Osculating Paths and Oscillating Tableaux

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Full details and further results in **math.CO/0701755**

Main Result

- There are bijections between certain tuples of *osculating paths* and certain *generalized oscillating tableaux*.

Motivation

- Generalize well-known bijections between certain tuples of *nonintersecting paths* and *semistandard Young tableaux*.

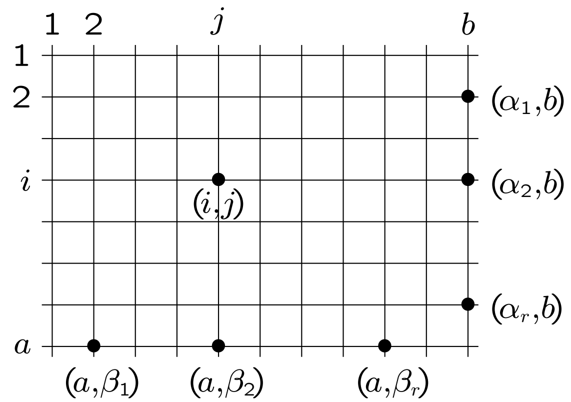
- Improve understanding of combinatorics of *alternating sign matrices*, e.g.,
  - explain combinatorially the appearance of determinants in enumeration formulae derivations
  - possibly find correspondences with certain *plane partitions*.

- Possibly improve understanding of *osculating walkers*.

Osculating Paths

Consider

- an a by b rectangle of lattice points with
  - rows labeled 1 to a from top to bottom
  - columns labeled 1 to b from left to right
  - the point in row i and column j labeled (i, j)
- r points on the lower boundary (a, β_1), ..., (a, β_r)
- r points on the right boundary (α_1, b), ..., (α_r, b)

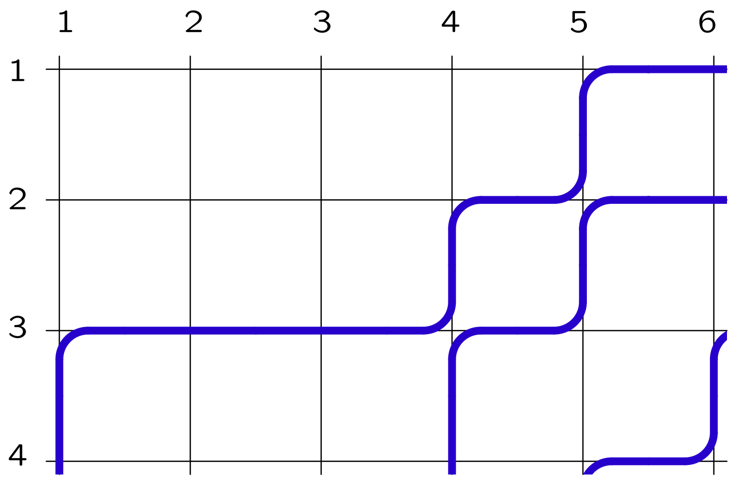


Denote $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ and $\beta = \{\beta_1, \ldots, \beta_r\}$.

Now let $\mathrm{OP}(a, b, \alpha, \beta)$ be the set of all r-tuples of paths in which

- The k-th path of a tuple starts at (a, β_k) and ends at (α_k, b)
- Each path of a tuple can take only unit steps upwards or rightwards
- Different paths within a tuple are allowed to share lattice points, but not to cross or share lattice edges, i.e., the paths are *osculating*.

Example of an element of $OP(4, 6, \{1, 2, 3\}, \{1, 4, 5\})$:



Alternating Sign Matrices

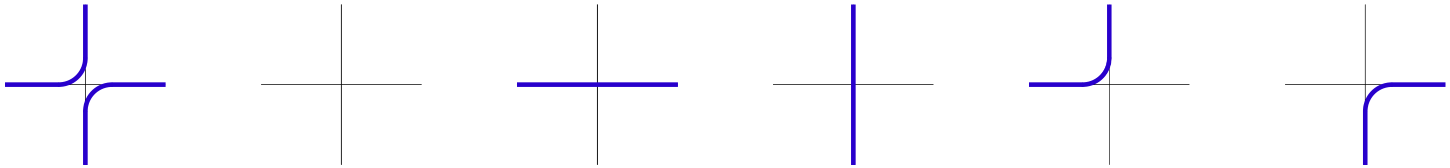
Define

$$\mathrm{ASM}(a,b,\alpha,\beta) := \{\, A \mid$$

- A is an $a\times b$ matrix with all entries in $\{-1,0,1\}$
- along each row and column of A the nonzero entries, if there are any, alternate in sign starting with a 1
- $\sum_{j=1}^{b} A_{ij} = \delta_{i\in\alpha}\,,\quad i=1,\ldots,a$
- $\sum_{i=1}^{a} A_{ij} = \delta_{j\in\beta}\,,\quad j=1,\ldots,b\,\}$

Therefore $\mathrm{ASM}(n,n,\{1,\ldots,n\},\{1,\ldots,n\}) = \{\text{standard } n\times n \text{ ASMs}\}$.

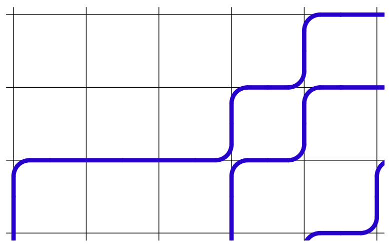
ASM – Osculating Path Correspondence



$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	0	0	0	-1	1

gives a bijection between OP(a, b, α, β) and ASM(a, b, α, β)

Example:



$$\longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Known Enumeration Formulae

- Standard ASMs: $|\mathrm{OP}(n,n,[n],[n])| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$ *(Zeilberger 1996, Kuperberg 1996)*

- Refined ASM: $|\mathrm{OP}(n,n+1,[n],[n+1]\setminus\{n+1-m\})| = \frac{(2n-m)!\,(n+m)!}{n!\,m!\,(n-m)!} \prod_{i=1}^{n} \frac{(3i-2)!}{(n+i)!}$

  (Zeilberger 1996, Fischer 2007)

- Related case: $|\mathrm{OP}(n,n+m,[n],[n-1]\cup\{n+m\})| =$

  $$\frac{1}{(n-1)!\,m!} \prod_{i=0}^{n-2} \frac{(3i+1)!}{(n+i)!} \sum_{i=0}^{n-1} \frac{(2n-2-i)!\,(n-1+i)!\,(m+i)!}{i!^2\,(n-1-i)!}$$ *(Fischer 2007)*

- Vertically Symmetric ASMs: $|\mathrm{OP}(n,2n-1,[n],\{1,3,\ldots,2n-1\})| = \prod_{i=1}^{n} \frac{(6i-2)!}{(2n+2i)!}$

  (Kuperberg 2002)

- Horizontally and Vertically Symmetric ASMs:

  $$|\mathrm{OP}(n,n,\{1,3,\ldots,2\lceil\tfrac{n}{2}\rceil-1\},\{1,3,\ldots,2\lceil\tfrac{n}{2}\rceil-1\})| = \frac{(\lfloor\frac{3n}{2}\rfloor+1)!}{3^{\lfloor\frac{n}{2}\rfloor}\,(2n+1)!\,\lfloor\frac{n}{2}\rfloor!} \prod_{i=1}^{n} \frac{(3i)!}{(n+i)!}$$

  (Okada 2006)

Partitions

For the a by b rectangle with boundary conditions α and β, define the partition

$$\lambda_{a,b,\alpha,\beta} \;:=\; [a]\times[b]\setminus(b-\beta_1,\ldots,b-\beta_r\,|\,a-\alpha_1,\ldots,a-\alpha_r)$$

using complement and Frobenius notation.

Examples:

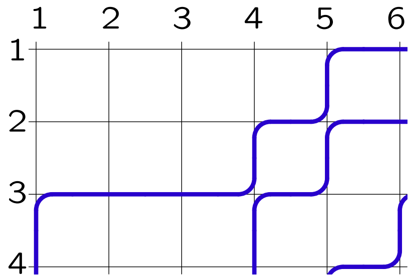
$a,\ b,\ \alpha,\ \beta$	$\lambda_{a,b,\alpha,\beta}$
$4,\ 6,\ \{1,2,3\},\ \{1,4,5\}$	$(3,2,2)$
$n,\ n,\ [n],\ [n]$	$\emptyset$
$n,\ n+1,\ [n],\ [n+1]\setminus\{n+1-m\}$	$(m)^t$
$n,\ n+m,\ [n],\ [n-1]\cup\{n+m\}$	(m)
$n,\ 2n-1,\ [n],\ \{1,3,\ldots,2n-1\}$	$(n-1,n-2,\ldots,1)$
$n,\ n,\ \{1,3,\ldots,2\lceil\frac{n}{2}\rceil-1\},\ \{1,3,\ldots,2\lceil\frac{n}{2}\rceil-1\}$	$(n-1,n-2,\ldots,1)$

Vacancies and Osculations

For a path tuple $P \in \mathrm{OP}(a, b, \alpha, \beta)$ define

- *vacancies*: points of the rectangle through which no path of P passes
- *osculations*: points of the rectangle through which two paths of P pass.

Example:



vacancies: $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,2)$, $(2,3)$, $(4,2)$, $(4,3)$

osculations: $(2,5)$, $(3,4)$

Lemmas: For each $P \in \mathrm{OP}(a,b,\alpha,\beta)$

- P is uniquely determined by its vacancies and osculations
- (number of vacancies in P) – (number of osculations in P) $= |\lambda_{a,b,\alpha,\beta}|$.

Define

$$\mathrm{OP}(a,b,\alpha,\beta,l) := \{\, P \in \mathrm{OP}(a,b,\alpha,\beta) \mid \text{(number of vacancies in } P) + \text{(number of osculations in } P) = l\,\}.$$

Oscillating Tableaux

For a partition λ and nonnegative integer l, an *oscillating tableau* of *shape* λ and *length* l is a sequence of $l+1$ partitions for which

- The first partition is $\emptyset$.
- The last partition is λ.
- The Young diagrams of successive partitions differ by the addition or deletion of a single square.

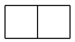
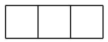
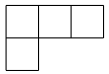
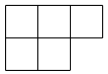
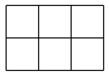
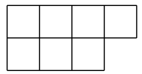
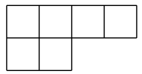
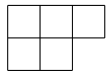
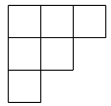
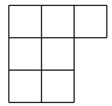
Let $\mathrm{OT}(\lambda, l)$ denote the set of all such oscillating tableaux.

For $\eta = (\eta_0, \eta_1, \ldots, \eta_l) \in \mathrm{OT}(\lambda, l)$, define the *profile* of η as

$$\Omega(\eta) \;:=\; (j_1 - i_1, \ldots, j_l - i_l)$$

where (i_k, j_k) is the position of the square by which the diagram of η_k differs from the diagram of η_{k-1}.

Example of an element η of $\mathrm{OT}((3,2,2), 11)$:

k	0	1	2	3	4	5	6	7	8	9	10	11
η_k	$\emptyset$											
$\Omega(\eta)_k$		0	1	2	-1	0	1	3	1	3	-2	-1

It follows that

- Each oscillating tableau is uniquely determined by its profile
- $\mathrm{OT}(\lambda, |\lambda|) \longleftrightarrow$ {standard Young tableaux of shape λ}.

Theorem: $|\mathrm{OT}(\lambda, l)| = \binom{l}{|\lambda|}(l-|\lambda|-1)!!\, f^{\lambda}$ *(Sundaram 1986)*

where $f^{\lambda} =$ number of SYT of shape λ.

Proof: Bijection between $\mathrm{OT}(\lambda, l)$ and certain pairs of matchings and standard Young tableaux.

Generalized Oscillating Tableaux

For an integer q and positive integer n, define the set of *generalized oscillating tableaux* $\mathsf{GOT}(n,q,\lambda,l)$ to be the set of pairs $((t_1,\ldots,t_l),\eta)$ in which

- t_k is an integer between 1 and n, for each $k=1,\ldots,l$
- η is an oscillating tableau of shape λ and length l
- $t_k < t_{k+1}$, or $t_k = t_{k+1}$ and $\Omega(\eta)_k \prec_q \Omega(\eta)_{k+1}$, for each $k=1,\ldots,l-1$,

  where $\prec_q$ is the *total strict order* on the integers defined by

  $z \prec_q z'$ if and only if $|z-q| > |z'-q|$ or $z-q = q-z' < 0$

  i.e., $\ldots \prec_q q-2 \prec_q q+2 \prec_q q-1 \prec_q q+1 \prec_q q$.

Bijection

Theorem: There is a bijection between OP(a,b,α,β,l) and GOT$(\min(a,b), b-a, \lambda_{a,b,\alpha,\beta}, l)$.

Given a path tuple P the corresponding generalized oscillating tableau (t,η) is obtained as follows.

(1) For any lattice point (i,j), define the *level* $L_{i,j} := \begin{cases} \max(i, j+a-b), & a \le b \\ \max(i-a+b, j), & a \ge b\,. \end{cases}$

(2) Order the l vacancies and osculations of P as $(i_1,j_1),\dots,(i_l,j_l)$ with $L_{i_k,j_k} < L_{i_{k+1},j_{k+1}}$, or $L_{i_k,j_k} = L_{i_{k+1},j_{k+1}}$ and $j_k - i_k \prec_{b-a} j_{k+1} - i_{k+1}$.

(3) Then $t = (L_{i_1,j_1},\dots,L_{i_l,j_l})$ and η is the oscillating tableau with profile $\Omega(\eta) = (j_1 - i_1, \dots, j_l - i_l)$.

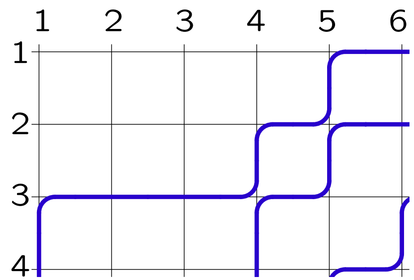
- If (i_k,j_k) is a vacancy or osculation of P, then the diagram of η_k is related to that of η_{k-1} by respectively the addition or deletion of a square.

Corollary: The number of osculating path tuples can be written as a sum over oscillating tableaux,

$$|\mathsf{OP}(a,b,\alpha,\beta,l)| = \sum_{\eta \in \mathsf{OT}(\lambda_{a,b,\alpha,\beta},l)} \binom{\min(a,b) + A_{b-a}(\eta)}{l},$$

where $A_q(\eta) = |\{k \mid \Omega(\eta)_k \prec_q \Omega(\eta)_{k+1})\}|$.

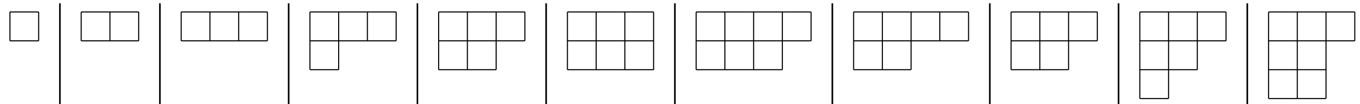
Example:



(1) $L_{i,j} = \max(i, j-2) \ \rightarrow \ \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix}.$

(2) The ordered list of vacancies and osculations is $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, $(2,2)$, $(2,3)$, $(1,4)$, $(3,4)$, $(2,5)$, $(4,2)$, $(4,3)$.

(3) $t = (1,1,1,2,2,2,2,3,3,4,4)$ and $\Omega(\eta) = (0,1,2,-1,0,1,3,1,3,-2,-1)$, so η is the previous example of an oscillating tableau:



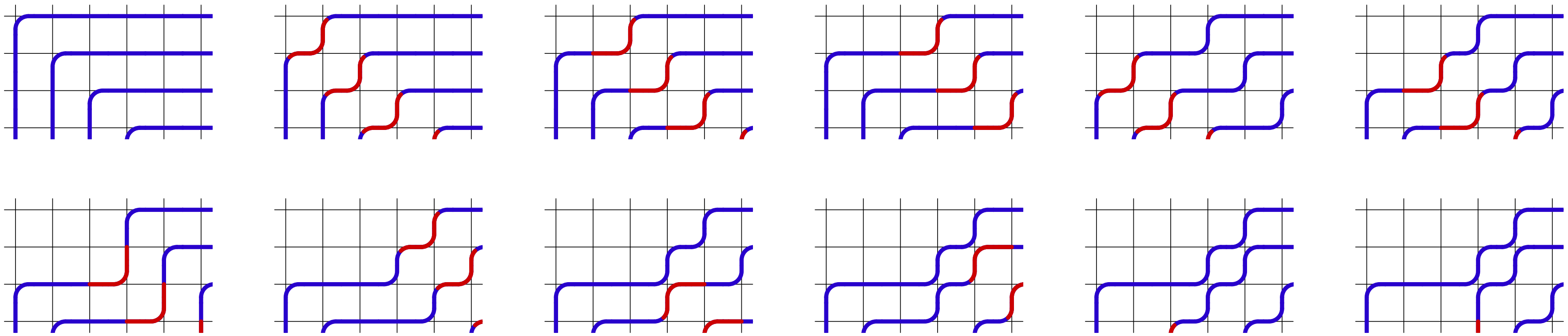
η_k	$\emptyset$											
$\Omega(\eta)_k$		0	1	2	-1	0	1	3	1	3	-2	-1

Part of Proof

Use a sequence of $l+1$ path tuples in the a by b rectangle in which

- The first tuple has no vacancies or osculations
- Each successive tuple has an additional vacancy or osculation
- The last tuple is P.

Example:



Further Work

- Use the osculating paths – oscillating tableaux bijection, other known bijections, and the Lindström-Gessel-Viennot theorem to obtain determinantal enumeration formulae and generating functions for ASMs, and possibly an ASM – descending plane partition bijection.

- Study osculating paths with other external configurations.

- Find representation theoretic interpretation of generalized oscillating tableaux.