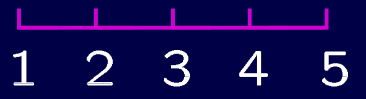




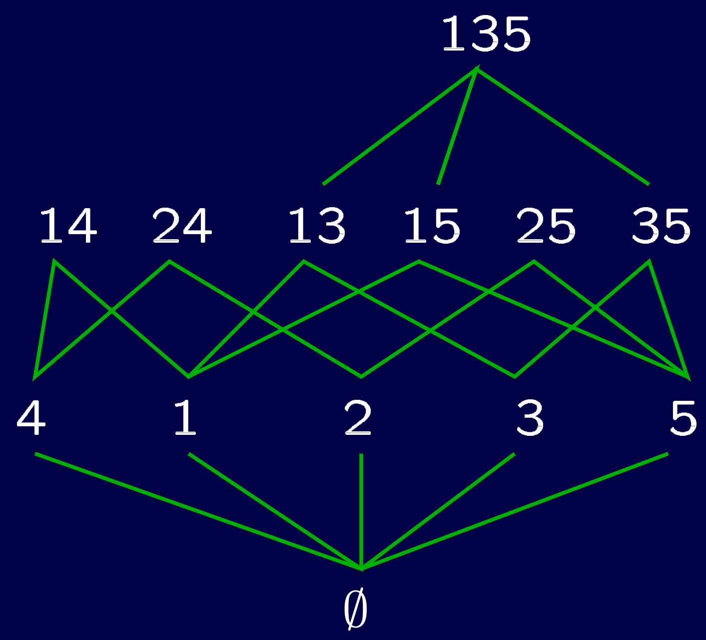
Independence complex $\Sigma(G)$

The topology of the independence complex

- Graph G



$$Z(G) = 1 - 5 + 6 - 1$$



Independence complex $\Sigma(G)$

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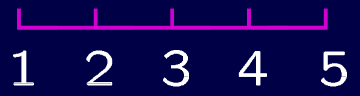
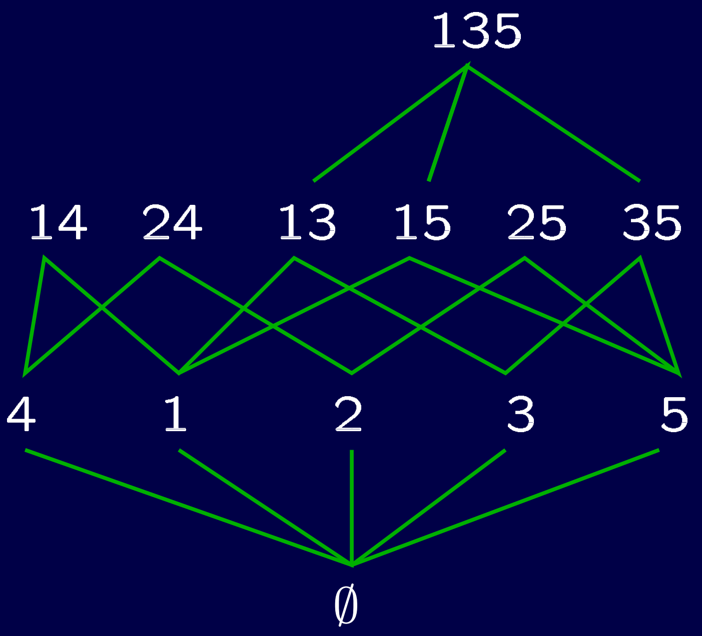


$$Z(G) = 1 - 5 + 6 - 1$$
$$= -\chi(\Sigma(G)) = 1$$

The reduced Euler characteristic of the complex



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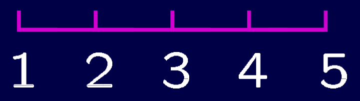

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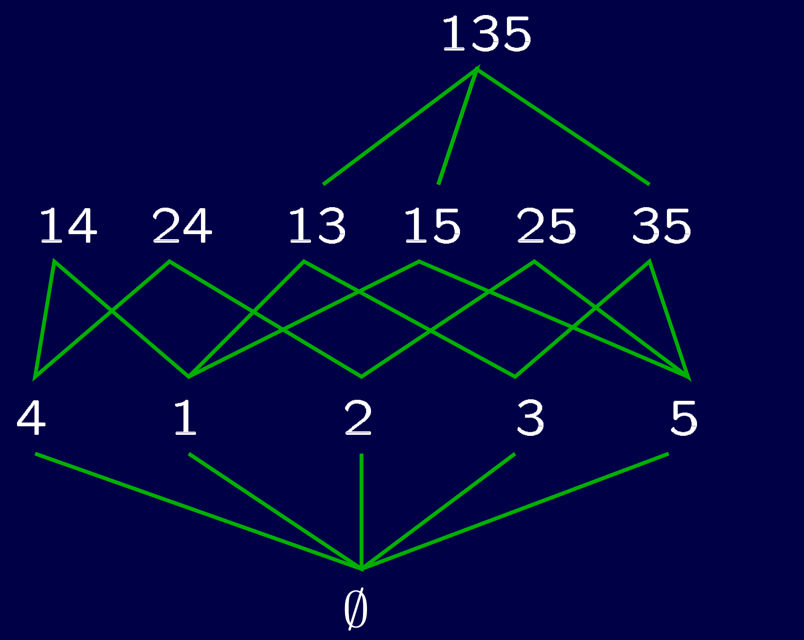
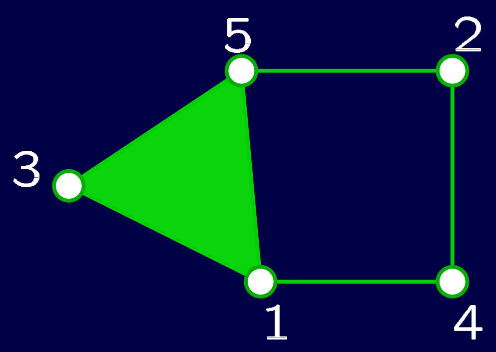


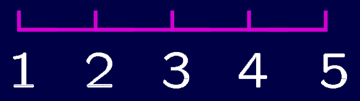
Independence complex $\Sigma(G)$

- Topological realization



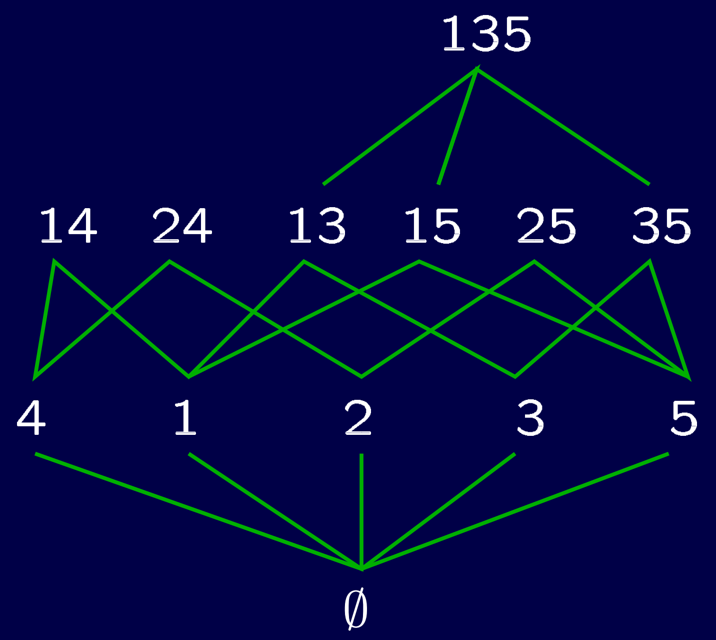
The topology of the independence complex

- Graph G



$$Z(G) = 1 - 5 + 6 - 1 = -\chi(\Sigma(G)) = 1$$

The reduced Euler characteristic of the complex



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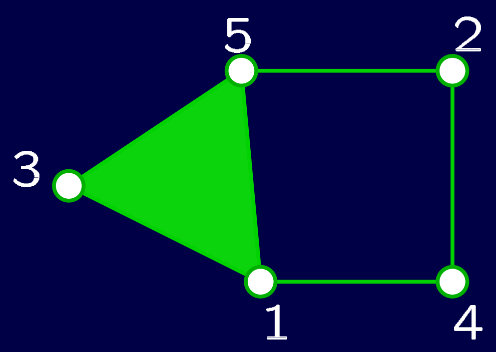


$\simeq$ a circle

$\chi(\Sigma(G)) = -1$

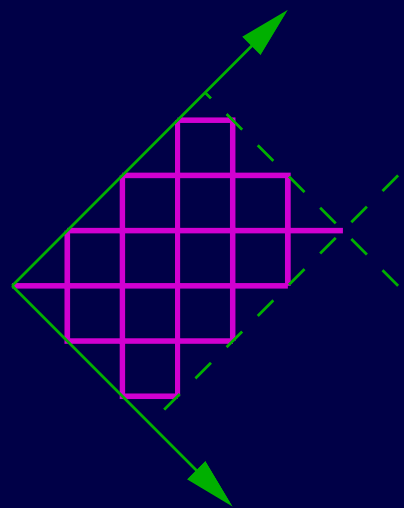
The independence complex of tilted rectangles

Let $\mathcal{R}(M,N)$ be the subgraph of $\mathbb{Z}^2$ induced by the points (x,y) satisfying

$$y \le x \le y + M - 1 \quad \text{and} \quad -y \le x \le -y + N - 1.$$

Theorem

- If $M \equiv_3 1$ or $N \equiv_3 1$, then $\Sigma(\mathcal{R}(M,N))$ is contractible and $Z_{\mathcal{R}}(M,N) = 0$.
- Otherwise, $\Sigma(\mathcal{R}(M,N))$ is homotopy equivalent to a sphere of dimension $mn-1$, and $Z_{\mathcal{R}}(M,N) = (-1)^{mn}$,



$\mathcal{R}(6,8)$

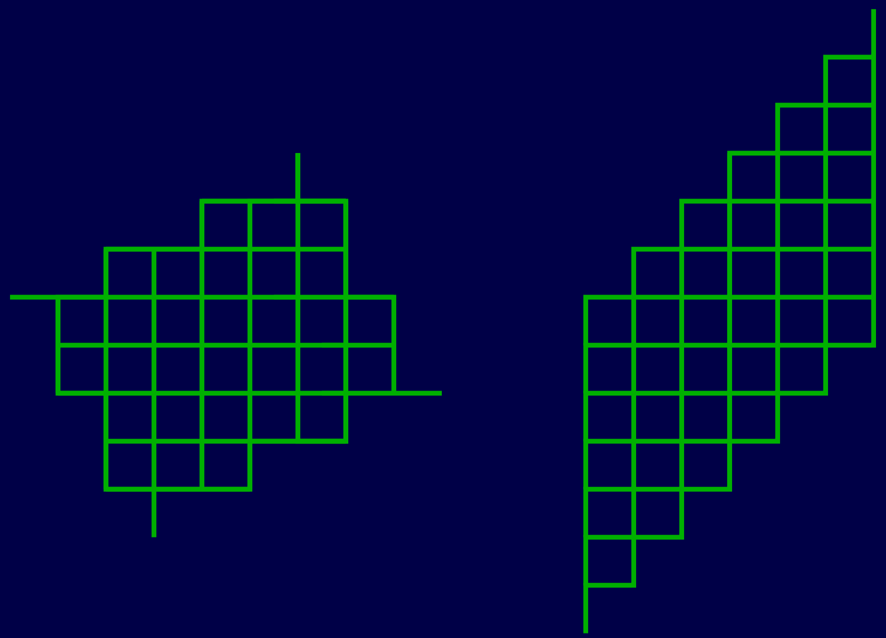
with $m = \lceil M/3 \rceil$ and $n = \lceil N/3 \rceil$.

Proof: Our involution defines a Morse matching of the complex $\Sigma(\mathcal{R}(M,N))$. Discrete Morse theory [Forman 95]

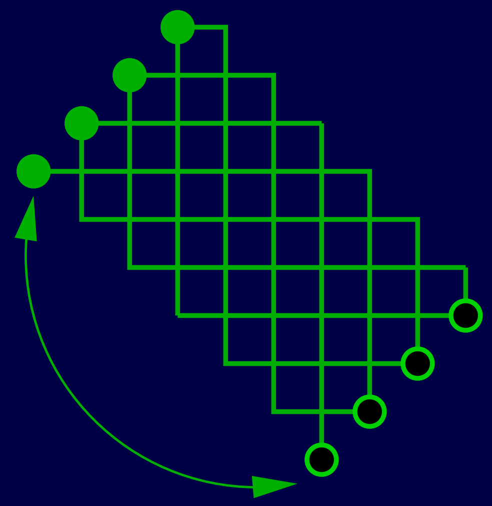
Extension to other shapes

Other shapes

- Shapes with lots of vertices of degree 2

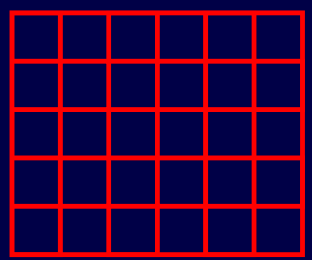


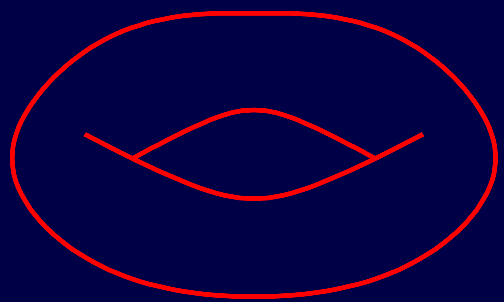
Open boundary conditions



Cylindric boundary conditions

- Does not work for:





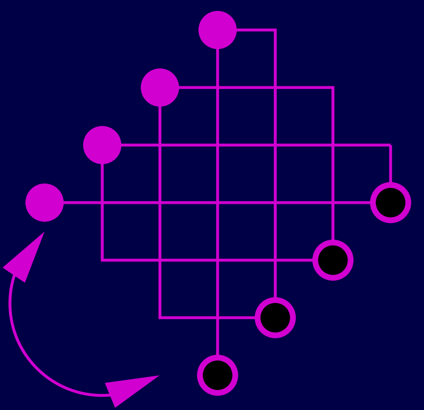
Tilted rectangles with cylindric boundary conditions

For M even, let $\mathcal{R}^c(M,N)$ be obtained by wrapping $\mathcal{R}(M+1,N)$ on a cylinder

Theorem

- If $N \equiv_3 1$, then $\Sigma(\mathcal{R}^c(M,N))$ is contractible and $Z^c_{\mathcal{R}}(M,N) = 0$.
- Otherwise,
  - If $M \equiv_3 0$, then $\Sigma(\mathcal{R}^c(M,N))$ is homotopy equivalent to a wedge of 2^n spheres of dimension $mn-1$, and $Z^c_{\mathcal{R}}(M,N) = 2^n$.
  - If $M \equiv_3 1$ or 2 then $\Sigma(\mathcal{R}^c(M,N))$ is homotopy equivalent to a single sphere of dimension $mn-1$, and $Z^c_{\mathcal{R}}(M,N) = (-1)^n$.

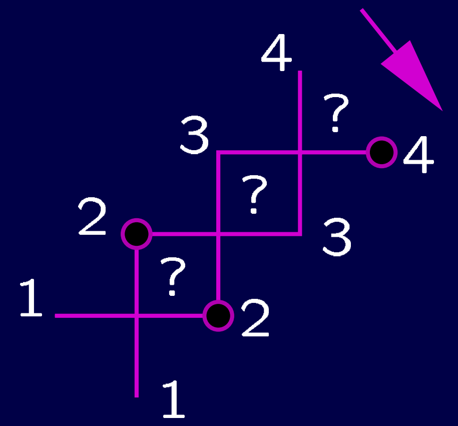
with $m = \lfloor \frac{M+1}{3} \rfloor$ and $n = \lceil N/3 \rceil$.



$\mathcal{R}^c(6,8)$

Diagonal transfer matrices

$$\mathrm{tr}(\mathbb{R}_N)^M = Z^c_{\mathcal{R}}(M,N)$$



$\mathbb{R}_7$

$$\mathbb{R}_N(C,D) = \begin{cases} (-1)^{|D|} & \text{if } {}^cC \cap^c D \cap^c (C-1) \cap^c (D-1) = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem [BM-L-N]

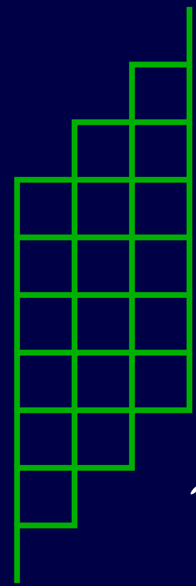
- If $N \equiv_3 1$, then $\mathbb{R}_N$ is nilpotent (all its eigenvalues are 0).
- Otherwise, $\mathbb{R}_N$ has eigenvalues:
  - 0 with multiplicity $d_N - 2^n$,
  - 1 with multiplicity $(2^n + 2(-1)^n)/3$,
  - j and j^2 with multiplicity $(2^n - (-1)^n)/3$,

where $d_N := 2^{\lceil N/2 \rceil}$, $n = \lceil N/3 \rceil$, and $j = e^{2i\pi/3}$.

Independent sets of parallelograms $\mathcal{P}(K,N)$

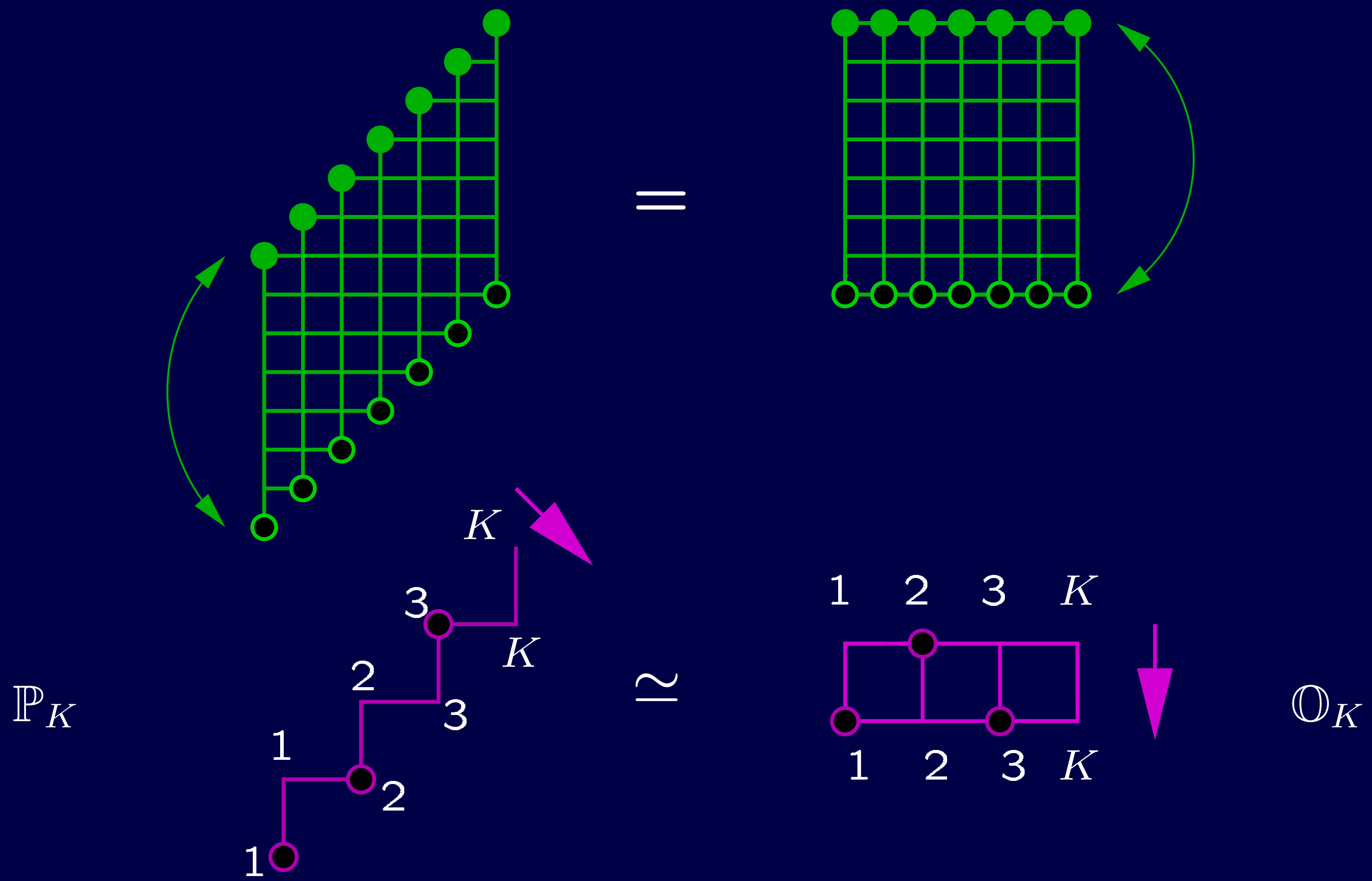
Theorem

- If $K \equiv_3 1$, then
  - if $N \equiv_3 1$ then $Z_{\mathcal{P}}(K,N) = 0$,
  - otherwise, $Z_{\mathcal{P}}(K,N) = (-1)^n$ with $n = \lceil N/3 \rceil$.
- If $K \equiv_3 2$, write $N = 2qK + r$, with $0 \le r \le 2K - 1$.
  - If $r \equiv_3 1, 2$, then $Z_{\mathcal{P}}(K,N) = 0$,
  - otherwise $Z_{\mathcal{P}}(K,N) = 1$.
- If $K \equiv_3 0$, write $N = 2q(K+1)+r$ with $0 \le r \le 2K+1$.
  - If $r \equiv_3 0$ with $r \ge 1$, or $r \equiv_3 1$ with $r \le 2K$, then $Z_{\mathcal{P}}(K,N) = 0$,
  - Otherwise, $Z_{\mathcal{P}}(K,N) = 1$.



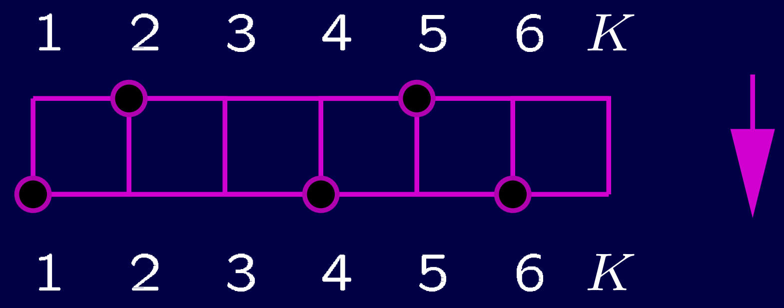
$\mathcal{P}(4,8)$

From parallelograms to rectangles



The matrices $\mathbb{P}_K$ and $\mathbb{O}_K$ have the same spectrum, except from the multiplicity of the null eigenvalue

Some eigenvalues of the ordinary transfer matrix $\mathbb{O}_K$



Theorem [BM-L-N]

- If $K \equiv_3 1$, then $e^{i\pi/3}$ and $e^{-i\pi/3}$ are eigenvalues of $\mathbb{O}_K$.
- If $K \equiv_3 2$, then all the $2K$th roots of unity, except maybe -1, are eigenvalues of $\mathbb{O}_K$.
- If $K \equiv_3 0$, then all the $(2K+2)$th roots of unity, except maybe -1 and, if K is odd, $\pm i$, are eigenvalues of $\mathbb{O}_K$.

Perspectives

(1) Range of application

- Other shapes?

- Fixed boundary conditions $\Rightarrow$ more eigenvalues of the transfer matrix $\mathbb{O}_K$?

(2) Comparison with J. Jonsson's methods

(3) JJ's conjecture: for the ordinary cylinder formed of K circles of N points each, with N odd,

$$Z_{\mathcal{C}}(K,N) = \begin{cases} -2 & \text{if } N \equiv_6 3 \text{ and } K \equiv_3 1 \\ 1 & \text{otherwise} \end{cases}$$