

“Angular” matrix integrals

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J. Stat. Phys. 2007, math-ph/0610049.

J.-B. Z. *On the large N limit of matrix integrals over the orthogonal group,*

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Matrix integrals

$$Z_G = \int_G D\Omega \exp N\beta \Re e (\text{tr} (\Omega J)) \quad (1)$$

$$Z^{(G)} = \int_G D\Omega \exp N\beta \Re e (\text{tr} (A\Omega B\Omega^\dagger)) \quad (2)$$

over a compact group G , are frequently encountered in physics (and in maths) : “Bessel matrix functions” or “angular matrix integrals”.

$G = O(N), U(N), Sp(N)$, with respectively $\beta = 1, 2, 4$.

Invariance under $J \mapsto \Omega_1 J \Omega_2$ and $A \mapsto \Omega_1 A \Omega_1^\dagger, B \mapsto \Omega_2 B \Omega_2^\dagger$, resp.

$\Rightarrow Z_G$ expressible as a sum of $\prod_i \text{tr} (J J^\dagger)^{p_i}$ and $Z^{(G)}$ as a sum of

$\prod_i \text{tr} A^{p_i} \prod_j \text{tr} B^{q_j}$

Matrix integrals

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over a compact group G , are frequently encountered in physics (and in maths) : “Bessel matrix functions”. Mostly studied for $G = U(N)$ ($\beta = 2$).

What happens for other groups, e.g. $G = O(N)$ ($\beta = 1$), $Sp(N)$ ($\beta = 4$)?

- If A and B are both real **skew-symmetric** (i.e. in the Lie algebra of $G = O(N)$), Z is known exactly from the work of **Harish-Chandra '57**.

Also correlation functions are known [**Eynard *et al***].

- If A and B are both real **symmetric**, much more complicated and elusive, [**Brézin & Hikami '02-06, Bergère & Eynard 08**].

- if they are neither, ...?

- Expect simplification as $N \rightarrow \infty$ [**Weingarten '78**]. Universality of (1), (2).

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Outline of this talk

- Review of (2) in the Harish-Chandra case (A and B in the Lie algebra)
- Correlation functions
- The integral (2) in the symmetric case
- The large N limit

1. The Harish-Chandra integral

For A and B in the *Lie algebra* \mathfrak{g} of G , in fact in a *Cartan algebra*

$$Z^{(G)} = \int_G D\Omega \exp N\beta \operatorname{tr} (A\Omega B\Omega^\dagger) = \operatorname{const.} \sum_{w \in \mathcal{W}} \frac{\exp N\beta \operatorname{tr} AB^w}{\Delta_G(A)\Delta_G(B^w)} \quad (3)$$

$\Delta_G(A) := \prod_{\alpha > 0} \langle \alpha, A \rangle$, a product over the positive roots, \mathcal{W} the Weyl group.

More concretely, for $G = U(N)$, take $A = \operatorname{diag} (a_i)$, $B = \operatorname{diag} (b_i)$

$$Z^{(U)} = \operatorname{const.} \frac{\det e^{\beta N a_i b_j}}{\prod_{i < j} (a_i - a_j)(b_i - b_j)} \quad \text{[Itzykson-Z '80]}$$

and for $G = O(N)$, take A and B both skew-symmetric, block-diagonal form

$$A = \operatorname{diag} \left(\begin{array}{cc} 0 & a_i \\ -a_i & 0 \end{array} \right)_{i=1, \dots, m}, \quad B \text{ likewise}$$

$$Z^{(O)} = \operatorname{const.} \frac{\det(2 \cosh 2N a_i b_j)}{\Delta_O(a)\Delta_O(b)}$$

for $O(N)$, $N = 2m$, with $\Delta_O(a) = \prod_{1 \leq i < j \leq m} (a_i^2 - a_j^2)$.

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for $O(N)$, $N = 2m + 1$ with $\Delta_O(a) = \prod_{i=1}^m a_i \prod_{1 \leq i < j \leq m} (a_i^2 - a_j^2)$.

Proofs of this H-C formula

– Heat kernel

$Z' = t^{-\frac{1}{2} \dim G} \int_G d\Omega e^{-\frac{1}{2t} N\beta \text{tr}(A - \Omega B \Omega^\dagger)^2}$ satisfies $(N\beta \frac{\partial}{\partial t} - \frac{1}{2} \Delta_A) Z' = 0$ and boundary cond $Z' \xrightarrow[t \rightarrow 0]{} \text{const} \int_G d\Omega \delta(A - \Omega B \Omega^\dagger)$. Rewrite in “radial coordinates” a_i using the expression of the Laplacian

$$\Delta_A = \Delta_G^{-2}(A) \sum_i \partial_i \Delta_G^2(A) \partial_i + \text{angular part} = \Delta_G^{-1}(A) \sum_i \partial_i^2 \Delta_G(A) + \text{ang.}$$

Thus $Z'' := \Delta_G(A) \Delta_G(B) Z'$ solution of $(N\beta \partial_t - \frac{1}{2} \sum_i \partial_i^2) Z'' = 0$ and is an alternate sum over the Weyl group of $\exp -\frac{1}{2t} N\beta (a_i - b_i^w)^2$. QED

– Character expansion ...

– Exact semi-classical expression [Duistermaat-Heckman theorem]

Stationary points of $\text{tr}(A \Omega B \Omega^\dagger)$ w.r.t Ω satisfy $[A, \Omega B \Omega^\dagger] = 0$ and are for generic A and B in \mathfrak{g} , (distinct eigenvalues), in 1-to-1 correspondence with the elements of \mathcal{W} , whence the numerator of the H-C formula. Then the Gaussian fluctuations around each of these stationary points yield the denominator of the H-C formula.

Correlation functions

What about the associated “correlation functions” of invariant traces

$$\int \mathbf{D}\Omega e^{-\text{tr}A\Omega B\Omega^\dagger} \prod \text{tr}(A^{p_1} \Omega B^{q_1} \Omega^\dagger A^{p_2} \dots) \quad ?$$

(still invariant under $A \rightarrow \Omega_1 A \Omega_1^\dagger$, $B \rightarrow \Omega_2 B \Omega_2^\dagger$)

Is there still some localization property?

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Is there still some localization property?

$$\int \mathbf{D}\Omega e^{-\text{tr}A\Omega B\Omega^\dagger} F(A, \Omega B\Omega^\dagger) = c_n \sum_{w \in W} \frac{e^{-\text{tr}AB^w}}{\Delta(A)\Delta(B^w)} \int_{\mathfrak{n}_+ = [\mathfrak{b}, \mathfrak{b}]} \mathbf{D}T e^{-\text{tr}TT^\dagger} F(A + T, B^w + T^\dagger)$$

with \mathfrak{b} the Borel subalgebra (upper triangular matrices), \mathfrak{n}_+ its “derived ideal” (generated by positive roots) (strictly upper triangular matrices), whence a *finite* number of correction terms to the semi-classical approximation.

[Eynard–Prats Ferrer ’04, P F–E–Di Francesco–Z ’06, Bertola–P F ’08] generalizing or making more explicit previous expressions [Morozov ’92, Shatashvili ’93].

2. The integral (2) in the symmetric case

$$Z^{(G)} = \int_G D\Omega \exp N\beta \operatorname{tr} (A\Omega B\Omega^\dagger)$$

for $A = A^\dagger$ and $B = B^\dagger$.

For $G = U(N)$, A and B hermitian rather than *anti*hermitian, no difference, HCIZ formula works.

For $G = O(N)$, A and B real symmetric, ??

For $G = Sp(N)$, A and B quaternionic self-dual. ??

A case much studied in the recent years [Guhr–Kohler '00, Brézin–Hikami '02–06, Bergère–Eynard '08, Collins–Guionnet–Maurel-Segala '08]

Many nice features

- finite (semi-classical) expansion and “ τ -expansion” for β an *even* integer

$$Z^{(G)} = \sum_{\sigma \in \mathfrak{S}_N} \frac{e^{N\beta a_i b_{\sigma(j)}}}{\Delta(a)^\beta \Delta(b^\sigma)^\beta} \mathcal{P}_{\beta, N}(A, B^\sigma) \quad (5)$$

with $\mathcal{P}_{\beta, N}(A, B)$ a *polynomial* of degree $\beta/2$ in each variable $\tau_{ij} := (a_i - a_j)(b_i - b_j)$,

$$\text{hence } Z^{(G)} = \sum_{\sigma \in \mathfrak{S}_N} \frac{e^{N\beta a_i b_{\sigma(j)}}}{\Delta(a)^{\beta/2} \Delta(b^\sigma)^{\beta/2}} P_{\beta, N}\left(\frac{1}{\tau^\sigma}\right).$$

- Differential equation [Bergère-Eynard '08]: take A and B diagonal

Let $K = \{(K)_{ij}\}$ be the *matrix* differential operator

$$K_{ii} = \frac{\partial}{\partial a_i} + \frac{\beta}{2} \sum_{j \neq i} \frac{1}{a_i - a_j} \quad \text{and for } i \neq j, \quad K_{ij} = -\frac{\beta}{2} \frac{1}{a_i - a_j} \quad \text{and let}$$

$M_{ij} := Z \langle |\Omega_{ij}|^2 \rangle$. Then $Z = \sum_i M_{ij} = \sum_j M_{ij}$ and

$$\sum_j K_{ij} M_{jk} = N\beta M_{ik} b_k \quad \text{no summation over } k$$

$\sum_k M_{ik} = \sum_i M_{ik} = Z$ and $\sum_j K_{ij} M_{jk} = (N\beta) M_{ik} b_k$. Can iterate that equation to get

$$\sum_j K_{ij}^p M_{jk} = M_{ik} (N\beta)^p b_k^p$$

and summing over i and k

$$\underbrace{\left(\sum_{ij} K_{ij}^p \right)}_{\text{a differential operator of order } p} Z = (N\beta)^p \text{tr} B^p Z. \quad (6)$$

a differential operator of order p

Two remarks

1. *This solves the following problem :*

Define the differential operator $D_p(\partial/\partial A)$ by

$$D_p(\partial/\partial A) e^{N \text{tr} AB} = N^p \text{tr} B^p e^{N \text{tr} AB}$$

If D_p acts on *invariant functions* $F(A) = F(\Omega A \Omega^\dagger)$, how to write it in terms

of $\partial/\partial a_i$? For $G = U(N)$,

$$D_p(\partial/\partial A) = \text{tr} \left(\frac{\partial}{\partial A} \right)^p := \sum_{i_1, \dots, i_p} \frac{\partial}{\partial A_{i_1 i_2}} \frac{\partial}{\partial A_{i_2 i_3}} \cdots \frac{\partial}{\partial A_{i_p i_1}}$$

and

$$D_p = \frac{1}{\Delta(a)} \sum_i \left(\frac{\partial}{\partial a_i} \right)^p \Delta(a) .$$

$\Delta(a) = \prod_{i < j} (a_i - a_j)$ (a non trivial calculation !) [Itzykson–Z ’80].

In general, “radial” expression of D_p is given by $D_p = \sum_{i,j} (K^p)_{ij}$

2. Connection with Calogero

Note that by construction the $D_p := \sum_{ij} K_{ij}^p$ commute.

Consider $H_p := \Delta(a)^{\beta/2} D_p \Delta(a)^{-\beta/2}$. $H_2 = \sum_i \partial_i^2 + \frac{\beta}{2} \left(1 - \frac{\beta}{2} \right) \sum_{i \neq j} \frac{1}{(a_i - a_j)^2}$ is the Calogero Hamiltonian, and the H_p are the higher conserved quantities.

3. Large N limit

Expect things to simplify as $N \rightarrow \infty$ [Weingarten '78]. Look at the “free energies” :

$$W_G(J.J^\dagger) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_G$$

and

$$F_G(A, B) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z^{(G)}$$

Then $W(X)$ and $F(A, B)$ are, *up to an overall factor*, independent of $G = O(N), U(N)!$

(Not true at finite N !)

More precisely,

$$W_{\text{O}}(J.J^\dagger) = \frac{1}{2} W_{\text{U}}(J.J^\dagger) \quad (7)$$

and

$$F_{\text{O}}(A, B) = \frac{1}{2} F_{\text{U}}(A, B) \quad (8)$$

Intuitively, counting of # of degrees of freedom : $\beta N^2/2$ real parameters in $\text{O}(N)$, $\text{U}(N)$.

Actual proof relies either on inspection of explicit formulae (“Harish-Chandra case”), or on the use of differential equations satisfied by \mathcal{Z} , resp. Z , which simplify in the $N \rightarrow \infty$ limit.

For $Z_0 = \int_{O(N)} \mathbf{D}O \exp N \text{tr}(J.O)$, follow the steps of [Brézin-Gross '80]: the trivial identity $\sum_j \frac{\partial^2 Z_0}{\partial J_{ij} \partial J_{kj}} = N^2 \delta_{ik} Z_0$ is reexpressed in terms of the eigenvalues λ_i of the real symmetric matrix $J.J^t$:

$$4\lambda_i \frac{\partial^2 Z_0}{\partial \lambda_i^2} + \sum_{j \neq i} \frac{2\lambda_j}{\lambda_j - \lambda_i} \left(\frac{\partial Z_0}{\partial \lambda_j} - \frac{\partial Z_0}{\partial \lambda_i} \right) + 2N \frac{\partial Z_0}{\partial \lambda_i} = N^2 Z_0 .$$

Writing as above $Z_0 = e^{N^2 W_0}$ and dropping subdominant terms in the large N limit, with W_0 and $W_i := N \partial W_0 / \partial \lambda_i$ of order 1, we get

$$4\lambda_i W_i^2 + 2W_i + \frac{1}{N} \sum_{j \neq i} \frac{2\lambda_j}{\lambda_j - \lambda_i} (W_j - W_i) = 0 \quad (9)$$

which is precisely the equation satisfied by $\frac{1}{2} W_U$ in [B-G]. This, together with appropriate boundary conditions, suffices to complete the proof of (7).

An explicit expression of W_U is known [O' Brien-Z '84]

$$W_U(J.J^\dagger) = \sum_{n=1}^{\infty} \sum_{\alpha \vdash n} W_\alpha \frac{\text{tr}_\alpha J.J^\dagger}{\prod_p (\alpha_p! p^{\alpha_p})}$$

$$W_\alpha = (-1)^n \frac{(2n + \sum \alpha_p - 3)!}{(2n)!} \prod_{p=1}^n \left(\frac{-(2p)!}{p!(p-1)!} \right)^{\alpha_p},$$

where $\alpha \vdash n$ denotes a partition of $n = \alpha_1.1 + \alpha_2.2 + \dots + \alpha_n.n$ and

$$\text{tr}_\alpha(X) := \prod_{p=1}^n \frac{1}{N} (\text{tr} X^p)^{\alpha_p}.$$

For $Z^{(O)} = \int_{O(N)} \mathbf{D}O \exp N \text{tr}(AOBO^t)$, take A and B both skew-symmetric, or both symmetric.

- A and B both skew-symmetric [Harish-Chandra]

block-diagonal form $A = \text{diag} \left(\left(\begin{array}{cc} 0 & a_i \\ -a_i & 0 \end{array} \right)_{i=1, \dots, m} \right)$, B likewise, recall

$$Z^{(O)} = \text{const.} \frac{\det(2 \cosh 2Na_i b_j)}{\Delta_O(a) \Delta_O(b)}$$

(for $O(N = 2m)$), with $\Delta_O(a) = \prod_{1 \leq i < j \leq m} (a_i^2 - a_j^2)$.

Regard A as $N \times N$ anti-Hermitian, eigenvalues $A_j = \pm ia_j$, B likewise. Easy to check that as $N \rightarrow \infty$,

$$Z^{(U)}(A, B) = \frac{\det(e^{2NA_i B_j})}{\Delta(A) \Delta(B)} \sim \left(\frac{(\det(e^{2Na_i b_j})_{1 \leq i, j \leq m})}{\Delta_O(a) \Delta_O(b)} \right)^2 = (Z^{(O)}(A, B))^2$$

- A and B both symmetric

Can take them in diagonal form $A = \text{diag } a_i$, $B = \text{diag } b_i$

Then Bergère-Eynard equation $D_p Z = (N\beta)^p \text{tr } B^p Z$ (6), **in the large N limit**, yields

$$\sum_i \left(\frac{N}{\beta} \frac{\partial F^{(G)}}{\partial a_i} + \frac{1}{2N} \sum_{j \neq i} \frac{1}{a_i - a_j} \right)^p = \text{tr } B^p \quad (10)$$

Hence $F^{(O)}$ ($\beta = 1$) satisfies same set of equations as $\frac{1}{2} F^{(U)}$ ($\beta = 2$), QED.

Another proof [Guionnet-Zeitouni '02] (for A and B symmetric) as a by-product of the construction of F_G as the unique solution of Matytsin '94 flow: β dependence is explicit. (also [Collins-Guionnet-Maurel-Segala '08])

A Conjecture $F^{(O)}(A, B) = \frac{1}{2}F^{(U)}(A, B)$ extends to A and B generic (neither symmetric, nor skew-symmetric). Some evidence from power expansion.

Origin of this universality? Diagrammatics ? Relation between

$$Z = \int_G d\Omega \exp N\beta \Re(\text{tr}(\Omega J)) \text{ and } Z = \int_G d\Omega \exp N\beta \Re(\text{tr}(A\Omega B\Omega^\dagger)) ?$$

In the case of $U(N)$, yes [P. Zinn-Justin-Zuber '03, Collins '03].

For $O(N)$?? no such simple relation ...

Heuristic argument: consider the two-matrix integrals

$$Z_{2RS} = \int_{\substack{\text{real} \\ \text{symmetric}}} dA dB e^{-N \text{tr}(V(A)+W(B)-AB)} \sim e^{-N^2 F_{2RS}} \quad (11)$$

$$Z_{2CH} = \int_{\substack{\text{complex} \\ \text{hermitian}}} dA dB e^{-2N \text{tr}(V(A)+W(B)-AB)} \sim e^{-N^2 F_{2CH}} , \quad (12)$$

over real symm., resp complex Hermitian, matrices. It is “well known” that for large N , same perturbative expansion of F , up to rescalings and a global factor 2 i.e.

$$F_{2CH} = 2F_{2RS} \quad (13)$$

On the other hand, if we diagonalize the matrices $A = \text{diag}(a_i)$, $B = \text{diag}(b_i)$, we see that (11-12) reduce to

$$Z_{2RS} = \int d\mathbf{a} d\mathbf{b} |\Delta(a)\Delta(b)| e^{-N \sum_i (V(a_i)+W(b_i))} \int DO e^{N \text{tr} A O B O^t} \quad (14)$$

$$Z_{2CH} = \int d\mathbf{a} d\mathbf{b} (\Delta(a)\Delta(b))^2 e^{-2N \sum_i (V(a_i)+W(b_i))} \int DU e^{2N \text{tr} A U B U^t} . \quad (15)$$

If integrals (14-15) dominated as $N \rightarrow \infty$ by a saddle point configuration, scaling $F_O(A, B) = \frac{1}{2} F_U(A, B)$ of angular part *consistent* with the scaling (13) of the integral.

Particular case where A is of finite *rank* r . Then in the expansion of $F = \sum_{p,q} \prod(\frac{1}{N} \text{tr} A^{p_i}) \prod(\frac{1}{N} \text{tr} B^{q_j})$, terms with a single trace of A dominate.

In the $U(N)$ case (and $N \rightarrow \infty$) ([IZ '80])

$$F^{(U)} \sim \sum_{p=1}^{\infty} \frac{1}{p} \left(\frac{1}{N} \text{tr} A^p \right) \psi_p(B)$$

where $\psi_p(B) = p$ -th “non-crossing cumulant” of B ([Brézin–Itzykson–Parisi–Z ’78, Speicher ’94]).

Application : ★ The β universality of F_G first pointed out in that finite rank case [Marinari, Parisi, Ritort, '94],

Spin glass Hamiltonian with n replicas of N Ising spins

$$\mathcal{H} = \sum_{i,j=1}^N \underbrace{\sum_{a=1}^n \sigma_i^a \sigma_j^a}_{\Omega_{ij}} O_{ij} \quad \Omega \text{ of rank } \leq n$$

with a coupling O_{ij} , a real, orthogonal, symmetric matrix with an equal number of ± 1 eigenvalues, $O = V^t . D . V$.

Have to compute $Z = \int_{O(N)} dV \exp \beta \text{tr} D V \Omega V^t$.

Now according to Marinari, Parisi, Ritort, pretend you integrate over the unitary group,

compute $\sum \frac{1}{p} \text{tr} \Omega^p \psi_p(D) =: \text{tr} G(\Omega)$

and (with some insight ...) the correct formula is $\frac{1}{2} G(2\Omega) ! \dots$

Proved later by **Collins, Collins and Sniady, Guionnet & Maida**

Conclusion and Open issues

- More explicit formulae for Z , F
- A priori argument for universality, graphical argument ?
- Relations with integrability: D-H localization, finite semi-classical expansions, Calogero, ...