Two non-holonomic lattice walks in the quarter plane

Collaboration with Andrew Rechnitzer

Marni Mishna

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Characterize lattice walks that have holonomic generating series.

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- Coefficients are readily computable, and easy to manipulate. (e.g. with tools from `gfun`)

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- In many cases, we can understand much of the story (eg. half plane, slit plane, quarter plane).
- They are nice candidates for the Kernel Method, a technique that still holds many mysteries.

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Is there *any hope*?

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Conjecture (Mishna 2005)

The generating function is holonomic iff the walk set $\mathcal{S}$ has any of the following four properties,

1. The quarter plane condition reduces to a half-plane condition;
2. $\mathcal{S}$ is $x-$ or $y-$ axis symmetric;
3. $\mathcal{S} = \text{reverse}(\mathcal{S})$ (path reversibility);
4. $\mathcal{S} = \text{reflect}(\text{reverse}(\mathcal{S}))$;
5. $\mathcal{S} = \{\text{N}, \text{E}, \text{SW}\}$ or $\mathcal{S} = \{\text{S}, \text{W}, \text{NE}\}$.

In proving the conjecture for walk sets of cardinality 3, we need to prove the non-holonomy of: walks with steps from $\mathcal{S} = \{\mathrm{NW}, \mathrm{NE}, \mathrm{SE}\}$ and $\mathcal{T} = \{\mathrm{NW}, \mathrm{N}, \mathrm{SE}\}$.

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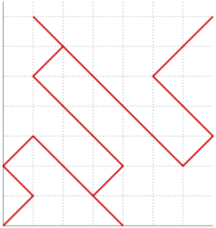

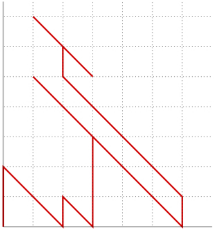

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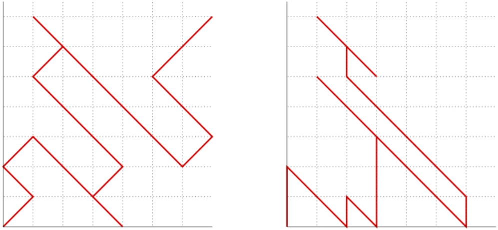

Theorem (M., Rechnitzer 2007)

The generating functions $S(t) = \sum_{w \in L(\mathcal{S})} t^{length(w)}$ *and* $T(t) = \sum_{w \in L(\mathcal{T})} t^{length(w)}$ *are not holonomic.*

An abundance of singularities

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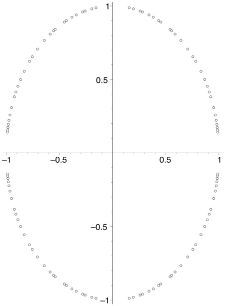

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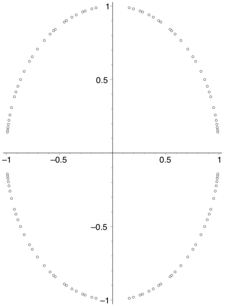
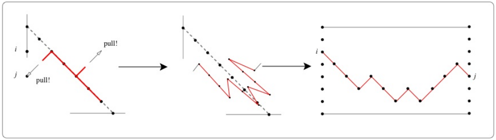
- $\implies \sum S_k(1;t)z^k$ not holonomic. (à la D-unsolvable)



We can give a recurrence that explains how the singularities arise.

A combinatoiral source of the singularities

A walk ending on $x+y=k$ is a walk ending on $x+y=k-2$, a NE step and then a directed walk in a strip of height k.



$$D_{i,j}^{(k)}\left(\frac{q}{1+q^2}\right) = q^{j-i+1}\frac{(1-q^{2i+2})(1-q^{2k-2j+2})}{(1-q^2)(1-q^{2k+4})}$$

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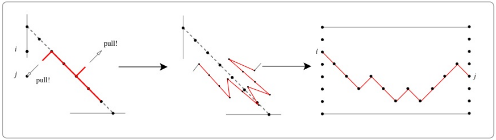

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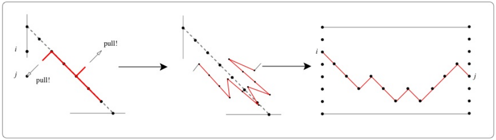

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Unless "a miracle occurs" we would not expect the singularities to cancel.

Enter Andrew: Analytic atta-a-ack!

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5. Strategy: We show that $S(1, 0; t)$ has an infinite number of singularities.

The kernel is infinite

$$\underbrace{\left(xy - tx^2y^2 - tx^2 - ty^2\right)}_{K(x,y)} S(x,y) = xy - tx^2 S(x,0;t) - ty^2 S(y,0;t).$$

Roots of $K(x,y)$:

$$Y_{\pm1}(x) = \frac{x}{2t(1+x^2)}\left(1 \mp \sqrt{1-4t^2(1+x^2)}\right).$$

- Define: $Y_n := Y_1(Y_{n-1}(x))$
- $Y_n(Y_m(x)) = Y_n \circ Y_m = Y_{n+m}$, identity $Y_0 = x$
- $\{Y_n | n \in \mathbb{Z}\}$ forms an infinite group
- *this is what distinguishes it from the holonomic cases*

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ITERATE

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Method based on Prellberg et. al 2006, and related to process of Bousquet-Mélou

Singularities spring eternal

Theorem

$$S(t) = (1-3t)^{-1}(1 - 2\sum_{n\geq 0}(-1)^n Y_n(1) Y_{n+1}(1)).$$

The set $\bigcup_n \text{poles}(Y_n(1))$ *is infinite, and is a subset of poles*$(S(t))$*. Consequently,* $S(t)$ *is not holonomic.*

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- Show singularities don't cancel.

The singularities don't cancel

As usual, the hardest part is showing that there is no massive cancellation. But we succeed!
Essentially, because of the nice recurrence

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We can show q_c is not a root of unity; Recall $Y_0(1) = 1$.

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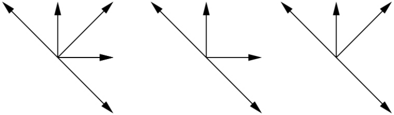
Which other walks reduce to $\mathcal{S}$-case?

Lots of non-holonomic classes?

Can proving the non-holonomy of these classes be far behind?



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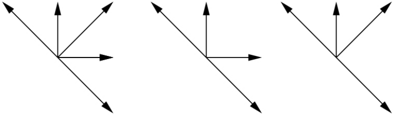

The gf of $\mathcal{S}$-walks in any wedge less than the half plane seems to satisfy an equation not unlike in the quarter plane, recall:

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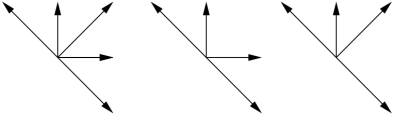

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Can we reduce this to the quarter plane case?

(work in progress with D. Laferrière)

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6. Prove that miracles don't exist. (or, rather understand when they do.)