Dyson's non-intersecting

Brownian motions

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1. Forced Non-intersecting Brownian motions

2. The Airy process $A(\tau)$

3. The Pearcey process $\mathcal{P}(\tau)$

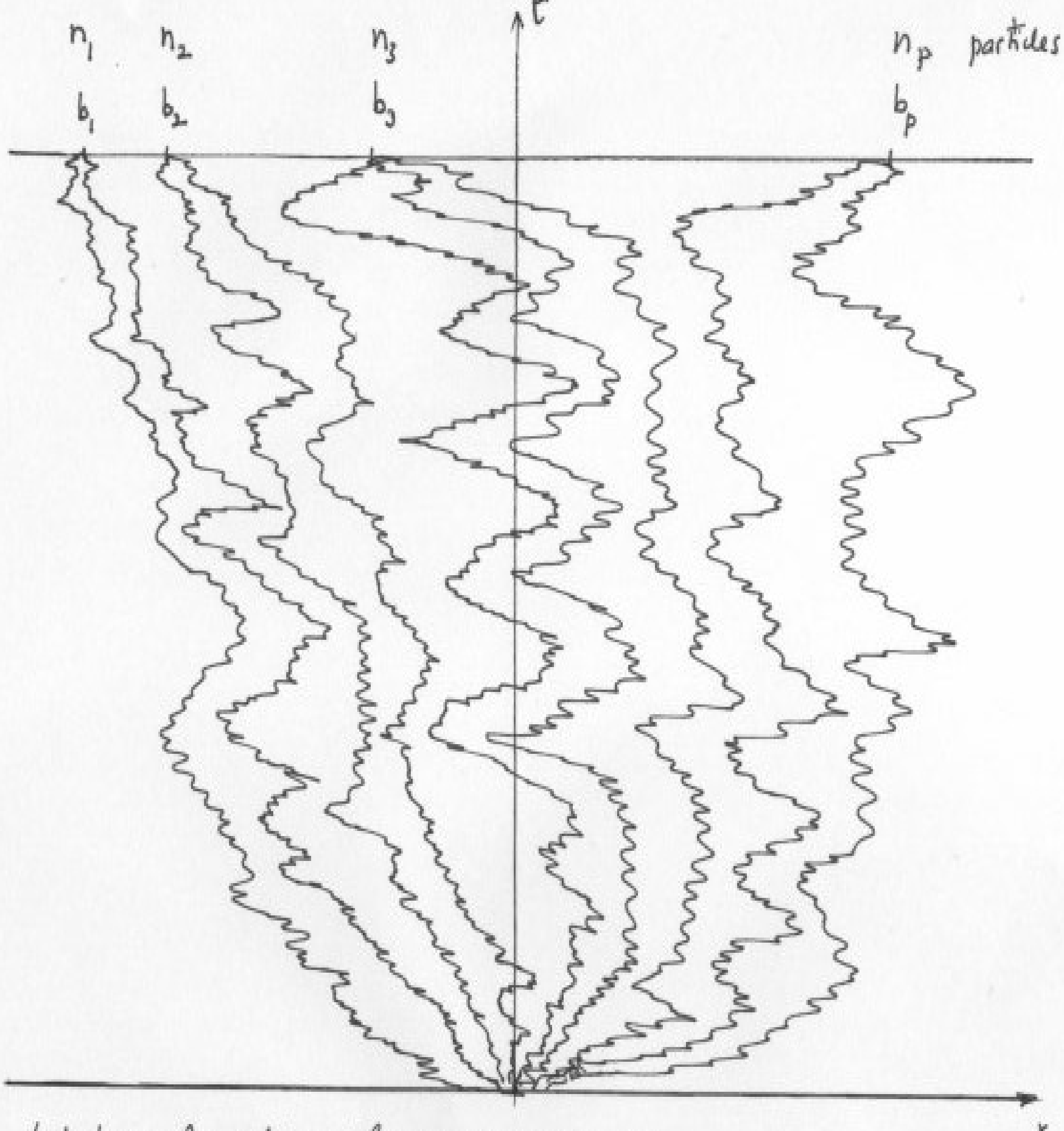
4. From Pearcey to Airy

5. The Airy process $\mathcal{A}^{(k)}(t)$ with k outliers

Joint work with MARK ADLER

1. Non-intersecting Brownian motions, leaving from a point and forced to one or several points

Karlin-McGregor '59, Dyson '62, Ueno-Takasaki '84, Grabiner '99, Johansson '01, Aptekarev-Bleher-Kuijlaars '04, Tracy-Widom '03 and '05, Bleher-Kuijlaars '04, Adler-PvM '05, Kuijlaars-Daems '05, Adler-PvM-Vanhaecke '06



total number of particles $n = n_1 + n_2 + \cdots + n_p$

• n non-intersecting Brownian motions $X_1(t), \ldots, X_n(t)$ $\quad (\sum_1^p n_i = n)$

forced to leave from 0

forced to arrive at $b := (\overbrace{b_1, b_1, \ldots, b_1}^{n_1}, \overbrace{b_2, b_2, \ldots, b_2}^{n_2}, \ldots, \overbrace{b_p, b_p, \ldots, b_p}^{n_p})$

The probability

$$\mathbb{P}_n^{(b_1,\ldots,b_p)} \left(\text{all } (x_1(t) < \ldots < x_n(t)) \in E \;\middle|\; \begin{array}{c} \text{all } x_j(0) = 0 \\ n_1 \text{ paths end up at } b_1 \text{ at } t = 1 \\ \vdots \\ n_p \text{ paths end up at } b_p \text{ at } t = 1 \end{array} \right)$$

has 4 very different descriptions !

I.

$$\mathbb{P}_n^{(b_1,\dots,b_p)}\left(\text{all } (x_1(t)<\dots<x_n(t))\in E \;\middle|\; \begin{array}{c}\text{all } x_j(0)=0\\ n_1 \text{ paths end up at } b_1 \text{ at } t=1\\ \vdots\\ n_p \text{ paths end up at } b_p \text{ at } t=1\end{array}\right)$$

$$= \lim_{\substack{\text{all } \gamma_i\to 0\\ \delta_1,\dots,\delta_{n_1}\to b_1\\ \vdots\\ \delta_{n_1+\dots+n_{p-1}+1},\dots,\delta_n\to b_p}} \int_{E^n}\frac{\prod_1^n dx_i}{Z_n}\det\Big(p(t;\gamma_i,x_j)\Big)_{1\le i,j\le n}\det(p(1-t;x_{i'},\delta_{j'}))_{1\le i',j'\le n},$$

$$= \frac{1}{Z_n}\det\left(\begin{array}{l}\left(\displaystyle\int_{\tilde{E}} x^{i+j}e^{-\frac{x^2}{2}+\tilde{b}_1x}dx\right)_{0\le i\le n_1-1,\ 0\le j\le n-1}\\ \vdots\\ \left(\displaystyle\int_{\tilde{E}} x^{i+j}e^{-\frac{x^2}{2}+\tilde{b}_px}dx\right)_{0\le i\le n_p-1,\ 0\le j\le n-1}\end{array}\right), \quad \left(\begin{array}{l}\textbf{block moment}\\ \textbf{matrix}\end{array}\right)$$

where $\tilde{E} = \frac{E}{\sqrt{t(1-t)}}, \quad \tilde{b}_\beta = \sqrt{\frac{t}{1-t}} b_\beta$

$$\mathbf{II}.\ \mathbb{P}_n^{(b_1,\dots,b_p)}\left(\text{all } (x_1(t)<\ldots<x_n(t))\in E \;\middle|\; \begin{array}{c}\text{all } x_j(0)=0\\ n_1 \text{ paths end up at } b_1 \text{ at } t=1\\ \vdots\\ n_p \text{ paths end up at } b_p \text{ at } t=1\end{array}\right)$$

$$=\lim_{\substack{\text{all }\ \gamma_i\to 0\\ \delta_1,\dots,\delta_{n_1}\to b_1\\ \vdots\\ \delta_{n_1+\ldots+n_{p-1}+1},\dots,\delta_n\to b_p}}\int_{E^n}\frac{\prod_1^n dx_i}{Z_n}\det\Big(p(t;\gamma_i,x_j)\Big)_{1\le i,j\le n}\det(p(1-t;x_{i'},\delta_{j'}))_{1\le i',j'\le n},$$

$$=\frac{1}{Z_n}\int_{\tilde{E}^n}\Delta_n(x_1,\dots,x_n)\prod_{\ell=1}^{p}\Delta_{n_\ell}(x_1^{(\ell)},\dots,x_{n_\ell}^{(\ell)})\prod_{j=1}^{n_\ell}e^{-\frac{1}{2}{x_j^{(\ell)}}^2+\tilde{b}_\ell x_j^{(\ell)}}dx_j^{(\ell)}\Bigg|_{\substack{\tilde{E}=E\sqrt{\frac{2}{t(1-t)}}\\ \tilde{b}_\ell=\sqrt{\frac{2t}{1-t}}b_\ell}}$$

$$=\frac{1}{Z_n}\int_{\mathcal{H}_n\left(E\sqrt{\frac{2}{t(1-t)}}\right)}dM e^{-\frac{1}{2}\operatorname{Tr}(M^2-2A_tM)}$$

(Integral over Hermitian matrices with spectrum in E: **Gaussian model with external potential**) with

$$\Delta_n(x_1, \ldots, x_n) = \text{Vandermonde determinant.}$$

and

$$A_t := \begin{pmatrix} \tilde{b}_1 & & & & & & & & \\ & \ddots & & & \mathbf{O} & & & & \\ & & \tilde{b}_1 & & & & & & \\ & & & \tilde{b}_2 & & & & & \\ & \mathbf{O} & & & \ddots & & & & \\ & & & & & \tilde{b}_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \tilde{b}_p & \\ & & & & & & & & \ddots \\ & & & & & & & & & \tilde{b}_p \end{pmatrix} \begin{array}{l} \\ \updownarrow n_1 \\ \\ \\ \updownarrow n_2 \\ \vdots \\ \\ \\ \updownarrow n_p \\ \\ \end{array} \quad \text{and} \quad \tilde{b}_i = b_i \sqrt{\frac{2t}{1-t}},$$

$$\textbf{III}.\ \mathbb{P}_n^{(b_1,\ldots,b_p)}\left(\text{all } (x_1(t)<\ldots<x_n(t))\in E \;\middle|\; \begin{array}{c}\text{all } x_j(0)=0\\ n_1 \text{ paths end up at } b_1 \text{ at } t=1\\ \vdots\\ n_p \text{ paths end up at } b_p \text{ at } t=1\end{array}\right)$$

$$= \lim_{\substack{\text{all } \gamma_i\to 0\\ \delta_1,\ldots,\delta_{n_1}\to b_1\\ \vdots\\ \delta_{n_1+\ldots+n_{p-1}+1},\ldots,\delta_n\to b_p}} \int_{E^n}\frac{\prod_1^n dx_i}{Z_n}\det\Big(p(t;\gamma_i,x_j)\Big)_{1\le i,j\le n}\det(p(1-t;x_{i'},\delta_{j'}))_{1\le i',j'\le n},$$

$$= \quad \det(I-H_n^{(t;b_1,\ldots,b_p)})_{E^c}, \quad (\textbf{Fredholm determinant})$$

with kernel

$$H_n^{(t;b_1,\ldots,b_p)}(x,y)dy$$
$$= -\frac{dy}{2\pi^2(1-t)}\int_{\mathcal{C}}dV\int_{L+i\mathbb{R}}dU\ e^{-\frac{tV^2}{1-t}+\frac{2xV}{1-t}+\frac{tU^2}{1-t}-\frac{2yU}{1-t}}\prod_{r=1}^{p}\left(\frac{U-b_r}{V-b_r}\right)^{n_r}\frac{1}{U-V}$$

$\mathcal{C}$ is a closed contour enclosing all the points b_r, which is to the left of the line $L+i\mathbb{R}$ by picking L large enough, guaranteeing $\Re e(U-V)>0$.

IV. Consider now the probability, as a function of

- the boundary points of E
- the target points b_i, with a linear dependence $\sum_1^p c_i b_i = 0$, with $\sum_1^p c_i = 1$

$$\log \mathbb{P}_n^{(b_1,\dots,b_p)} \left(\text{all } (x_1(t) < \dots < x_n(t)) \in E \;\middle|\; \begin{array}{c} \text{all } x_j(0) = 0 \\ n_1 \text{ paths end up at } b_1 \text{ at } t = 1 \\ \vdots \\ n_p \text{ paths end up at } b_p \text{ at } t = 1 \end{array} \right)$$

satisfies a non-linear PDE in the boundary points of the interval E and in the target points b_i(*nearly a Wronskian!!*)

$$\det \begin{pmatrix} F_1 & F_2 & F_3 & \dots & F_p & 0 \\ \nabla F_1 & \nabla F_2 & \nabla F_3 & \dots & \nabla F_p & G_1 \\ \nabla^2 F_1 & \nabla^2 F_2 & \nabla^2 F_3 & \dots & \nabla^2 F_p & G_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \nabla^p F_1 & \nabla^p F_2 & \nabla^p F_3 & \dots & \nabla^p F_p & G_p \end{pmatrix} = 0, \qquad \nabla := \left\{ \begin{array}{l} \textit{sum of} \\ \textit{partials} \\ \textit{in the} \\ \textit{boundary} \\ \textit{points of } E \end{array} \right\}$$

(Adler-Vanhaecke-PvM CMP '08)

where

$$
\begin{aligned}
F_\ell &:= \left(\nabla_\ell^{(b)} + c_\ell \nabla\right) \nabla \log \mathbb{P}_n + n_\ell \\
G_{\ell+1} &:= \nabla G_\ell \quad + \sum_{j=1}^{p} (\nabla^\ell F_j) \left(\nabla \frac{H_j^{(1)}}{F_j} - \nabla_j^{(b)} \frac{H_j^{(2)}}{F_j} \right), \qquad G_0 = 0
\end{aligned}
$$

with (remember $\sum_1^p c_i b_i = 0$, so that $\sum_{\ell=1}^p \nabla_\ell^{(b)} = 0$)

$$
\nabla_\ell^{(b)} := c_\ell \sum_1^{p-1} \frac{\partial}{\partial b_i} - \frac{\partial}{\partial b_\ell} (1 - \delta_{\ell p})
$$

Proof: The matrix integral (deformed with additional variables)

$$\tau_{n_1,\dots,n_p}(E;t;s^{(1)},\dots,s^{(p)}) = \int_{E^n} I_{n_1,\dots,n_p}(x_1,\dots,x_n)\prod_1^n dx_i$$

$$= \frac{1}{\prod_1^p n_\ell!}\int_{E^n}\Delta_n(x^{(1)},\dots,x^{(p)})\prod_{\ell=1}^{p}\prod_{j=1}^{n_\ell} e^{\sum_{i=1}^\infty t_i (x_j^{(\ell)})^i}$$

$$\prod_{\ell=1}^{p}\left(\Delta_{n_\ell}(x^{(\ell)})\prod_{j=1}^{n_\ell} e^{-\frac{1}{2}x_j^{(\ell)2}+b_\ell x_j^{(\ell)}+\beta_\ell x_j^{(\ell)2}-\sum_{i=1}^\infty s_i^{(\ell)} x_j^{(\ell)i}}dx_j^{(\ell)}\right)\Bigg|_{\substack{\sum_1^p c_i b_i=0\\ \sum_1^p c_i\beta_i=0}}$$

satisfies

1. **linear PDE's (Virasoro) for** $\tau_{n_1,\dots,n_p}(E)$:

$$\begin{aligned}\nabla\tau_{n_1,\dots,n_p}(E) &= \int_{E^n}\left(\sum_1^n \frac{\partial I_{n_1,\dots,n_p}}{\partial x_i}\right)\prod_1^n dx_i\\ &= \left(\Sigma-\frac{\partial}{\partial t_1}+\sum_1^p n_\ell(b_\ell+t_1-s_1^{(\ell)})+2\sum_1^p\beta_\ell\frac{\partial}{\partial b_\ell}\right)\tau_n,\end{aligned}$$

where

$$\nabla := \left\{\begin{array}{l}\textit{sum of partials in the}\\ \textit{boundary points of } E\end{array}\right\},\qquad \Sigma = \sum_{i\geq 2}\left(it_i\frac{\partial}{\partial t_{i-1}}+\sum_{\ell=1}^p is_i^{(\ell)}\frac{\partial}{\partial s_{i-1}^{(\ell)}}\right)$$

Proof: The matrix integral (deformed with additional variables)

$$\tau_{\overrightarrow{n}} = \tau_{n_1,\dots,n_p}(E; t; s^{(1)}, \dots, s^{(p)})$$

$$= \det \begin{pmatrix} \left(\int_E x^{i+j} e^{-\frac{x^2}{2} + b_1 x + \beta_1 x^2} e^{\sum_1^\infty (t_k - s_k^{(1)}) x^k} dx \right)_{0 \le i \le n_1 - 1,\ 0 \le j \le n-1} \\ \vdots \\ \vdots \\ \left(\int_E x^{i+j} e^{-\frac{x^2}{2} + b_p x + \beta_p x^2} e^{\sum_1^\infty (t_k - s_k^{(p)}) x^k} dx \right)_{0 \le i \le n_p - 1,\ 0 \le j \le n-1} \end{pmatrix},$$

satisfies

1. linear PDE's (Virasoro constraints)
2. $p+1$-**component KP hierarchy** (described by Ueno-Takesaki). In particular:

$$\frac{\partial}{\partial t_1} \log \frac{\tau_{\overrightarrow{n}+e_\ell}}{\tau_{\overrightarrow{n}-e_\ell}} = \frac{\frac{\partial^2}{\partial t_2 \partial s_1^{(\ell)}} \log \tau_{\overrightarrow{n}}}{\frac{\partial^2}{\partial t_1 \partial s_1^{(\ell)}} \log \tau_{\overrightarrow{n}}}, \qquad \frac{\partial}{\partial s_1^{(\ell)}} \log \frac{\tau_{\overrightarrow{n}+e_\ell}}{\tau_{\overrightarrow{n}-e_\ell}} = -\frac{\frac{\partial^2}{\partial t_1 \partial s_2^{(\ell)}} \log \tau_{\overrightarrow{n}}}{\frac{\partial^2}{\partial t_1 \partial s_1^{(\ell)}} \log \tau_{\overrightarrow{n}}}.$$

Bilinear relations for the $p+1$-component KP hierarchy involve a matrix

$\Longleftrightarrow$

$(p+1)\times(p+1)$ Riemann-Hilbert matrix for multiple orthogonal polynomials (Deift, Bleher-Kuijlaars).

Use the

1. Virasoro PDE's

2. $p+1$-component KP hierarchy

to eliminate unwanted partials,
leading to (after a lot of miracles!)

$$\det\begin{pmatrix} F_1 & F_2 & F_3 & \dots & F_p & 0 \\ F_1' & F_2' & F_3' & \dots & F_p' & G_1 \\ F_1'' & F_2'' & F_3'' & \dots & F_p'' & G_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_1^{(p)} & F_2^{(p)} & F_3^{(p)} & \dots & F_p^{(p)} & G_p \end{pmatrix} = 0, \qquad {}' := \nabla$$

What happens to this probability, when $\#\{\text{particles}\} = n \to \infty$?

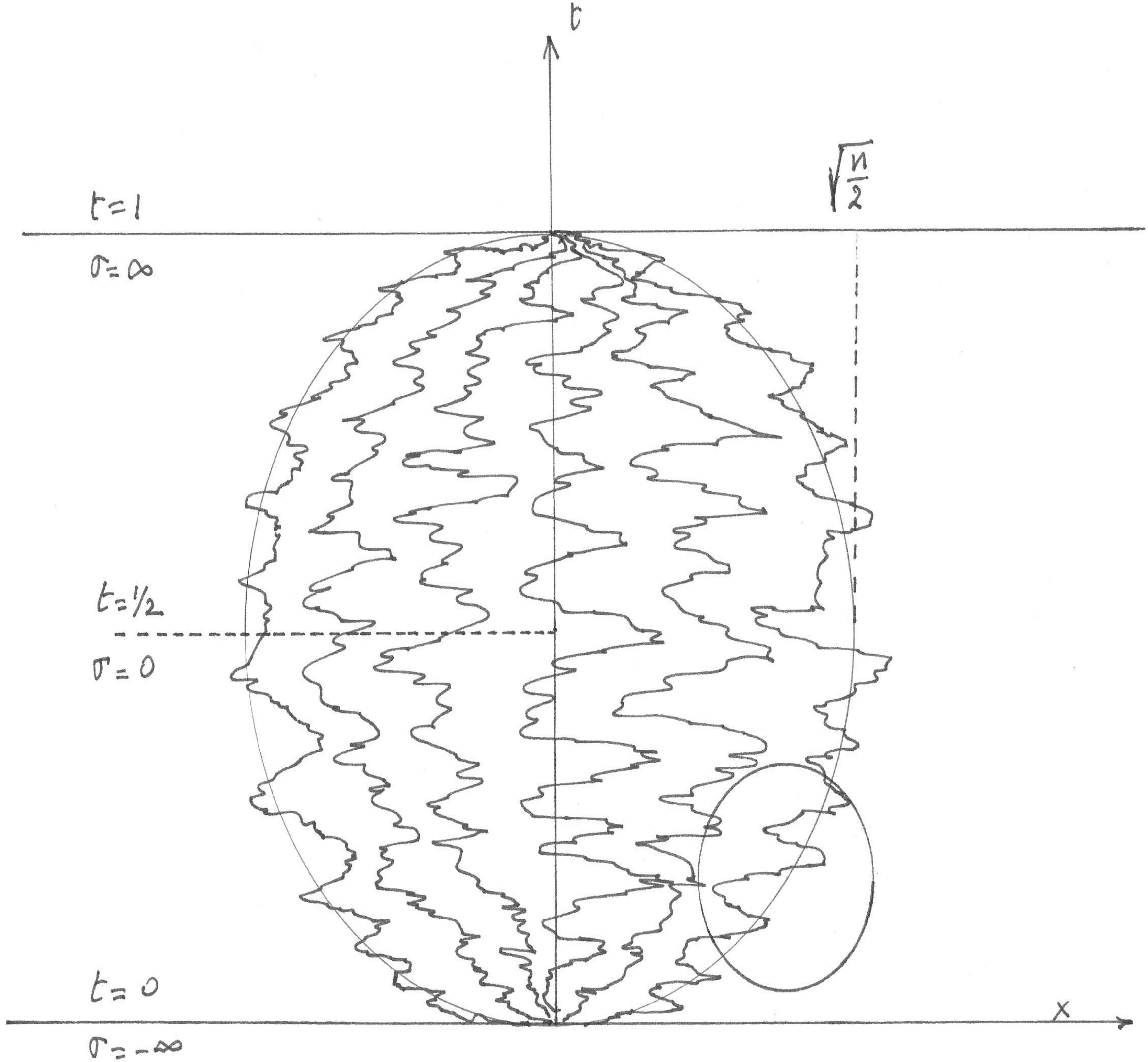
- Airy process
- Pearcey process
- Airy process, with outliers
- Pearcey process with inliers,. . .

2. The Airy process $A(\tau)$

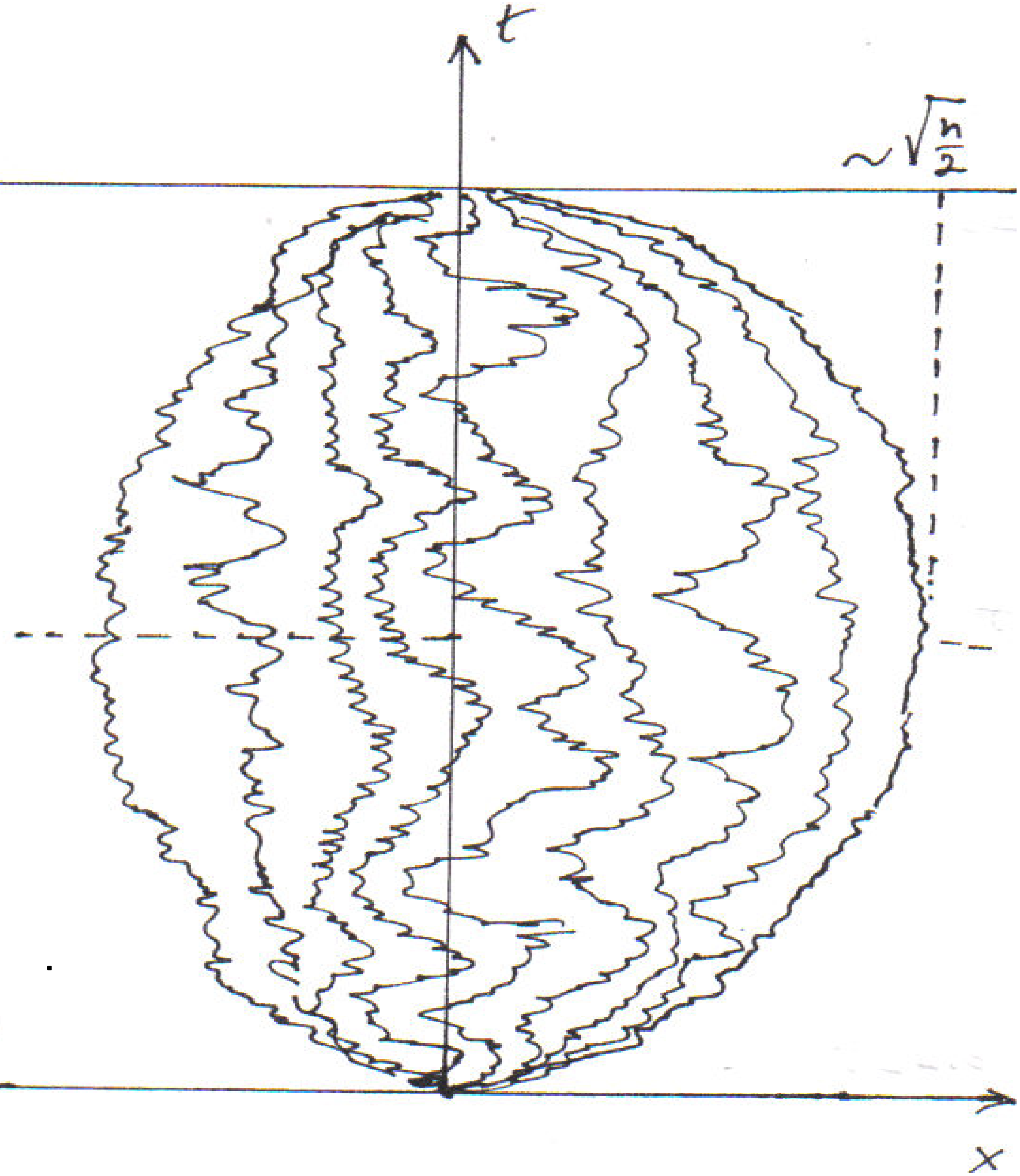
(Prähofer-Spohn '02, Johansson '03, '05, Tracy-Widom '04, Adler-PvM '05),

$$\mathbb{P}^0_{Br}\left(\text{all } x_j(t) \in E^c\right) := \mathbb{P}\left(\text{all } x_j(t) \in E^c \text{ for } 1 \leq j \leq n \;\middle|\; \begin{array}{l} \text{all } x_j(0) = 0 \\ \\ \text{all } x_j(1) = 0 \end{array}\right).$$

Let $n \to \infty$**:**



t
$\sqrt{\frac{n}{2}}$
t = 1
σ = ∞
t = 1/2
σ = 0
t = 0
σ = −∞
x



curve: $x = \sqrt{2nt(1-t)}$

parametrized by

$$x = \frac{\sqrt{2n}}{2\cosh\sigma}, \quad t = \frac{e^{\sigma}}{2\cosh\sigma}$$

$$\begin{array}{lll} t = 1 & x = 0 & \sigma = \infty \\ t = 1/2 & x = \sqrt{\frac{n}{2}} & \sigma = 0 \\ t = 0 & x = 0 & \sigma = -\infty \end{array}$$

Look through a microscope at **the fluctuations of the paths** about any point on the curve, for very large n:

$$(x,t) = \left(\frac{\sqrt{2n}}{2\cosh\sigma}, \quad \frac{e^{\sigma}}{2\cosh\sigma} \right) \in \text{curve}$$

New process $\mathcal{A}(\tau)$ in the neighborhood of the point $(x,t) \in$ curve :

$$\lim_{n\to\infty} \mathbb{P}_{Br}\left(\text{all } x_i \left(\frac{1}{1+e^{-2(\sigma+\tau n^{-1/3})}} \right) \in \frac{\sqrt{2n} + \frac{E^c}{\sqrt{2}n^{1/6}}}{2\cosh(\sigma+\tau n^{-1/3})} \right)$$

$$= \det(I - \mathbf{A})_E \qquad \text{(independent of } \sigma \text{ and } \tau\text{)}$$

$$= \mathbb{P}(\mathcal{A}(\tau) \cap E = \emptyset) \qquad \text{(stationary process!!)}$$

$$\text{Airy kernel} = \mathbf{A}(u,v) = \int_0^{\infty} dw A(w+u)A(w+v) = \frac{A(u)A'(v) - A'(u)A(v)}{u-v}$$

$$A(u) = \int_{-\infty}^{\infty} e^{\frac{1}{3}ia^3 + iau} \frac{da}{2\pi} = \text{Airy function} \qquad \text{(independent of } \tau\text{!!!)}$$

Remember $\nabla = \sum_1^{2r} \frac{\partial}{\partial x_i}$ and $\mathcal{E} = \sum_1^{2r} x_i \frac{\partial}{\partial x_i}$ for $E = \cup_1^r [x_{2i-1}, x_{2i}] \subset \mathbb{R}$

$$f := \nabla \log \mathbb{P}(\mathcal{A}(\tau) \cap E = \emptyset),$$

satisfies

$$\left(\nabla^3 - 4\left(\mathcal{E} - \frac{1}{2}\right)\right) f + 6(\nabla f)^2 = 0. \qquad \text{(PDE-version of Painlevé II)}$$

For $E = (x, \infty)$,

$$\mathbb{P}(\sup \mathcal{A}(\tau) \leq x) = e^{-\int_x^\infty (\alpha - x) g^2(\alpha) d\alpha} =: \mathcal{F}(x) \qquad \text{(Tracy-Widom distribution}$$

where $g(\alpha)$ is the Hastings-MacLeod solution of Painlevé II:

$$g'' = \alpha g + 2g^3, \quad \text{with} \quad g(\alpha) \cong \frac{e^{-\frac{2}{3}\alpha^{\frac{3}{2}}}}{2\sqrt{\pi}\alpha^{1/4}} \quad \text{for} \quad \alpha \nearrow \infty.$$

$$\lim_{n\to\infty} \mathbb{P}^0_{Br} \left(\begin{array}{l} \text{all } x_i \left(\frac{1}{1+e^{-2(\sigma+\tau_1 n^{-1/3})}} \right) \in \frac{\sqrt{2n}+\frac{E_1^c}{\sqrt{2}n^{1/6}}}{2\cosh(\sigma+\tau_1 n^{-1/3})} \\ \\ \text{all } x_i \left(\frac{1}{1+e^{-2(\sigma+\tau_2 n^{-1/3})}} \right) \in \frac{\sqrt{2n}+\frac{E_2^c}{\sqrt{2}n^{1/6}}}{2\cosh(\sigma+\tau_2 n^{-1/3})} \end{array} \right)$$

$$= \mathbb{P}\left(\mathcal{A}(\tau_1) \cap E_1 = \emptyset, \quad \mathcal{A}(\tau_2) \cap E_2 = \emptyset\right)$$

$$= \det\left(I - (\chi_{E_i} \mathbf{A}_{\tau_i \tau_j} \chi_{E_j})_{1 \le i,j \le 2}\right), \qquad \left(\begin{array}{l} \text{Prähofer-Spohn 02, Johansson 03} \\ \text{Tracy-Widom '03, Widom '03} \end{array} \right)$$

Then

$$\mathbb{Q}(s;x,y) := \log \mathbb{P}\left(\sup \mathcal{A}(\tau_1) \le x+y, \quad \sup \mathcal{A}(\tau_2) \le x-y\right), \quad s = \frac{\tau_2 - \tau_1}{2} > 0$$

satisfies the PDE ($\{\ ,\ \}$ is a Wronskian)

$$2s \frac{\partial^3 \mathbb{Q}}{\partial s \partial x \partial y} = \left(2s^2 \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) \left(\frac{\partial^2 \mathbb{Q}}{\partial y^2} - \frac{\partial^2 \mathbb{Q}}{\partial x^2}\right) + \left\{\frac{\partial^2 \mathbb{Q}}{\partial x \partial y}, \frac{\partial^2 \mathbb{Q}}{\partial x^2}\right\}_x .$$

(Adler-PvM, '05),

with "initial" condition: $\lim_{s\to\infty}\mathbb{Q}(s;x,y) = \log\mathcal{F}(x+y) + \log\mathcal{F}(x-y)$.

Hence

$$P(A(\tau_1)\le u, A(\tau_2)\le v) = \mathcal{F}(u)\mathcal{F}(v) + \frac{\mathcal{F}'(u)\mathcal{F}'(v)}{(\tau_2-\tau_1)^2} + \frac{\Phi(u,v)+\Phi(v,u)}{(\tau_2-\tau_1)^4} + O\left(\frac{1}{(\tau_2-\tau_1)^6}\right),$$

2-time Airy kernel:

$$\mathbf{A}_{s,t}(x,y) = \int_0^\infty e^{-\lambda(s-t)} A(x+\lambda)A(y+\lambda)\,d\lambda$$
$$-\begin{cases} 0 & \text{for } t-s\le 0 \\ \frac{1}{\sqrt{4\pi(t-s)}} e^{\frac{1}{12}(t-s)^3 - \frac{(x-y)^2}{4(t-s)} - \frac{1}{2}(t-s)(y+x)} \sqrt{dx\,dy} & \text{for } \quad t-s>0 \end{cases}$$

3. The Pearcey process $\mathcal{P}(\tau)$

$-\infty < b < a < \infty$

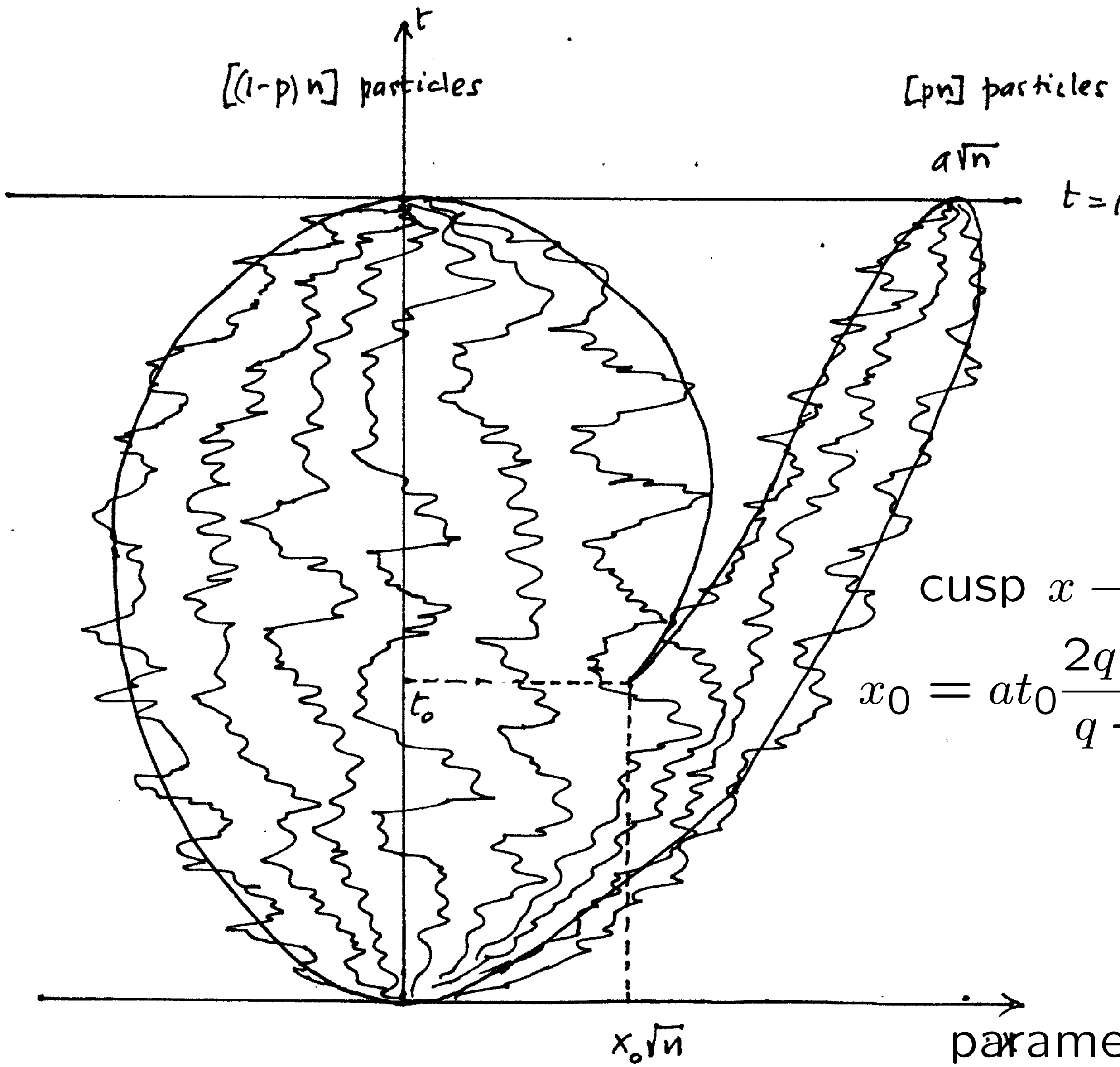
$$\mathbb{P}^{ba}_{Br}\left(\text{all } x_j(t) \in E^c\right)$$

$$:= \mathbb{P}\left(\text{all } x_j(t) \in E^c \;\middle|\; \begin{array}{c} \text{all } x_j(0) = 0 \\ [(1-p)n] \text{ paths end up at } b\sqrt{n} \text{ at } t = 1 \\ [pn] \text{ right paths end up at } a\sqrt{n} \text{ at } t = 1 \end{array}\right).$$

Let $n \to \infty$:

Pastur '72, Brézin-Hikami '96-98, Zinn-Justin '97-98, Johansson '01, Bleher-Kuijlaars '04, Tracy-Widom '05, Okounkov-Reshetikhin '05, Adler-PvM '05, Adler-Orantin-PvM '08

Set $b = 0$:



cusp $x - x_0 = 2\left(\frac{t-t_0}{3}\right)^{3/2}$ at

$$x_0 = at_0\frac{2q-1}{q+1}, \quad t_0 = \frac{1}{1+2\left(\frac{ar}{q+1}\right)^2}.$$

parametrization:

$$q^3 := \frac{1-p}{p}, \quad r := \sqrt{q^2 - q + 1}.$$

Look through a microscope at **the fluctuations of the paths** about the cusp, for very large n: (take $b=0$)

$$\textbf{cusp}: \qquad x - x_0 = 2\left(\frac{t-t_0}{3}\right)^{3/2}$$

New process $\mathcal{P}(\tau)$ **in the neighborhood of the cusp** : (universality!)

$$\lim_{n\to\infty} \mathbb{P}_n^{(0,a\sqrt{n})}\left(\text{all } x_j\left(t_0+(c_0\mu)^2\frac{2\tau}{n^{1/2}}\right) \in x_0 n^{1/2} + c_0 A\tau + c_0\mu\frac{E^c}{n^{1/4}}\right)$$

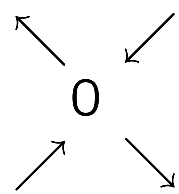
$$= \det\left(I - \mathbf{P}_\tau\right)_E \qquad \text{(independent of } pn \text{ and } a\text{)},$$

$$= \mathbb{P}^{\mathcal{P}}\left(\mathcal{P}(\tau)\cap E = \emptyset\right) \qquad (\textit{Pearcey process})$$

$$\mu^2 = \frac{r}{\sqrt{q}}, \qquad c_0 := \frac{art_0}{q+1}, \qquad A = \sqrt{q} - \frac{(2q-1)t_0}{\sqrt{q}}$$

with Pearcey kernel:

$$\mathbf{P}_\tau(\xi,\eta) := -\frac{1}{4\pi^2}\int_X dV \int_{-i\infty}^{i\infty} dU \quad e^{-\frac{U^4}{4}+\frac{\tau U^2}{2}-U\eta} \quad e^{\frac{V^4}{4}-\frac{\tau V^2}{2}+V\xi}\ \frac{1}{U-V}$$

contour $X =$ 

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$$\lim_{n\to\infty} \mathbb{P}_n^{(0,a\sqrt{n})}\left(\text{all } x_j\left(t_0+(c_0\mu)^2\frac{2\tau}{n^{1/2}}\right) \in x_0 n^{1/2} + c_0 A\tau + c_0\mu \frac{E^c}{n^{1/4}}\right)$$

$$= \det\left(I - \mathbf{P}_\tau\right)_E \qquad \text{(independent of } pn \text{ and } a\text{!)},$$

$$= \mathbb{P}^{\mathcal{P}}\left(\mathcal{P}(\tau) \cap E = \emptyset\right) \qquad (\textit{Pearcey process})$$

Then $\mathbb{Q}(\tau;E) = \log \mathbb{P}^{\mathcal{P}}\left(\mathcal{P}(\tau)\cap E = \emptyset\right)$ satisfies (Adler-PvM '06, Adler-Orantin-PvM '08)

$$\frac{\partial^3 \mathbb{Q}}{\partial t^3} + \frac{1}{8}\left(\mathcal{E} - 2t\frac{\partial}{\partial t} - 2\right)\nabla^2\mathbb{Q} - \frac{1}{2}\left\{\nabla^2\mathbb{Q}, \nabla\frac{\partial\mathbb{Q}}{\partial t}\right\}_{\nabla} = 0.$$

with an *"initial condition" at* $t=\infty$, given by the Airy process.(see later)

Proof: I. Use the method of stationary phase

$$\frac{tU^2}{(1-t)} - \frac{2Uy}{(1-t)} + n_2 \log(U-b) + n_1 \log(U-a) \Bigg| \begin{array}{l} n_1 = np, \quad n_2 = (1-p)n \\ U = \frac{c_0}{t_0} u\sqrt{n} \\ t = t_0 + \frac{2c_0^2\mu^2\tau}{n^{1/2}} \\ a \mapsto a\sqrt{n}, \quad b \mapsto b\sqrt{n} \\ y = x_0\sqrt{n} + c_0 A\tau + c_0\mu\frac{\eta}{n^{1/4}} \\ u = u_0 + \frac{U'}{\mu n^{1/4}} \end{array}$$

$$= nF(u_0) - \frac{(\mu u_0)^2}{2} n^{1/2}\tau - n^{1/4} u_0 \eta\mu - \underbrace{\left(\frac{U'^4}{4} - \frac{\tau U'^2}{2} + \eta U'\right)}_{\text{Pearcey quartic}} - \frac{t_0 \mu^4 u_0^2}{2}\tau^2 + O(n^{-1/4}).$$

Note the Taylor expansion of the function $F(u)$ about $u = u_0 = \frac{q}{r}$,

$$\begin{aligned} F(u) &:= \frac{u^2}{2} - u\frac{x_0}{c_0} + p\log(u - \frac{at_0}{c_0}) + (1-p)\log u \\ &= F(u_0) - \frac{r^2}{4q}(u-u_0)^4 + \mathbf{O}(u-u_0)^5. \end{aligned}$$

II. Then, setting $' = \nabla_{E'}$ and $\nabla = \sum \frac{\partial}{\partial y_i}$ for $E = (y_1, y_2)$, $z := (r/n)^{1/4}$:

$$\det \left(\begin{array}{ccc} F_1 & F_2 & 0 \\ F_1' & F_2' & F_1 F_2 \left(\frac{H_1}{F_1} + \frac{H_2}{F_2}\right) \\ F_1'' & F_2'' & F_1 F_2 \left(\frac{H_1'}{F_1} + \frac{H_2'}{F_2}\right) \end{array} \right) \Bigg|_{scaling} = -2^4 q^{\frac{13}{2}} \frac{q-1}{(q+1)^5} \left\{ \nabla^3 \log \mathbb{P}^{\mathcal{P}}, \mathbb{X} \right\}_{\nabla} z^{-18}$$

$$+ \left(\left\{ \frac{\partial}{\partial t} \nabla^2 \log \mathbb{P}^{\mathcal{P}}, \mathbb{X} \right\}_{\nabla} + O(q-1) \right) z^{-17} + O(z^{-16})$$

where

$$\mathbb{X} := \frac{\partial^3 \mathbb{Q}}{\partial t^3} + \frac{1}{8} \left(\mathcal{E} - 2t \frac{\partial}{\partial t} - 2 \right) \nabla^2 \mathbb{Q} - \frac{1}{2} \left\{ \nabla^2 \mathbb{Q}, \nabla \frac{\partial \mathbb{Q}}{\partial t} \right\}_{\nabla}, \qquad \text{(independent of } q\text{)}$$

II. Then, setting $' = \nabla_{E'}$ and setting $\nabla = \sum \frac{\partial}{\partial y_i}$ for $E = (y_1, y_2)$, $z^4 = r/n$:

$$\det \left(\begin{array}{ccc} F_1 & F_2 & 0 \\ F_1' & F_2' & F_1 F_2 \left(\frac{H_1}{F_1} + \frac{H_2}{F_2}\right) \\ F_1'' & F_2'' & F_1 F_2 \left(\frac{H_1'}{F_1} + \frac{H_2'}{F_2}\right) \end{array} \right) \Bigg|_{scaling} = -2^4 q^{\frac{13}{2}} \frac{q-1}{(q+1)^5} \left\{ \nabla^3 \log \mathbb{P}^{\mathcal{P}}, \mathbb{X} \right\}_{\nabla} z^{-18}$$

$$+ \left(\left\{ \frac{\partial}{\partial t} \nabla^2 \log \mathbb{P}^{\mathcal{P}}, \mathbb{X} \right\}_{\nabla} + O(q-1) \right) z^{-17} + O(z^{-16})$$

- For $q \neq 1$ ($p \neq 1/2$), the lead term z^{-18} gives a PDE for $\mathbb{P}^{\mathcal{P}}$ ($\mathcal{P}(\tau) \cap E = \emptyset$),

$$\left\{ \nabla^3 \log \mathbb{P}^{\mathcal{P}}, \mathbb{X} \right\}_{\nabla} = 0. \qquad (1)$$

- For $q = 1$ ($p = 1/2$), the lead term z^{-17} gives a (different) PDE:

$$\left\{ \nabla^2 \frac{\partial}{\partial t} \log \mathbb{P}^{\mathcal{P}}, \mathbb{X} \right\}_{\nabla} = 0. \qquad (2)$$

By the **universality**, this means that $\mathbb{Q} := \log \mathbb{P}^{\mathcal{P}}$ satisfies two equations:

$$\left\{\nabla^3 \mathbb{Q}\ ,\ \mathbb{X}\right\}_\nabla = 0 \quad \text{and} \quad \left\{\nabla^2 \frac{\partial}{\partial t}\mathbb{Q}\ ,\ \mathbb{X}\right\}_\nabla = 0.$$

$$\Downarrow$$

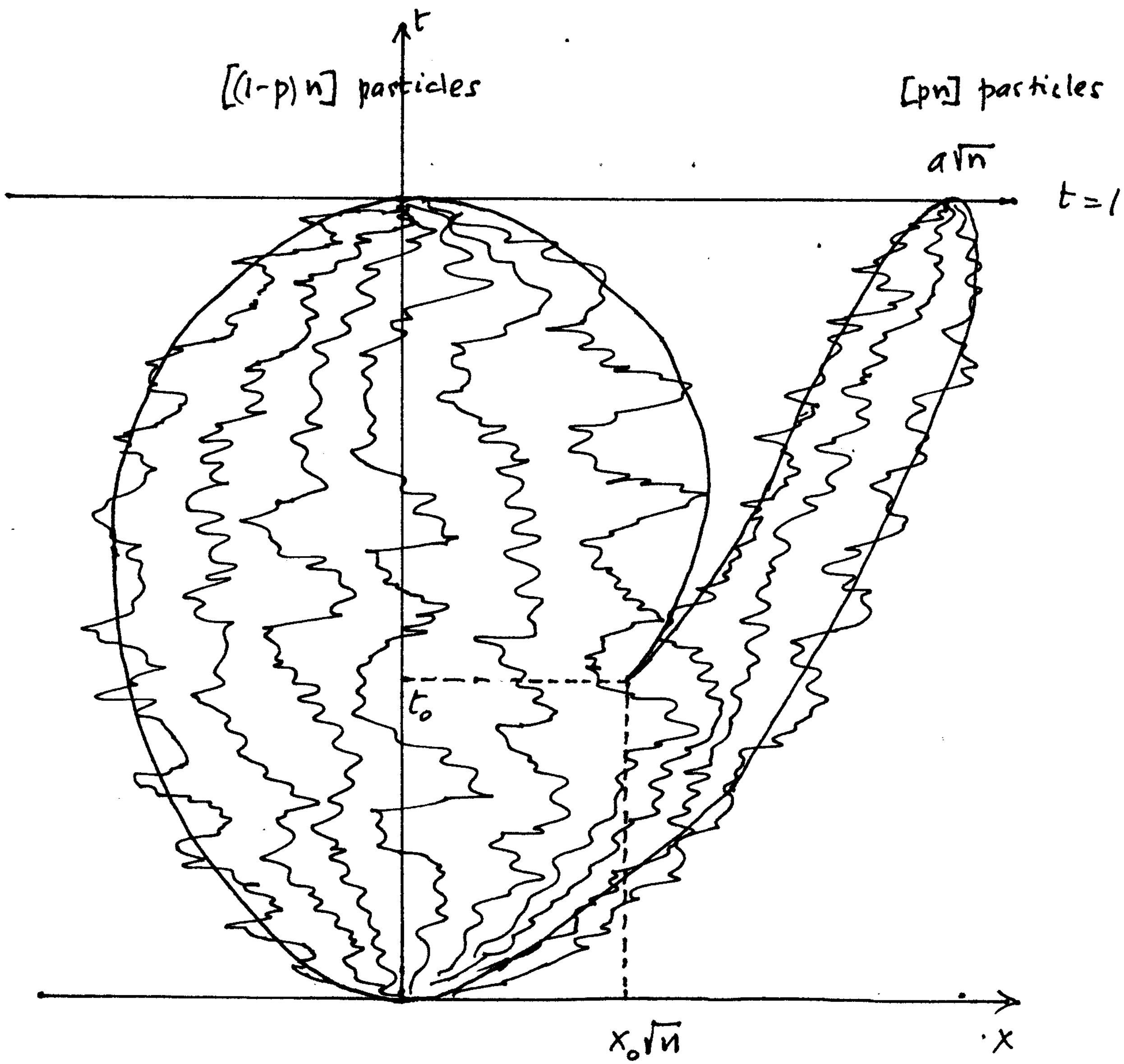
$$\mathbb{X} := \frac{\partial^3 \mathbb{Q}}{\partial t^3} + \frac{1}{8}\left(\mathcal{E} - 2t\frac{\partial}{\partial t} - 2\right)\nabla^2 \mathbb{Q} - \frac{1}{2}\left\{\nabla^2 \mathbb{Q}, \nabla \frac{\partial \mathbb{Q}}{\partial t}\right\}_\nabla = 0$$

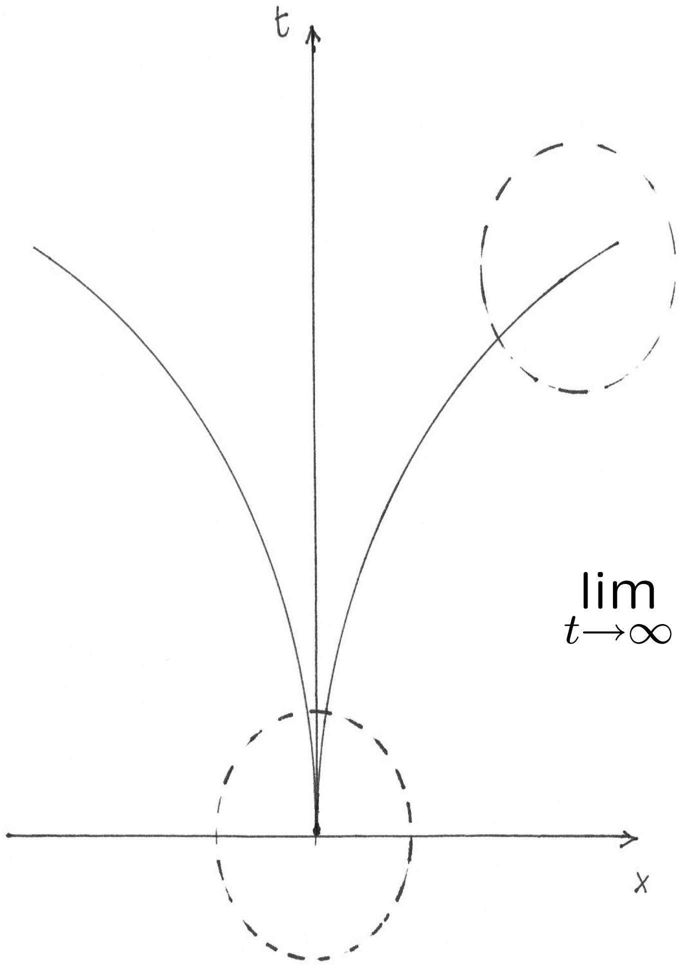
Argument: for $E = (x, y)$ and $\mathbb{Q}(t; x, y) := \log \mathbb{P}^{\mathcal{P}}(t; x, y)$, there exists a function $F(x) \neq 0$ such that

$$\left\{\nabla^3 \mathbb{Q}(t; x, y)\ ,\ \nabla^2 \frac{\partial}{\partial t}\mathbb{Q}(t; x, y)\right\}_\nabla = F(x) O(y - x)^2 \neq 0,$$

4. Limit from Pearcey to Airy

by moving along the cusp $x - x_0 = 2\left(\frac{t-t_0}{3}\right)^{3/2}$,





$$\lim_{t\to\infty} \mathbb{P}^{\mathcal{P}} \left(\frac{\mathcal{P}(t + \frac{8\tau^3}{t}) - 2\left(\frac{t}{3}\right)^{3/2}}{(3t)^{1/6}\left(1 + \frac{8\tau}{(3t)^{2/3}}\right)} \cap E = \emptyset \right)$$

Limit for large t and $E=(\xi_1,\xi_2)$: (Adler-PvM '08)

$$\lim_{t\to\infty}\mathbb{P}^{\mathcal{P}}\left(\frac{\mathcal{P}(t+\frac{8\tau^3}{t})-2\left(\frac{t}{3}\right)^{3/2}}{(3t)^{1/6}\left(1+\frac{8\tau}{(3t)^{2/3}}\right)}\cap -E=\emptyset\right) \;=\; \mathbb{P}^{\mathcal{A}}\left(\mathcal{A}(\tau)\cap E=\emptyset\right)$$

Limit for large t and s,

$$\lim_{\substack{s\\ t\to\infty}}\mathbb{P}^{\mathcal{P}}\left(\frac{\mathcal{P}(s+\frac{8\sigma^3}{s})-2\left(\frac{s}{3}\right)^{3/2}}{(3s)^{1/6}\left(1+\frac{8\sigma}{(3s)^{2/3}}\right)}\cap -E_1=\emptyset,\quad \frac{\mathcal{P}(t+\frac{8\tau^3}{t})-2\left(\frac{t}{3}\right)^{3/2}}{(3t)^{1/6}\left(1+\frac{8\tau}{(3t)^{2/3}}\right)}\cap -E_2=\emptyset\right)$$

$$=\mathbb{P}^{\mathcal{A}}(\mathcal{A}(\sigma)\cap E_1=\emptyset,\quad \mathcal{A}(\tau)\cap E_2=\emptyset)$$

with $t>s$ going to ∞, such that the difference behaves as

$$\frac{t-s}{2(\tau-\sigma)}=(3s)^{1/3}+\frac{9\tau+7\sigma}{(3s)^{1/3}}+2(7\tau^2+10\sigma\tau+7\sigma^2)\frac{1}{s}\quad+\quad O\left(\frac{1}{s^{5/3}}\right).$$

Approximation of Airy by Pearcey for large t:

$$\mathbb{P}^{\mathcal{P}}\left(\frac{\mathcal{P}(t+\frac{8\tau^3}{t})-2\left(\frac{t}{3}\right)^{3/2}}{(3t)^{1/6}\left(1+\frac{8\tau}{(3t)^{2/3}}\right)}\cap -E=\emptyset\right) \;=\; \mathbb{P}^{\mathcal{A}}\left(\mathcal{A}(\tau)\cap E=\emptyset\right)+O\Big(\frac{1}{t}\Big)^{2/3}$$

Better approximation of Airy by Pearcey for large t:

$$\mathbb{P}^{\mathcal{P}}\left(\frac{\mathcal{P}(t+\frac{8\tau^3}{t})-2\left(\frac{t}{3}\right)^{3/2}}{(3t)^{1/6}\left(1+\frac{8\tau}{(3t)^{2/3}}\right)}\cap -E=\emptyset\right)$$

$$=\mathbb{P}^{\mathcal{A}}\left(\mathcal{A}(\tau)\cap\left[8\tau^3\Big(\frac{1}{3t}\Big)^{2/3}+E\Big(1+8\tau\Big(\frac{1}{3t}\Big)^{2/3}\Big)\right]=\emptyset\right)+O\Big(\frac{1}{t}\Big)^{4/3}$$

Proof: Use saddle point method, combined with the Pearcey PDE. ∎

Pearcey kernel

$$K^{\mathcal{P}}_{ST}(X,Y)dY = \frac{dY}{4\pi^2 i}\int_{\mathcal{X}} dV \int_{-\infty}^{\infty} dU e^{-\frac{U^4}{4}-\frac{TU^2}{2}-iUY} e^{\frac{V^4}{4}-\frac{SV^2}{2}+VX} \frac{1}{iU-V}$$

$$-\begin{cases} 0 & \text{for } T-S \le 0 \\ \frac{dY}{\sqrt{2\pi(T-S)}} e^{-\frac{(X-Y)^2}{2(T-S)}} & \text{for } T-S > 0 \end{cases}$$

Airy kernel

$$K^{\mathcal{A}}_{s,t}(x,y)dy \;=\; dy\int_0^{\infty} e^{-\lambda(s-t)} \mathbf{A}(x+\lambda)\mathbf{A}(y+\lambda)d\lambda$$

$$-\begin{cases} 0 & \text{for } \quad t-s \le 0 \\ \frac{dy}{\sqrt{4\pi(t-s)}} e^{\frac{1}{12}(t-s)^3 - \frac{(x-y)^2}{4(t-s)} - \frac{1}{2}(t-s)(y+x)} & \text{for } \quad t-s > 0 \end{cases}$$

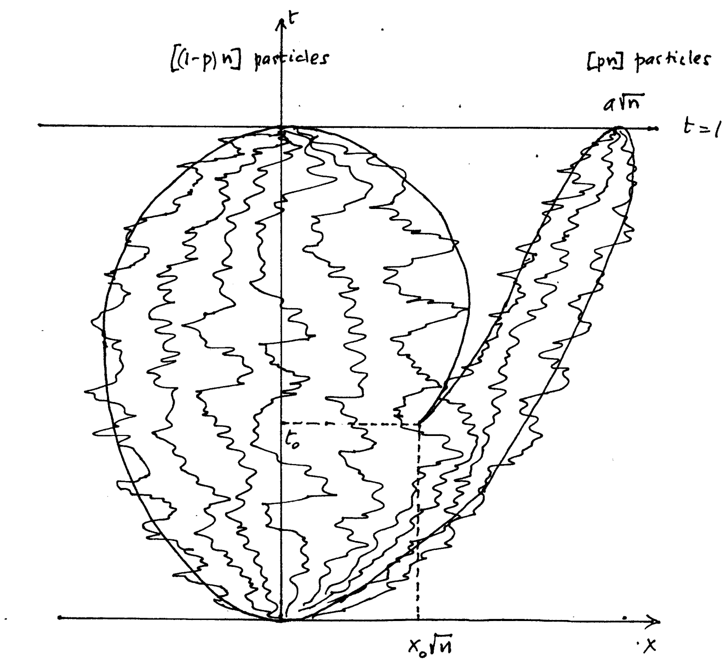
$$-\frac{U^4}{4} - \frac{TU^2}{2} - iUY \Bigg|_{\substack{T = \frac{1}{9z^6}\left(2 + e^{18tz^4}\right) \\ Y = \frac{2}{27}(3T)^{3/2} - (3T)^{1/6}y - 8\frac{(t^2+y)t}{(3T)^{1/2}} \\ U = \frac{ue^{3z^4 t}}{\sqrt{3}z^3} \\ u = \frac{1}{\sqrt{3}}\left(3\omega_u z^4 - i\right)}}$$

$$= \underbrace{\frac{i}{3}\left({\omega_u}^2 + 3y\right)\omega_u}_{\text{Airy cubic}} + Z_0(t,y) + O\left(z^4\right),$$

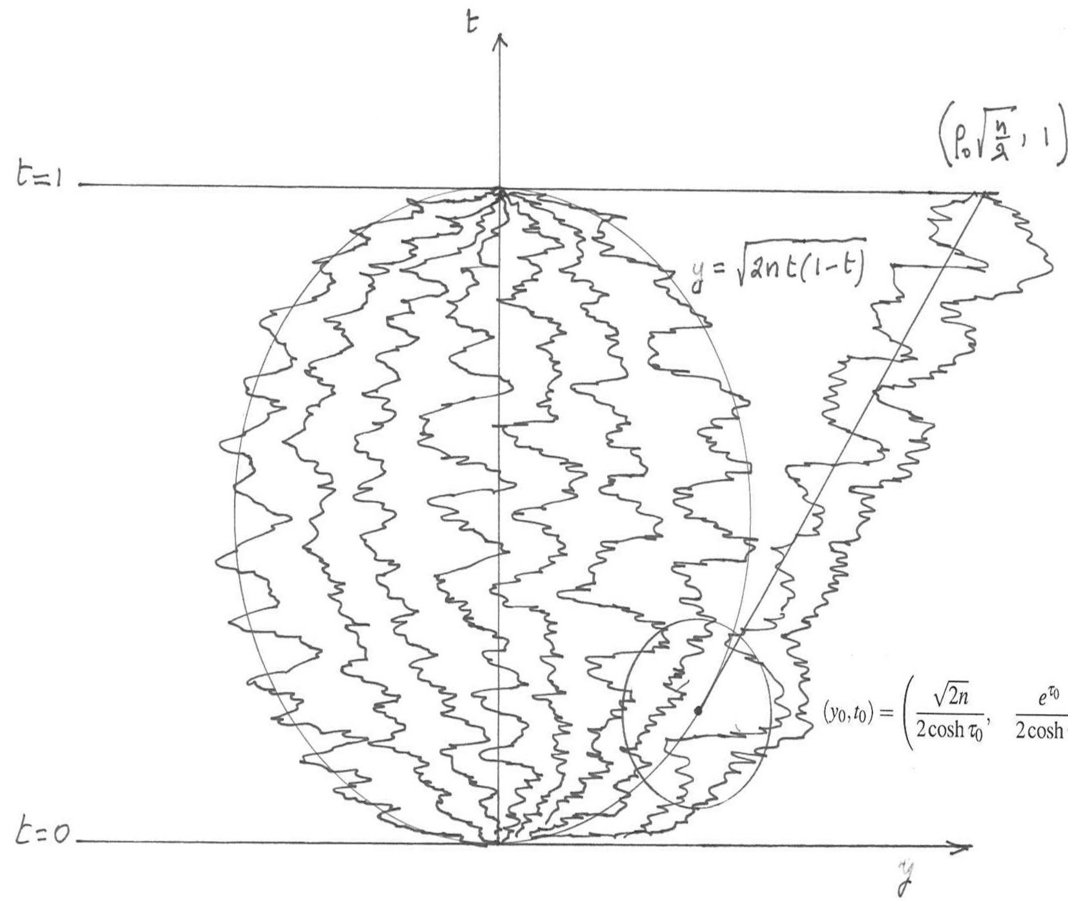
where

$$Z_0(t,y) := -\frac{1}{108}z^{-12} - \frac{t}{9}z^{-8} + \frac{1}{3}(y - 4t^2)z^{-4} + \frac{4t}{3}(3y - 7t^2).$$

Remark: Letting $p \to 0$, $n \to \infty$, such that $[pn] =$ fixed k:



$\Rightarrow$

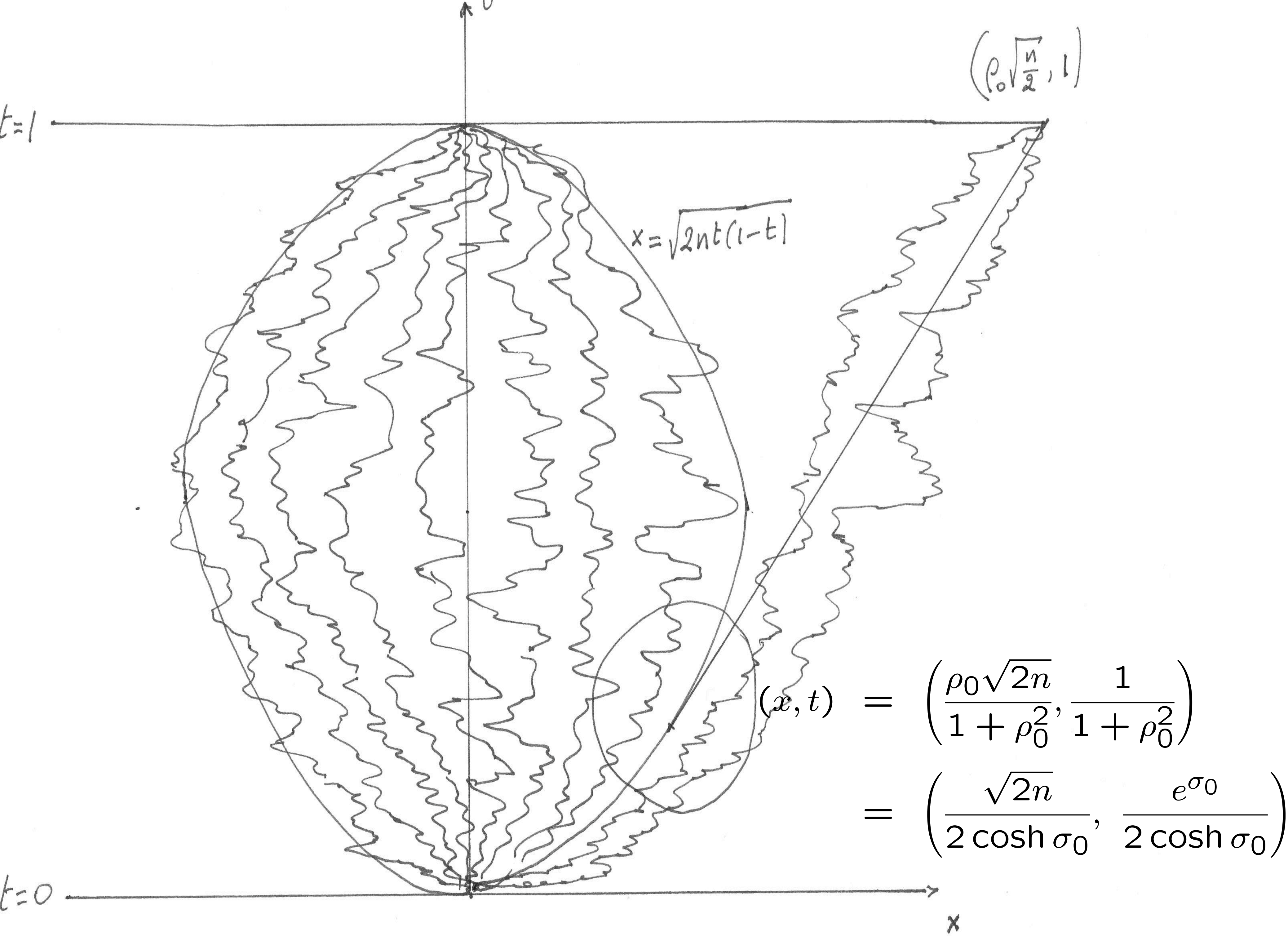


5. The Airy process $\mathcal{A}^{(k)}(t)$ with k outliers

$$\mathbb{P}^{0a}_{Br}\left(\text{all } x_j(t) \in E^c\right) := \mathbb{P}\left(\text{all } x_j(t) \in E^c \;\middle|\; \begin{array}{c} \text{all } x_j(0) = 0 \\ k \text{ right paths end up at } a \text{ at } t = 1 \\ (n-k) \text{ paths end up at } 0 \text{ at } t = 1 \end{array}\right).$$

Keep k fixed and let $n \to \infty$:

(Adler, Delépine, PvM: CPAM '08.)
(related to work of Baik, Ben Arous, Péché '04 and Baik '05 on estimating the covariance matrix of two Gaussian populations.)



$$(x,t) = \left(\frac{\rho_0\sqrt{2n}}{1+\rho_0^2}, \frac{1}{1+\rho_0^2}\right)$$
$$= \left(\frac{\sqrt{2n}}{2\cosh\sigma_0},\ \frac{e^{\sigma_0}}{2\cosh\sigma_0}\right)$$

$$\rho_0 = e^{-\sigma_0}$$

Look through the "Airy" microscope at **the fluctuations of the paths** about the point on the curve, for very large n:

$$(x,t) = \left(\frac{\sqrt{2n}}{2\cosh\sigma}, \frac{e^{\sigma}}{2\cosh\sigma} \right) \in \text{ curve} \quad \text{ for fixed } \left\{ \begin{array}{l} \sigma \leq \sigma_0 \\ \text{equivalently} \\ t \leq \frac{1}{1+\rho_0^2} \end{array} \right\}$$

$\Longrightarrow$ **Phase transition to new process** $\mathcal{A}^{(k)}(\tau)$: (Airy process with k outliers)

$$\lim_{n\to\infty} \mathbb{P}_{Br}^{(0,\rho_0\sqrt{n/2})} \left(\text{all } x_i \left(\frac{1}{1+e^{-2(\sigma+\tau n^{-1/3})}} \right) \in \frac{\sqrt{2n} + \frac{E^c}{\sqrt{2}n^{1/6}}}{2\cosh(\sigma+\tau n^{-1/3})} \right)$$

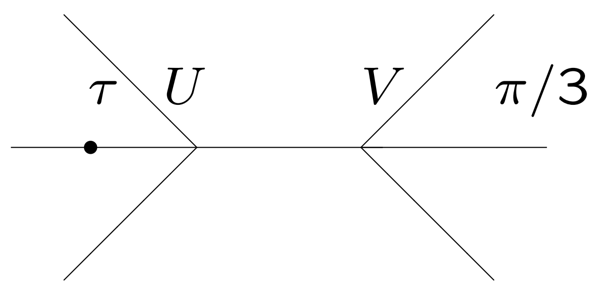
$$= \begin{cases} \mathbb{P}(\mathcal{A}(\tau) \cap E = \emptyset) = \det(I - \mathbf{A})_E & \text{for} \quad 0 \leq \sigma < \sigma_0 \\ \qquad\qquad (\textit{Airy process}) & \\ \mathbb{P}(\mathcal{A}^{(k)}(\tau) \cap E = \emptyset) = \det\left(I - \mathbf{A}_{\tau}^{(k)}\right)_E & \text{for} \quad \sigma = \sigma_0 = \left\{ \begin{array}{l} \text{point of} \\ \text{tangency} \end{array} \right. \end{cases}$$

(*Baik, Ben Arous, Péché kernel: Airy process with k outliers*) where

$$\mathbf{A}(x,y) = \int_0^\infty dw A(w+x)A(w+y) = \frac{A(x)A'(y) - A'(x)A(y)}{x-y}$$

$$\mathbf{A}_\tau^{(k)}(x,y) = \int_0^\infty dw A_k^-(w+x;\tau)A_k^+(w+y;\tau)$$

$$= \frac{1}{(2\pi i)^2}\int dU \int dV e^{-\frac{U^3}{3}+xU+\frac{V^3}{3}-yV} \frac{1}{U-V}\left(\frac{V-\tau}{U-\tau}\right)^k.$$



with (C is a path running from $\infty e^{5i\pi/6}$ to $\infty e^{i\pi/6}$, passing under $-i\tau$)

$$A(u) \ := \ \int_C e^{\frac{1}{3}ia^3+iau}\frac{da}{2\pi} \quad \text{and} \quad A_k^\pm(u;\tau) := \int_C e^{\frac{1}{3}ia^3+iau}(\mp ia-\tau)^{\pm k}\frac{da}{2\pi}$$

Then

$$\mathbb{Q}(\tau,x) := \log \mathbb{P}(\sup \mathcal{A}^{(k)}(\tau) \le x) = \log\det\left(I - K_\tau^{(k)}\right)_{(x,\infty)}$$

satisfies a PDE ($\{f,g\}_x$ is the Wronskian with regard to x.),

$$\left\{\frac{\partial^3\mathbb{Q}}{\partial\tau\partial x^2}, \left[\begin{array}{l} \left(k-\frac{\partial^2\mathbb{Q}}{\partial\tau\partial x}\right)^2\left(\frac{\partial^3\mathbb{Q}}{\partial x^3}\right)+\left(k-\frac{\partial^2\mathbb{Q}}{\partial\tau\partial x}\right)\frac{\partial}{\partial\tau}\left(2\tau\frac{\partial^2\mathbb{Q}}{\partial x\partial\tau}+\frac{\partial^2\mathbb{Q}}{\partial\tau^2}\right) \\ +\frac{\partial^3\mathbb{Q}}{\partial\tau\partial x^2}\left(2k\frac{\partial^2\mathbb{Q}}{\partial x^2}+2\frac{\partial\mathbb{Q}}{\partial\tau}-xk\right)+\frac{\partial}{\partial x}\left(\frac{1}{4}(\frac{\partial^2\mathbb{Q}}{\partial\tau^2})^2+\tau\frac{\partial^2\mathbb{Q}}{\partial\tau^2}\frac{\partial^2\mathbb{Q}}{\partial\tau\partial x}\right) \end{array}\right]\right\}_x$$

$$-\frac{1}{2}\left(\frac{\partial^3\mathbb{Q}}{\partial\tau\partial x^2}\right)^2\left(\frac{\partial^3\mathbb{Q}}{\partial\tau^3}-4\frac{\partial^2\mathbb{Q}}{\partial\tau\partial x}\frac{\partial^3\mathbb{Q}}{\partial x^3}\right)=0,$$

Limits to the Airy process and to the Pearcey process:

(I) to the Airy process For $\tau \to -\infty$, one has

$$\begin{aligned}
&\mathbb{P}(\sup \mathcal{A}^{(k)}(\tau) \le x) \\
&= \mathbb{P}\left(\sup \mathcal{A}(\tau) \le x\Big(1 + \frac{k}{3\tau^3}\Big) + \frac{k}{\tau}\left(1 + \frac{7k}{12\tau^3}\right)\right) + O\left(\frac{1}{\tau^5}\right).
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(\sup \mathcal{A}^{(k)}(\tau)) &= \mathbb{E}(\sup \mathcal{A}(\tau))\left(1 - \frac{k}{3\tau^3}\right) - \frac{k}{\tau} - \frac{k^2}{4\tau^4} + O\left(\frac{1}{\tau^5}\right) \\
\mathit{var}(\sup \mathcal{A}^{(k)}(\tau)) &= \mathit{var}(\sup \mathcal{A}(\tau))\left(1 - \frac{2k}{3\tau^3}\right) + O\left(\frac{1}{\tau^5}\right).
\end{aligned}$$

Proof: Use the k-Airy PDE. ■

(II) to the Pearcey process When the number of outliers $k \to \infty$: Mark Adler's lecture.