Formulas and Asymptotics for the Asymmetric Simple Exclusion Process

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ASEP: Particles are at integer sites on the line. Each particle waits exponential time, then

(1) with probability p it moves one step to the right if the site is unoccupied, otherwise it stays put;

(2) with probability $q = 1 - p$ it moves one step to the left if the site is unoccupied, otherwise it stays put.

T(totally)ASEP: $p = 1$. Random matrix connection (Johansson, 2000): Initial configuration $\mathbb{Z}^-$. Probability that at time t the particle initially at $-m$ has moved at least n times is a probability in $\beta = 2$ Laguerre ensemble.

N-particle ASEP. Possible configuration

$$X = \{x_1, \ldots, x_N\}, \quad x_1 < \cdots < x_N, \quad x_i \in \mathbb{Z}.$$

Initial configuration $Y = \{y_1, \ldots, y_N\}$. We obtain integral formulas for

$P_Y(X;t) =$ probability that at time t the system is in configuration X.

$\mathbb{P}(x_m(t) = x) =$ probability that at time t the mth particle from the left is at x.

For TASEP: Schütz (1997): $P_Y(X;t) = N \times N$ determinant.

Differential Equation

In ASEP there is a differential equation with boundary conditions and an initial condition whose solution gives $P_Y(X;t)$.

(Idea goes back to H. Bethe (1931).)

Unknown function $u(X;t)$ or $u(X)$, $\;X=(x_1,\ldots,x_N)\in\mathbb{Z}^N$.

Differential equation (master equation):

$$\frac{d}{dt}\,u(X;t)$$

$$=\sum_i [p\,u(\ldots,x_i-1,\ldots)+q\,u(\ldots,x_i+1,\ldots)-u(X)].$$

Boundary conditions ($i = 1, \ldots, N-1$):

$$u(\ldots, x_i, x_i + 1, \ldots)$$

$$= p\, u(\ldots, x_i, x_i, \ldots) + q\, u(\ldots, x_i + 1, x_i + 1, \ldots).$$

Initial condition

$$u(X; 0) = \delta_Y(X) \quad \text{when } x_1 < \cdots < x_N.$$

Equation + boundary conditions + initial condition $\Rightarrow$

$u(X; t) = P_Y(X; t)$ when $x_1 < \cdots < x_N$.

Bethe Ansatz Solution

Define

$$\varepsilon(\xi) = p\,\xi^{-1} + q\,\xi - 1.$$

For any $\xi_1, \ldots, \xi_N \in \mathbb{C}\backslash\{0\}$ a solution of the equation is

$$\prod_i \left(\xi_i^{x_i}\, e^{\varepsilon(\xi_i)\,t} \right).$$

For any $\sigma \in \mathbb{S}_N$, another solution is

$$\prod_i \xi_{\sigma(i)}^{x_i} \prod_i e^{\varepsilon(\xi_i)\,t}$$

or any linear combination of these, or any integral of a linear combination.

Bethe Ansatz:

$$u(X;t) = \int \sum_{\sigma \in \mathbb{S}_N} F_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i e^{\varepsilon(\xi_i)\,t}\, d^N\xi.$$

Want the boundary conditions to be satisfied.

Look for F_σ such that the integrand satisfies the boundary conditions pointwise.

Definition: $T_i\sigma$ differs from σ by an interchange of the ith and $(i+1)$st entries. If $\sigma = (2\ 3\ 1\ 4)$ then $T_2\sigma = (2\ 1\ 3\ 4)$.

Sufficient conditions

$$\frac{F_{T_i\sigma}}{F_\sigma} = -\frac{p + q\xi_{\sigma(i)}\xi_{\sigma(i+1)} - \xi_{\sigma(i+1)}}{p + q\xi_{\sigma(i)}\xi_{\sigma(i+1)} - \xi_{\sigma(i)}}.$$

General solution

$$F_\sigma(\xi) = \operatorname{sgn}\sigma \prod_{i<j} (p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)}) \times \varphi(\xi).$$

If we define

$$A_\sigma = \operatorname{sgn}\sigma \, \frac{\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j}(p + q\xi_i\xi_j - \xi_i)}$$

then

$$u(X;t) = \sum_\sigma \int A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \left(\xi_i^{-y_i-1} \, e^{\varepsilon(\xi_i)\,t}\right) d^N\xi \qquad (1)$$

satisfies the master equation + boundary conditions.

The $\sigma = id$ summand satisfies initial condition if contours go around $\xi_i = 0$.

$A_{id} = 1$; other A_σ have poles, so contours matter.

Schütz (1997), $N = 2$:

$$P_Y(X;t) = \text{2dim integral} + \text{ 1dim integral.}$$

T-W (2007), general N:

Want contours so that when $x_1 < \cdots < x_N$

$$\sum_{\sigma \neq id} \int A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \xi_i^{-y_i - 1} \, d^N\xi = 0.$$

$N = 2$: If both contours are small enough get Schütz's formula.

$N = 3$: Take all contours small. Three integrals are zero, other two are negatives.

So for $N = 2,\ 3$ $P_Y(X;t)$ equals (1) with integrals over $\mathcal{C}_r$ with small r.

General N: Some integrals

$$\int A_\sigma(\xi)\,\prod_i \xi_{\sigma(i)}^{x_i}\,\prod_i \xi_i^{-y_i-1}\,d^N\xi$$

with $\sigma \neq id$ are zero, the rest come in pairs whose sum is zero.

Theorem: If $p \neq 0$ and r is small enough then

$$P_Y(X;t) = \sum_\sigma \int_{\mathcal{C}_r^N} A_\sigma(\xi)\,\prod_i \xi_{\sigma(i)}^{x_i}\,\prod_i \left(\xi_i^{-y_i-1}\,e^{\varepsilon(\xi_i)\,t}\right)d^N\xi.$$

When $p = 1$ get Schütz's determinant.

Formulas for $\mathbb{P}(x_1(t) = x)$

Since $x_1 < \cdots < x_N$ set

$$x_1 = x,\ x_2 = x + z_1,\ x_3 = x + z_1 + z_2, \ldots,$$

$$x_N = x + z_1 + z_2 + \cdots z_{N-1}.$$

When $r < 1$ can sum the formula for $P_Y(X;t)$ over $z_i > 0$. Integrand becomes

$$\frac{\prod_i (\xi_i^{x-y_i-1} e^{\varepsilon(\xi_i)t})}{\prod_{i<j}(p + q\xi_i\xi_j - \xi_i)} \cdot \sum_\sigma \operatorname{sgn}\sigma \left(\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)}) \right.$$

$$\left. \times \frac{\xi_{\sigma(2)}\xi_{\sigma(3)}^2 \cdots \xi_{\sigma(N)}^{N-1}}{(1 - \xi_{\sigma(2)}\xi_{\sigma(3)} \cdots \xi_{\sigma(N)})(1 - \xi_{\sigma(3)} \cdots \xi_{\sigma(N)}) \cdots (1 - \xi_{\sigma(N)})} \right).$$

A miracle! The sum equals

$$p^{N(N-1)/2}\,\frac{(1-\xi_1\cdots\xi_N)\,\prod_{i<j}(\xi_j-\xi_i)}{\prod_i(1-\xi_i)}.$$

Define

$$I(x, Y, \xi) = \prod_{i<j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \, \frac{1 - \xi_1 \cdots \xi_N}{\prod_i (1 - \xi_i)} \, \prod_i (\xi_i^{x - y_i - 1} e^{\varepsilon(\xi_i)t}).$$

Theorem: When $p \neq 0$ and r is small enough

$$\mathbb{P}(x_1(t) = x) = p^{N(N-1)/2} \int_{\mathcal{C}_r^N} I(x, Y, \xi) \, d^N \xi.$$

To show that

$$\sum_{x=-\infty}^{\infty} \mathbb{P}(x_1(t) = x) = 1,$$

or to compute

$$\mathbb{E}(x_1(t)) = \sum_{x=-\infty}^{\infty} x \, \mathbb{P}(x_1(t) = x),$$

or to let $N \to \infty$, need integrals over $\mathcal{C}_R$ with R large.

Expand contours. Another miracle: only poles at $\xi_i = 1$ contribute.

For $S \subset \{1, \ldots, N\}$, $I(x, Y_S, \xi)$ is $I(x, Y, \xi)$ but only indices $i \in S$.

Theorem: When $q \neq 0$ and R is large enough

$$\mathbb{P}(x_1(t) = x) = \sum_{S} c_S \int_{\mathcal{C}_R^{|S|}} I(x, Y_S, \xi) \, d^{|S|}\xi,$$

where c_S are certain constants.

For initial configuration

$$y_1 < y_2 < \cdots \to +\infty$$

sum over all finite subsets of $\mathbb{Z}^+$.

Formulas for $\mathbb{P}(x_m(t) = x)$

Use the miraculous formula twice and another one,

$$\sum_{|S|=m} \prod_{i\in S,\, j\in S^c} \frac{p+q\xi_i\xi_j-\xi_i}{\xi_j-\xi_i} \cdot \big(1-\prod_{j\in S^c}\xi_j\big)$$

$$= C_{N,m}\,\big(1-\prod_{j=1}^{N}\xi_j\big).$$

$$C_{N,m} = q^{m(N-m+1)} \begin{bmatrix} N-1 \\ m \end{bmatrix}_{p/q},$$

$$\begin{bmatrix} N-1 \\ m \end{bmatrix}_{\tau} = \prod_{k=1}^{m} \frac{1-\tau^{N-k}}{1-\tau^{k}}, \quad (\tau-\text{binomial coefficient}).$$

Theorem: When $p \neq 0$

$$\mathbb{P}(x_m(t) = x) = \sum_{|S^c| < m} c_{N,m,S} \int_{\mathcal{C}_r^{|S|}} I(x, Y_S, \xi)\, d^{|S|}\xi.$$

When $q \neq 0$

$$\mathbb{P}(x_m(t) = x) = \sum_{|S| \geq m} c_{m,S} \int_{\mathcal{C}_R^{|S|}} I(x, Y_S, \xi)\, d^{|S|}\xi.$$

Second representation holds for $N = \infty$.

The case $Y = \mathbb{Z}^+$ (step initial condition)

Define the k-dimensional integrand

$$J_k(x,\xi) = \prod_{i \neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_i \frac{\xi_i^x \, e^{\varepsilon(\xi_i)t}}{(1-\xi_i)\,(q\xi_i - p)}.$$

Corollary. When $Y = \mathbb{Z}^+$ and $q \neq 0$

$$\mathbb{P}(x_m(t) \le x) = \sum_{k \ge m} c_{m,k} \int_{\mathcal{C}_R^k} J_k(x,\xi)\, d^k\xi.$$

When $p = 0$ (left-moving TASEP) only $c_{m,m}$ survives. Get $m \times m$ Toeplitz determinant

$$\mathbb{P}(x_m(t) \le x) = \det \left(\int_{\mathcal{C}_R} \xi^{i-j+x-1} \, (\xi - 1)^{-m} \, e^{(\xi-1)t} \, d\xi \right).$$

Obtained by Rákos-Schütz (2005) and used it used to get Johansson's result.

In the formula of the Corollary use the identity

$$\prod_{i\neq j}\frac{\xi_j-\xi_i}{p+q\xi_i\xi_j-\xi_i}=(-1)^k\,(pq)^{-k(k-1)/2}\prod_i(1-\xi_i)(q\xi_i-p)$$

$$\times\det\left(\frac{1}{p+q\xi_i\xi_j-\xi_i}\right)_{1\leq i,j\leq k}.$$

Get $(\tau=p/q<1)$

$$\mathbb{P}(x_m(t)\leq x)=(-1)^m\,\tau^{m(m-1)/2}\sum_{k\geq m}\begin{bmatrix}k-1\\k-m\end{bmatrix}_\tau\left(\frac{p}{\tau^m}\right)^k$$

$$\times\frac{(-1)^k}{k!}\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_R}\det K(\xi_i,\,\xi_j)_{1\leq i,j\leq k}\,d\xi_1\cdots d\xi_k,$$

where

$$K(\xi,\,\xi')=\frac{\xi^x\,e^{\varepsilon(\xi)t}}{p+q\xi\xi'-\xi}.$$

Last factor is coefficient of λ^k in the expansion of $\det(I - \lambda K)$, where K acts on $\mathcal{C}_R$, so it has an integral representation. Interchange sum and integral and use the τ-binomial theorem. Find that

$$\mathbb{P}\left(x_m(t) \le x\right) = \int \frac{\det(I - \lambda q K)}{\prod_{k=0}^{m-1}(1 - \lambda\,\tau^k)}\,\frac{d\lambda}{\lambda}.$$

Contour encloses all singularities of the integrand. In particular

$$\mathbb{P}\left(x_1(t) > x\right) = \det(I - qK).$$

Asymptotic analysis of $\det(I - \lambda q K)$ leads to three asymptotic results for $p < q$ as $t \to \infty$. One with m and x fixed; one with m fixed and x large; one with m and x large.

Set $\gamma = q - p$. Second one has special case

$$\lim_{t\to\infty} \mathbb{P}\left(x_1(t/\gamma) > -t - \gamma^{1/2}\, s\, t^{1/2}\right) = \det\big(I - \hat{K}\,\chi_{(s,\,\infty)}\big),$$

where

$$\hat{K}(z,\, z') = \frac{q}{\sqrt{2\pi}}\; e^{-(p^2+q^2)\,(z^2+z'^2)/4+pq\,zz'}.$$

A new distribution function?

Third one:

$$m/t = \sigma, \quad c_1 = -1 + 2\sqrt{\sigma}, \quad c_2 = \sigma^{-1/6}\,(1 - \sqrt{\sigma})^{2/3}.$$

When $0 \le p < q$ we have

$$\lim_{t\to\infty} \mathbb{P}\left(x_m(t/\gamma) \le c_1\, t + c_2\, s\, t^{1/3}\right) = F_2(s)$$

uniformly for σ in a compact subset of (0, 1).

Coefficient c_1 was known (Liggett 1985). When $p = 0$ get asymptotics for TASEP obtained by Johansson.

VERY rough outline of derivation.

Using lemmas on stability of spectrum we show that there are operators K_1 and K_2 acting on a different contour such that

$$\det(I - \lambda q K) = \det(I - \lambda(K_1 - K_2)),$$

where

$$\det(I - \lambda K_1) = \prod_{k=0}^{\infty}(1 - \lambda\tau^k).$$

We factor this out in

$$\int \frac{\det(I - \lambda K_1 + \lambda K_2))}{\prod_{k=0}^{m-1}(1 - \lambda\,\tau^k)}\,\frac{d\lambda}{\lambda}.$$

Get

$$\int \prod_{k=m}^{\infty} (1 - \lambda\,\tau^k)\,\det(I + \lambda K_2\,(I + R))\,\frac{d\lambda}{\lambda},$$

where R is the resolvent kernel of K_1.

For the third asymptotics set $\lambda = \mu\,\tau^{-m}$ with $|\mu|$ fixed. Above becomes

$$\int \prod_{k=0}^{\infty} (1 - \mu\,\tau^k)\,\det(I + \mu\,\tau^{-m} K_2\,(I + R))\,\frac{d\mu}{\mu}. \qquad (2)$$

Define

$$\varphi(\eta) = (1 - \eta)^{-x}\, e^{\frac{\eta}{1-\eta}\,t},$$

$$f(\mu,\, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k \mu}\, z^k.$$

Using stability of spectrum lemmas we show that

$$\det(I+\mu\,\tau^{-m}K_2\,(I+R))=\det(I+\mu J),$$

where

$$J(\eta,\,\eta')=\int\frac{\varphi(\zeta)}{\varphi(\eta')}\,\frac{\zeta^m}{(\eta')^{m+1}}\,\frac{f(\mu,\zeta/\eta')}{\zeta-\eta}\;d\zeta,$$

acting on circle with radius a little smaller than one, integral in J a circle with radius a little greater than one. Steepest descent and scaling give

$$\det(I+\mu\,J)\to\det\left(I-K_{\mathrm{Airy}}\,\chi_{(s,\,\infty)}\right)=F_2(s)$$

for all μ. (c_1 is chosen so that the two saddle points concide.)

Therefore the integral (2) has limit $F_2(s)$.