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Random matrix theory solutions to mysteries in number theory

August 2008

Research supported by:

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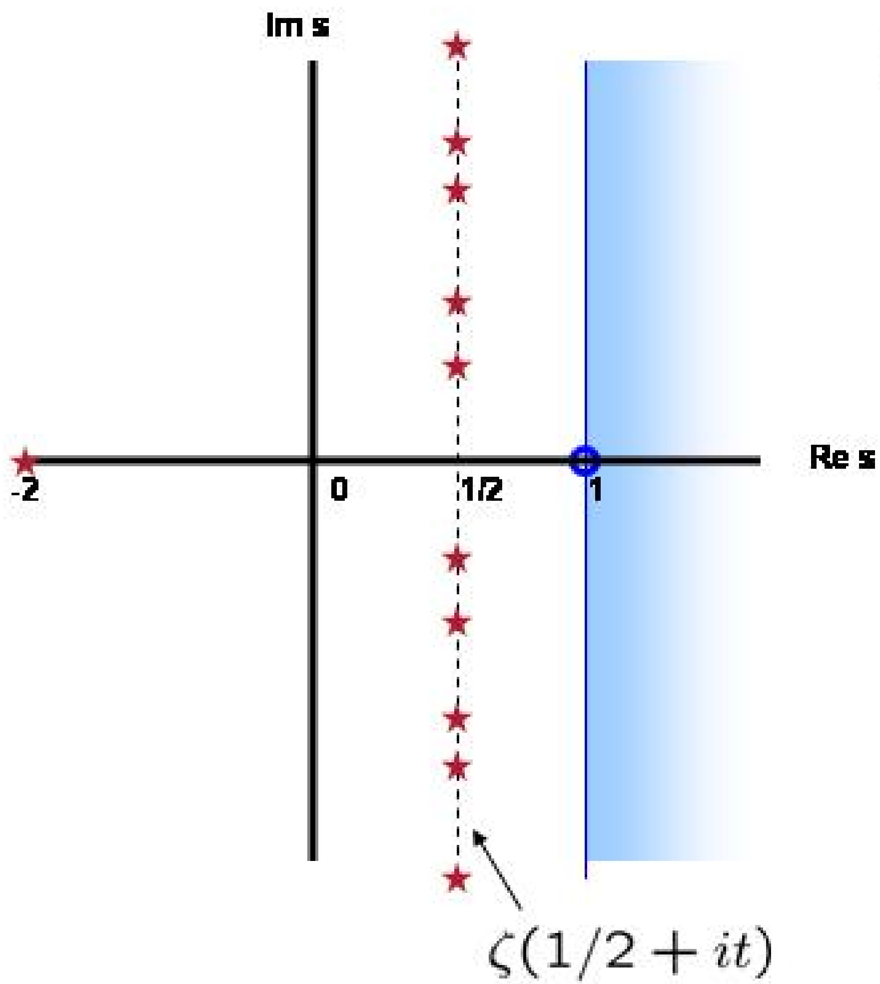
EPSRC Engineering and Physical Sciences Research Council

Riemann zeta function



For $\mathrm{Re}\, s > 1$:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

and

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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Density of zeros:

$$d(t) \sim \frac{1}{2\pi} \log \frac{t}{2\pi}$$

$$w_n = t_n \frac{1}{2\pi} \log \frac{t_n}{2\pi}, \quad t_n = \text{n}^{\text{th}} \text{ Riemann zero}$$

$$\Rightarrow \; < w_{n+1} - w_n >_n = 1$$

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Conjecture (Montgomery 1973)

$$\lim_{N\to\infty} \frac{1}{N} \#\{w_n, w_m \in [0, N] : \alpha \le w_n - w_m \le \beta\}$$
$$= \int_\alpha^\beta R_2(x)dx$$

where

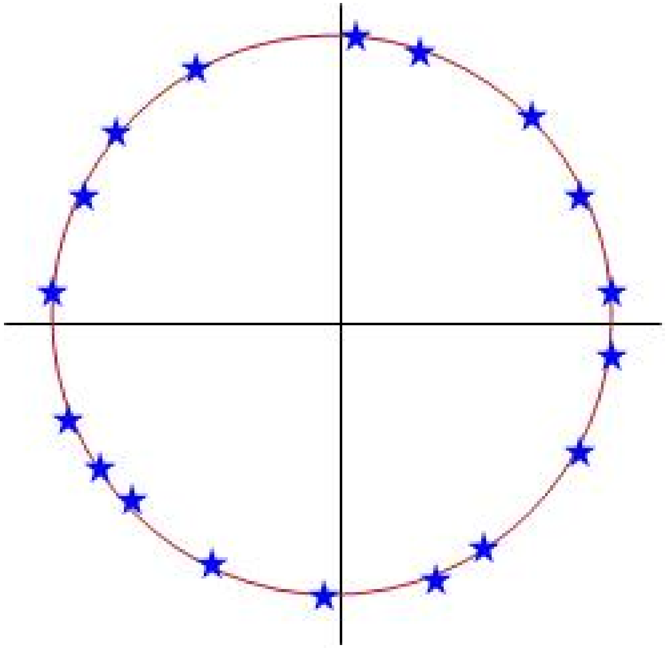
$$R_2(x) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2 + \delta(x)$$

Random Unitary Matrices

$N \times N$ unitary matrix

$e^{i\theta_n}$ - eigenvalues

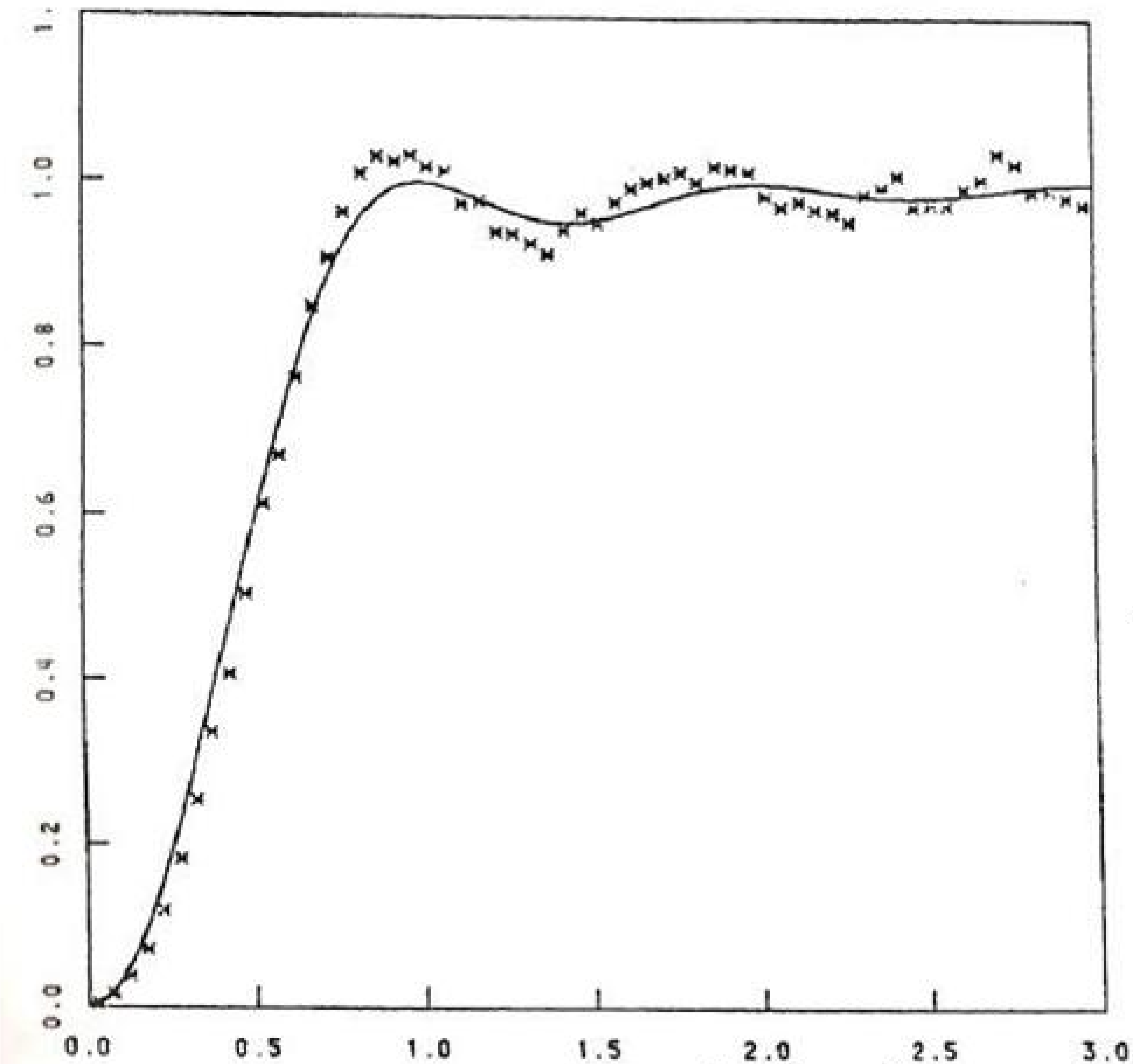
chosen randomly with respect to Haar measure on U(N)



$$\phi_n = \theta_n \frac{N}{2\pi}$$
$$\Rightarrow < \phi_{n+1} - \phi_n >_n = 1$$

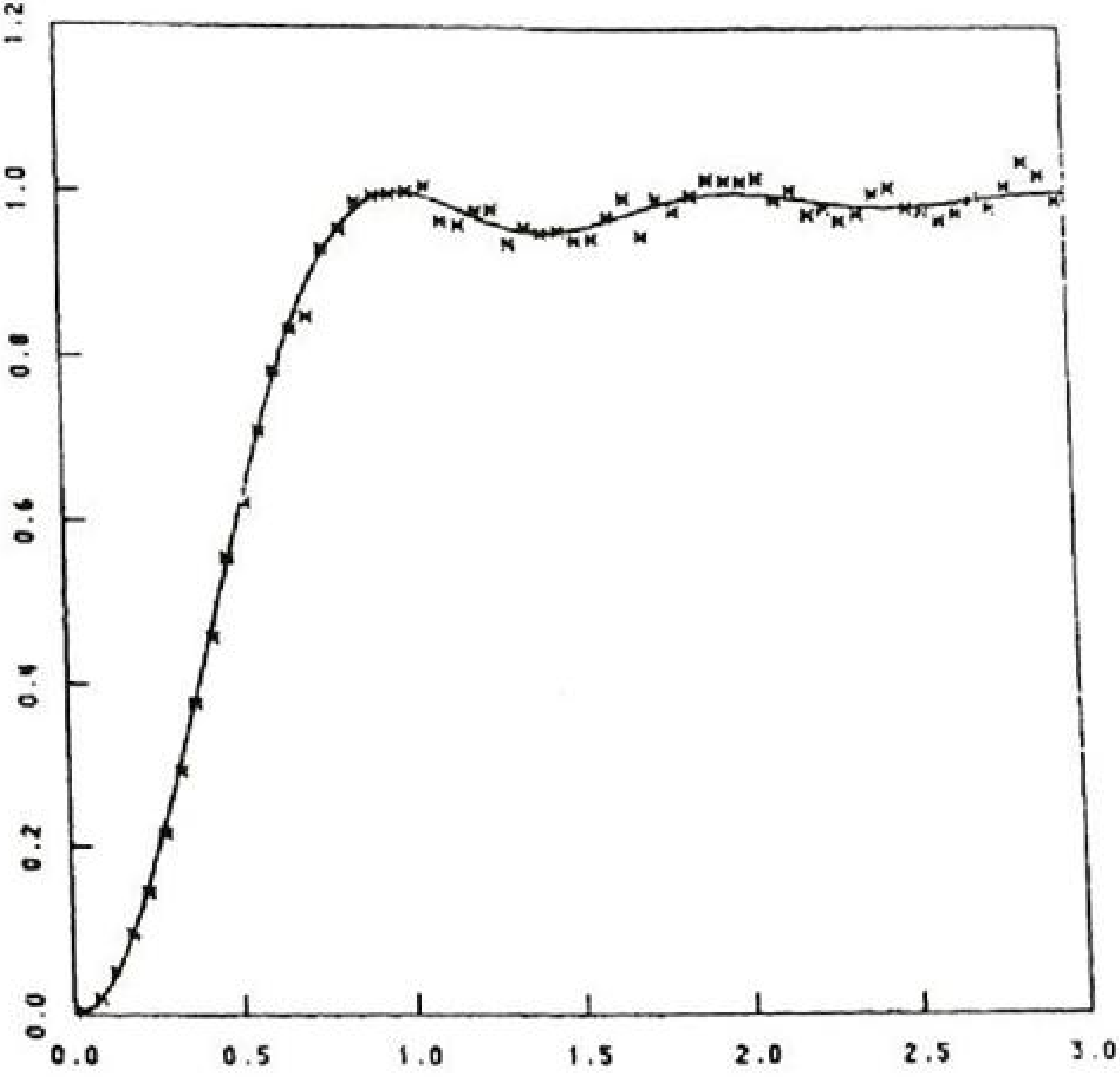
Theorem (Dyson 1963)

$$\lim_{N\to\infty} \frac{1}{N} \langle \#\{\phi_n, \phi_m : \alpha \le \phi_n - \phi_m \le \beta\}\rangle_{U(N)}$$
$$= \int_\alpha^\beta R_2(x)dx$$



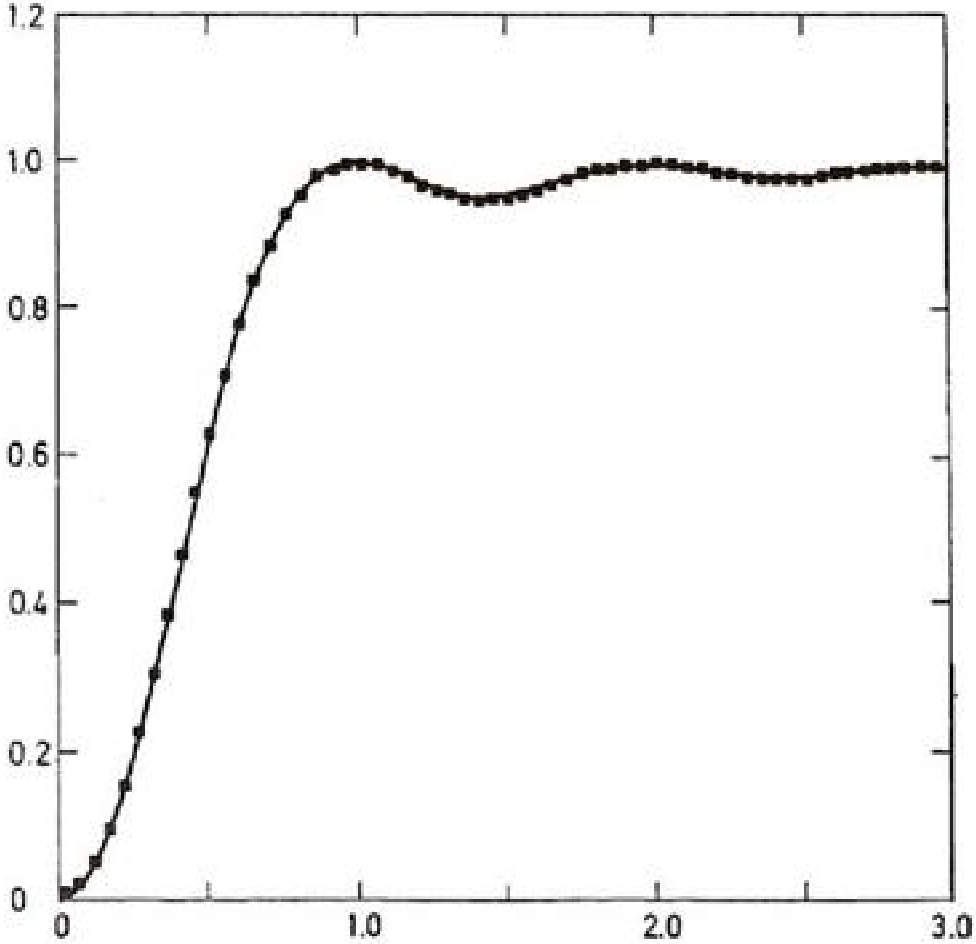
First 100000 zeros

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$



10^5 zeros around the 10^{12} th zero

Mystery 0: why are these statistics the same?!



Picture by A. Odlyzko

79 million zeros around the 10^{20} th zero

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$

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Distribution of log ζ (Selberg) :

For a rectangle E in $\mathbb{R}^2$

$$\lim_{T\to\infty} \frac{1}{T}\left|\left\{t : T \le t \le 2T, \frac{\ln \zeta(1/2+it)}{\sqrt{1/2 \ln\ln T}} \in E\right\}\right|$$
$$= \frac{1}{2\pi}\int\int_E e^{-(x^2+y^2)/2} dx\, dy$$

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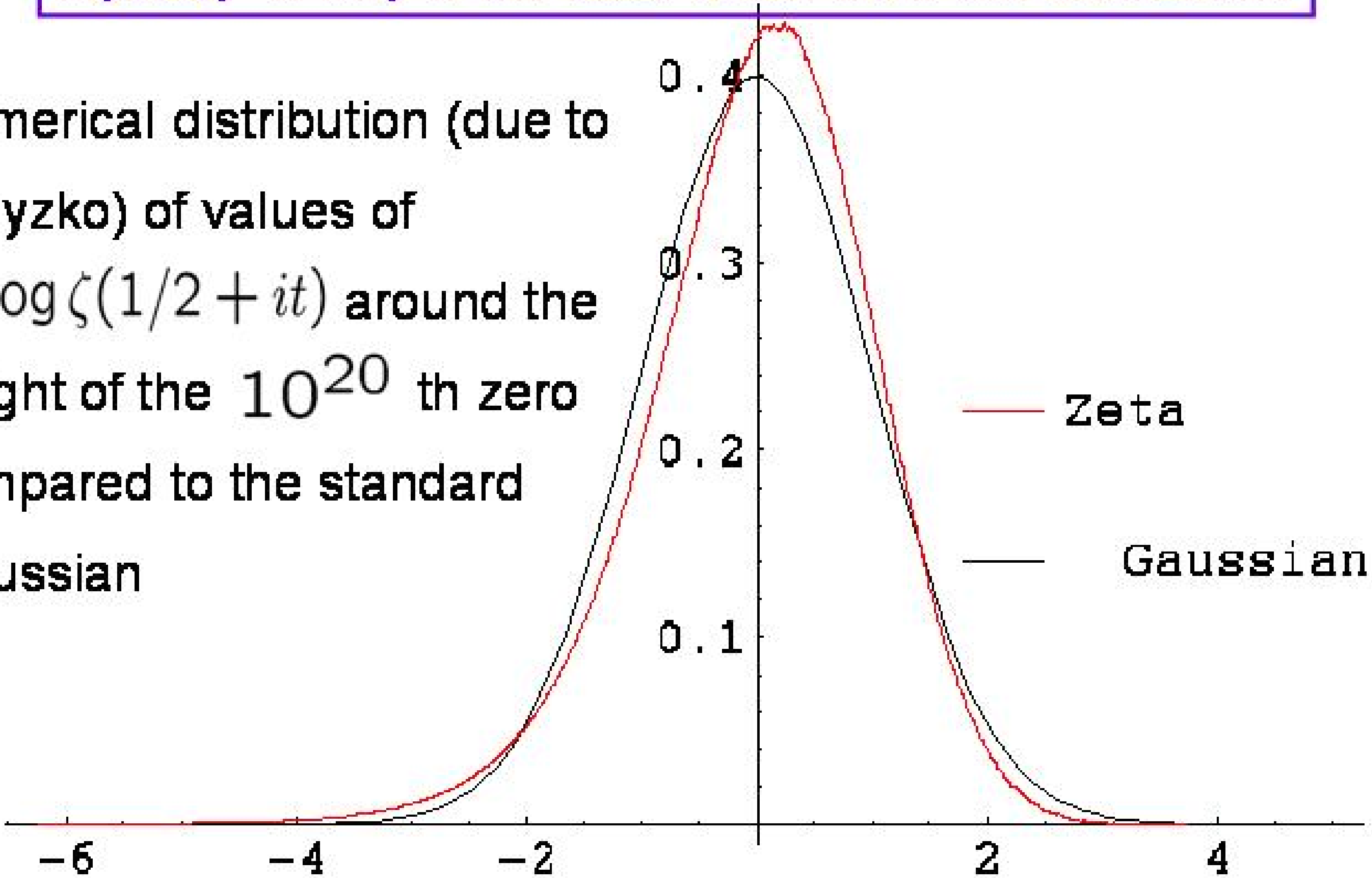
Mystery 1: why is the data so far from the Gaussian?

Numerical distribution (due to Odlyzko) of values of $\operatorname{Re}\log\zeta(1/2+it)$ around the height of the 10^{20} th zero compared to the standard Gaussian



Characteristic polynomial:

$$\Lambda_A(s) = \prod_{n=1}^{N} \left(1 - se^{-i\theta_n}\right) = \det\left(I - A^* s\right)$$

Equate densities of zeros:

$$\overset{\text{RZF}}{\frac{1}{2\pi}\log\frac{T}{2\pi}} = \overset{\text{RMT}}{\frac{N}{2\pi}}$$

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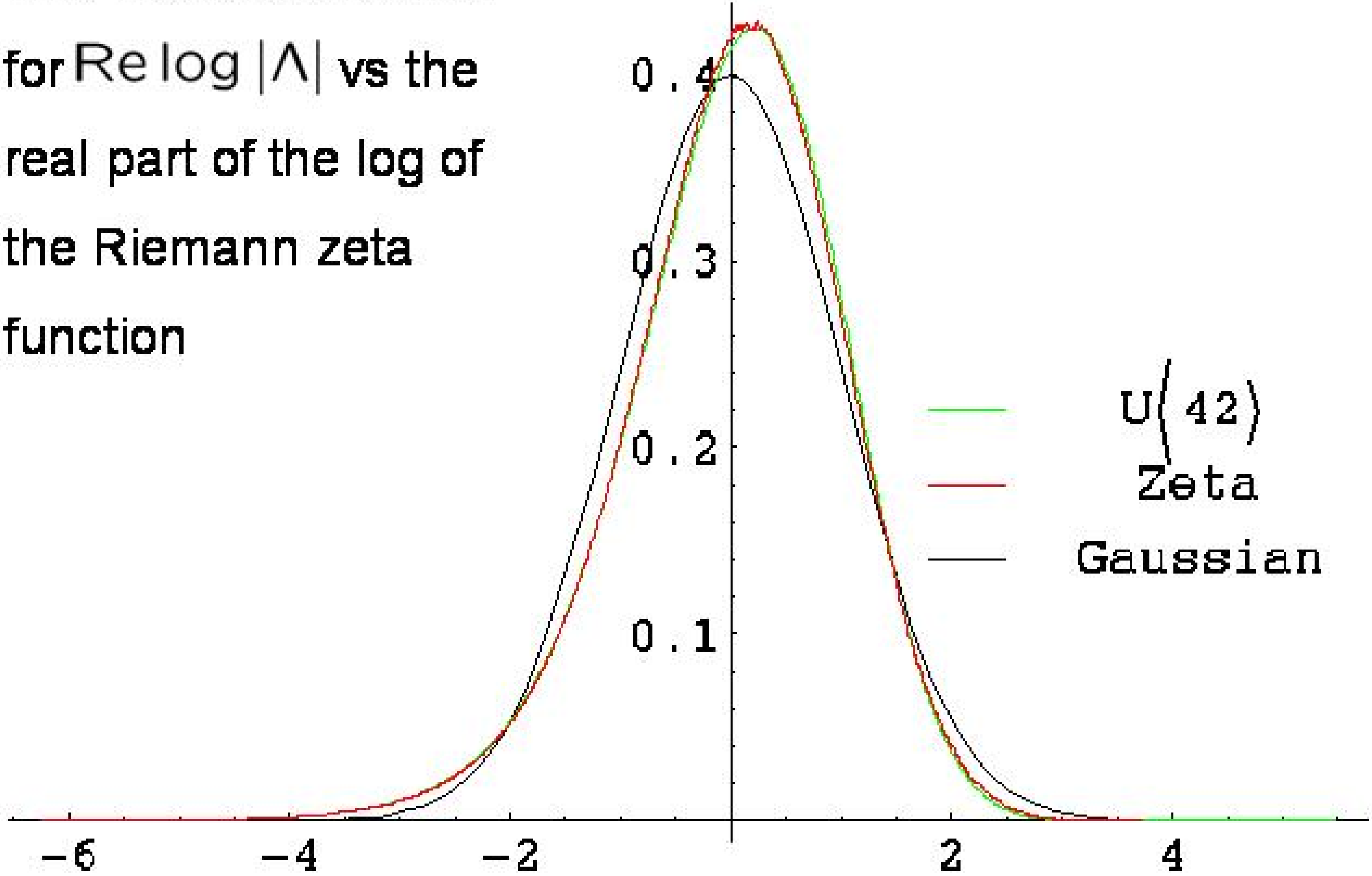
The value distribution for $\mathrm{Re}\log|\Lambda|$ vs the real part of the log of the Riemann zeta function



Keating and Snaith (2000)

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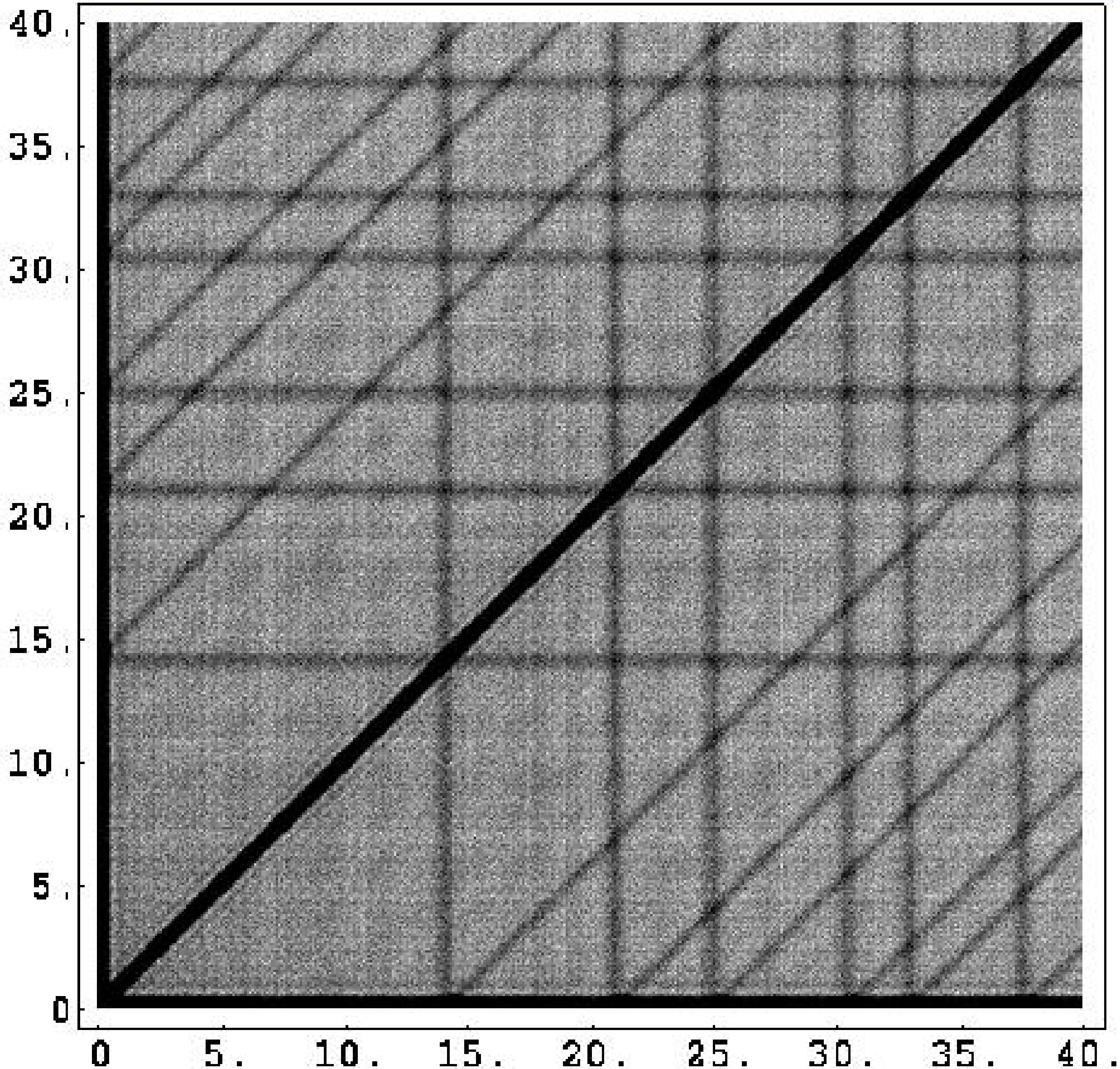
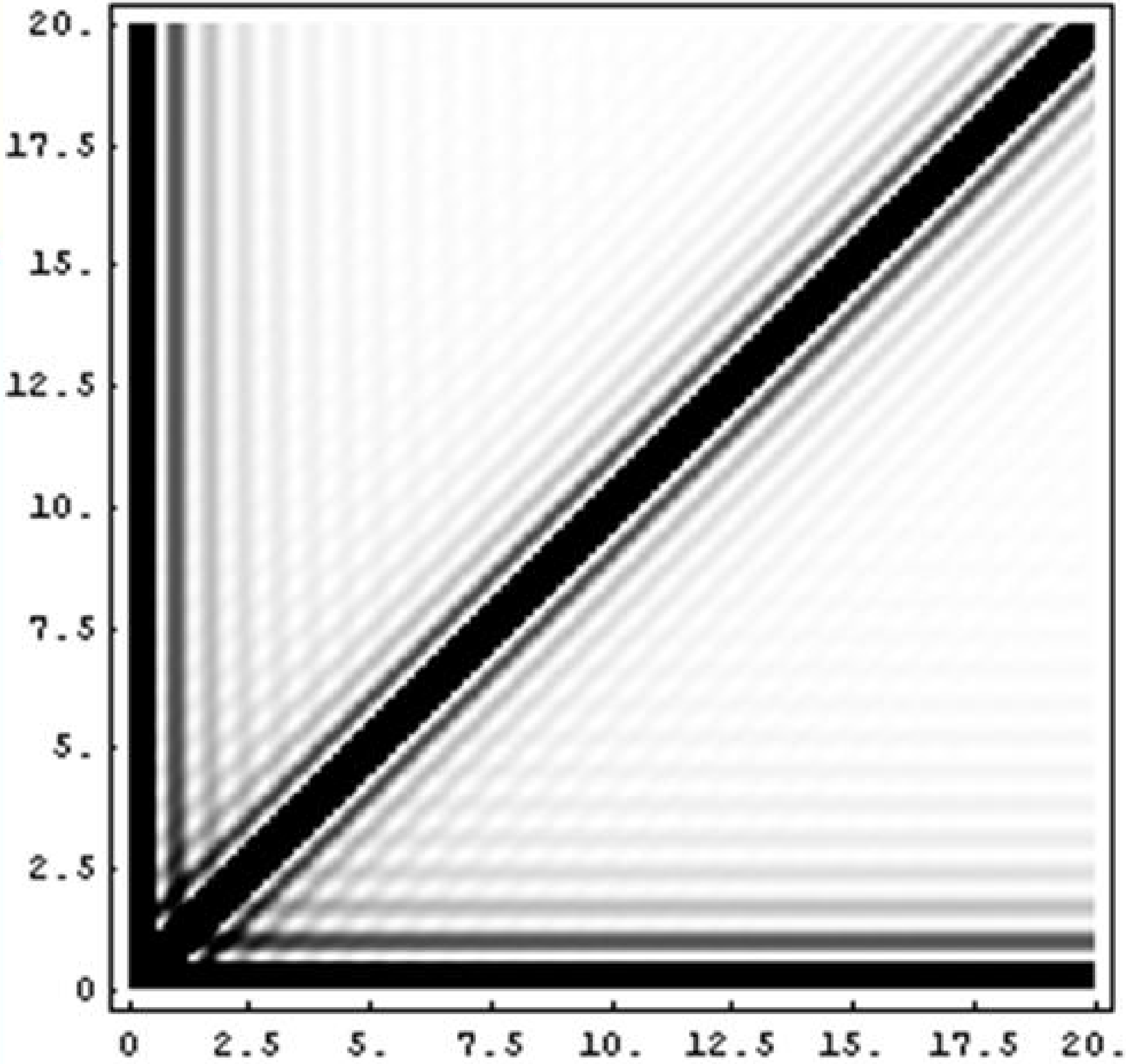
Mystery 2: how do you explain this pattern?

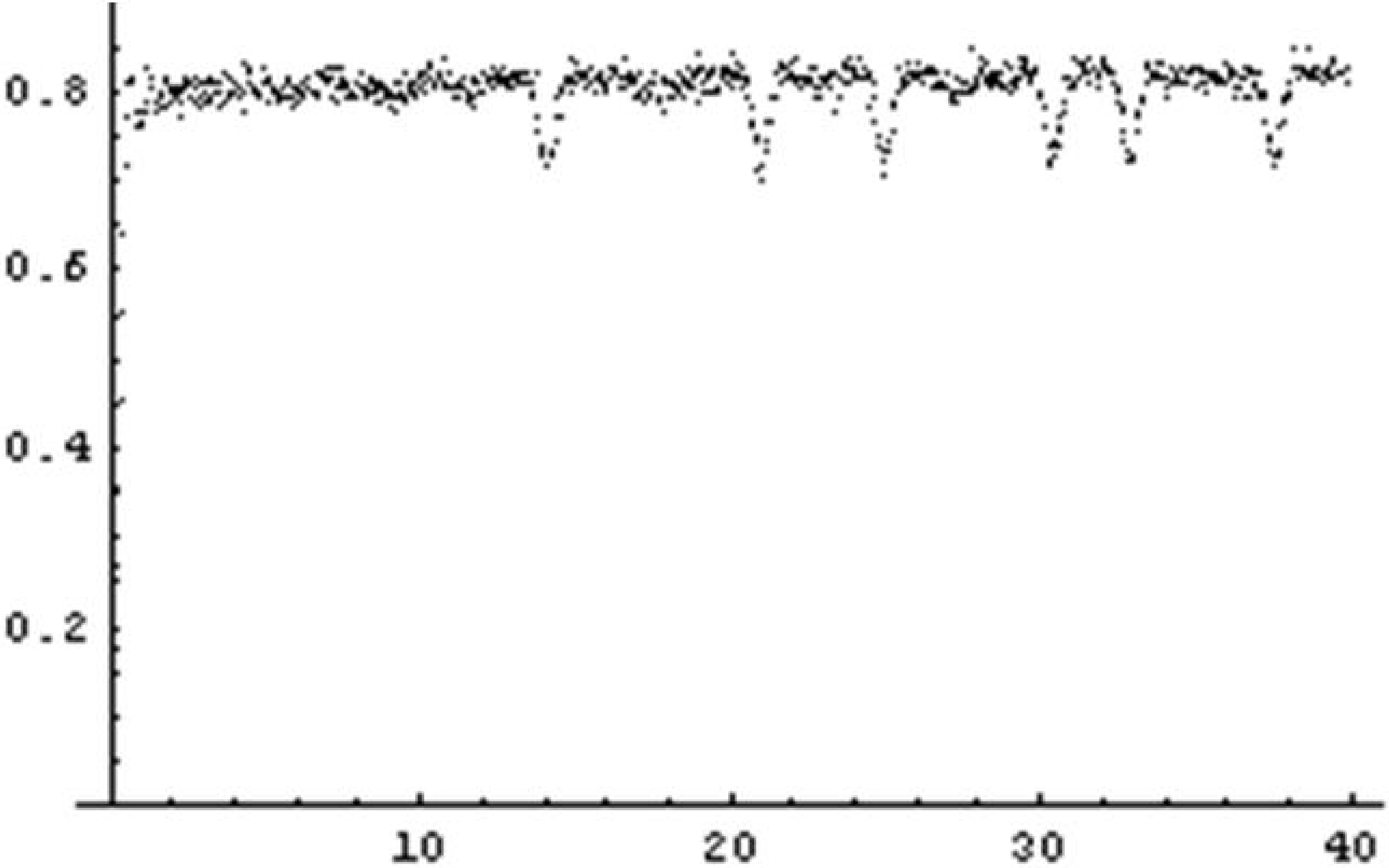


Conrey and Snaith (2007)

Triple correlation of the first 100000 Riemann zeros



Triple correlation of eigenvalues of matrices from U(N) with N=9



2-point correlation of the Riemann zeros

Moment Conjecture:

Conrey, Farmer, Keating, Rubinstein and Snaith (2005)

$$\frac{1}{T}\int_0^T \zeta(1/2+it+\alpha_1)\cdots\zeta(1/2+it+\alpha_k)\zeta(1/2-it-\alpha_{k+1})\cdots\zeta(1/2-it-\alpha_{2k})dt$$

Ratio Conjecture:

Conrey, Farmer, Zirnbauer (preprint)

$$\frac{1}{T}\int_0^T \frac{\prod_{k=1}^{K}\zeta(1/2+it+\alpha_k)\ \prod_{\ell=K+1}^{K+L}\zeta(1/2-it-\alpha_\ell)}{\prod_{q=1}^{Q}\zeta(1/2+it+\gamma_q)\ \prod_{r=1}^{R}\zeta(1/2-it+\delta_r)}dt$$

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Conjecture (CFZ):

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt = \int_0^T \left(\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)}A_\zeta(\alpha,\beta;\gamma,\delta)\right.$$
$$\left. + e^{-\log\frac{t}{2\pi}(\alpha+\beta)}\frac{\zeta(1-\alpha-\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)}A_\zeta(-\beta,-\alpha;\gamma,\delta)\right)\,dt$$
$$+O\left(T^{1/2+\epsilon}\right)$$

Theorem (CFZ):

$$\int_{U(N)} \frac{\Lambda_X(e^{-\alpha})\Lambda_{X^*}(e^{-\beta})}{\Lambda_X(e^{-\gamma})\Lambda_{X^*}(e^{-\delta})}dX = \frac{z(\alpha+\beta)z(\gamma+\delta)}{z(\alpha+\delta)z(\beta+\gamma)}$$
$$+e^{-N(\alpha+\beta)}\frac{z(-\beta-\alpha)z(\gamma+\delta)}{z(-\beta+\delta)z(-\alpha+\gamma)}$$

$$z(x) = (1-e^{-x})^{-1}$$

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$2k$	$\int_0^\infty \lvert\zeta(1/2+it)\rvert^{2k} e^{-t/10000} dt$	conjecture	% error
2	79499.9312635346822	79496.7897046730342	0.004
4	5088332.55512	5088336.43654	0.00007
6	708967359.4	708965694.5	0.0002
8	143638308513.0	143628911646.6	0.006

Computed by Michael Rubinstein

Assuming the ratios conjecture (Conrey and Snaith, 2007):

(Previously by a different method Bogomolny and Keating 1996)

$$\sum_{\gamma,\gamma'\leq T} f(\gamma-\gamma') = \frac{1}{(2\pi)^2}\int_0^T \Bigg(2\pi f(0)\log\frac{t}{2\pi}$$

$$+\int_{-T}^{T} f(r)\Bigg(\log^2\frac{t}{2\pi}+2\Bigg(\left(\frac{\zeta'}{\zeta}\right)'(1+ir)$$

$$+\left(\frac{t}{2\pi}\right)^{-ir}\zeta(1-ir)\zeta(1+ir)A(ir)-B(ir)\Bigg)\Bigg)\,dr\Bigg)\,dt$$

$$+O(T^{1/2+\epsilon})$$

with

$$A(\eta)=\prod_p \frac{(1-\frac{1}{p^{1+\eta}})(1-\frac{2}{p}+\frac{1}{p^{1+\eta}})}{(1-\frac{1}{p})^2}$$

$$B(\eta)=\sum_p\left(\frac{\log p}{(p^{1+\eta}-1)}\right)^2$$

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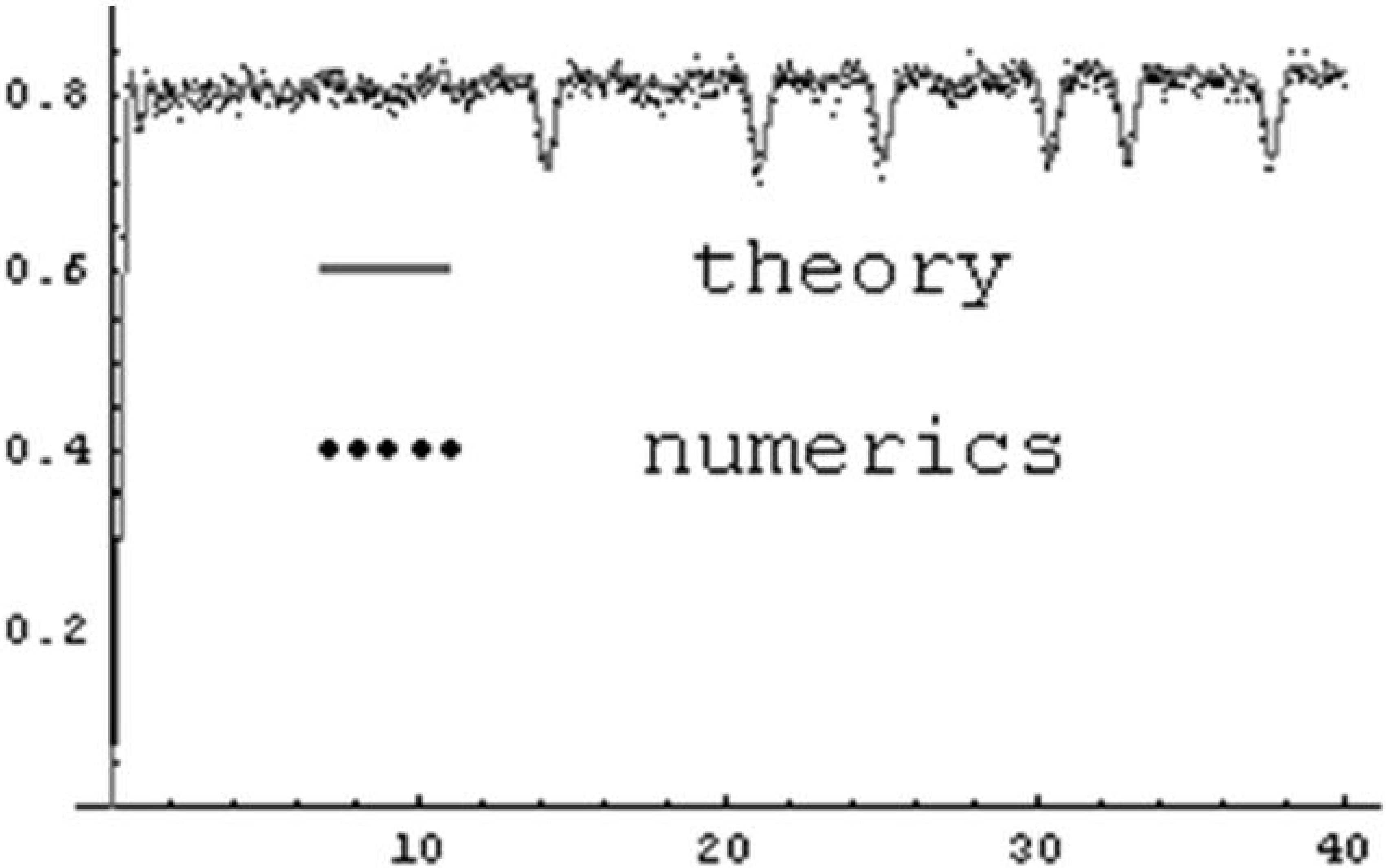
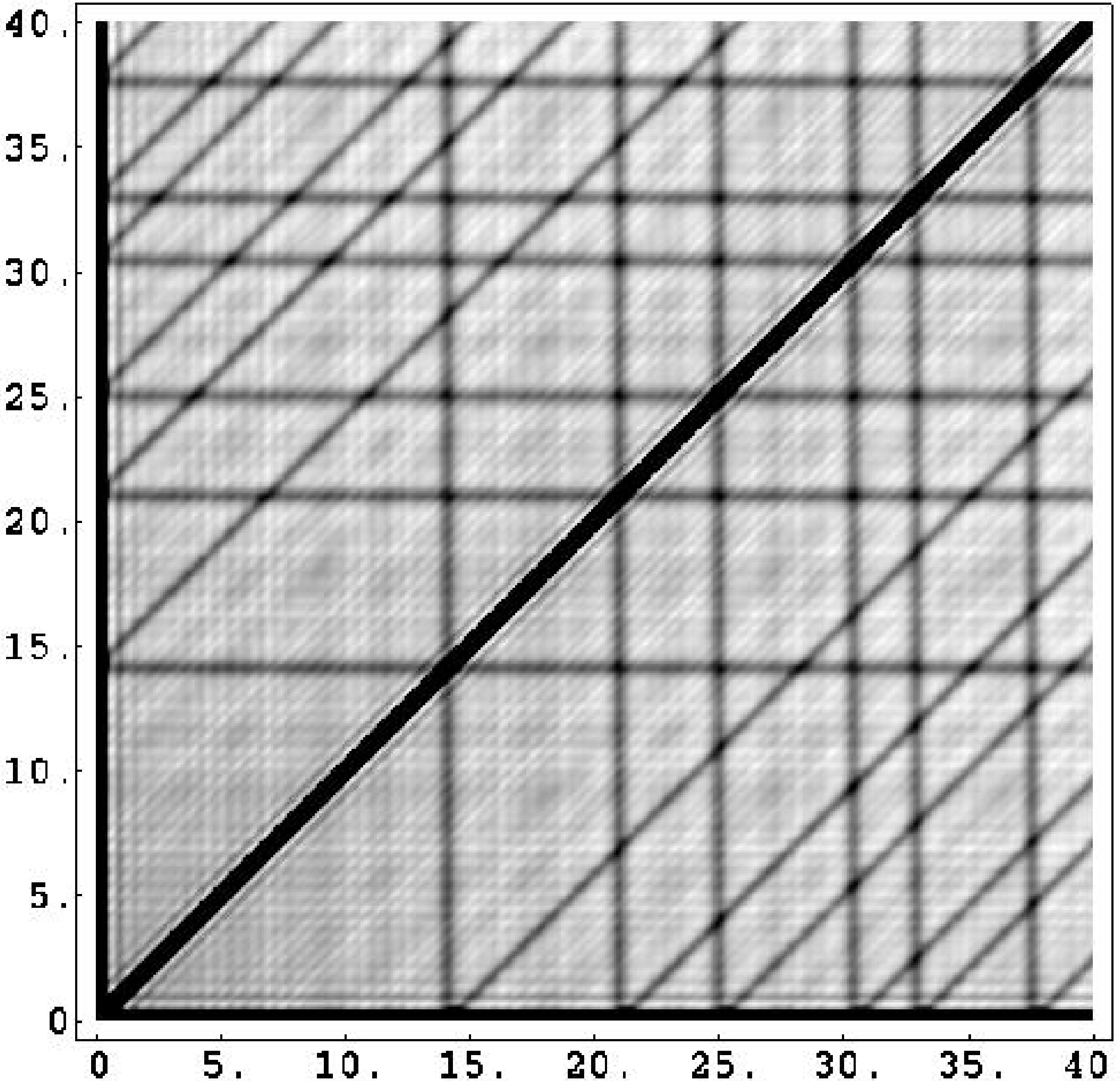

0.8
0.6
0.4
0.2
theory
numerics
10
20
30
40



Conrey and Snaith (2007)

Triple correlation of the Riemann zeros up to height T=75000

Mystery 3: what is the true size of ζ?

Lindelöf Hypothesis:

for every $\epsilon > 0$, $\zeta(\frac{1}{2} + it) = O(t^{\epsilon})$

Assuming the Riemann Hypothesis:

$$\zeta(\tfrac{1}{2} + it) = O\Big(\exp\Big(C\frac{\log t}{\log\log t}\Big)\Big)$$

$$|\zeta(\tfrac{1}{2} + it)| = \Omega\Big(\exp\Big(C'\sqrt{\frac{\log t}{\log\log t}}\Big)\Big)$$

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Hybrid formula, Gonek, Hughes and Keating (2007)

For $s = \sigma + it$, $0 \le \sigma \le 1$, $|t| \ge 2$, $x > 2$, k positive integer

$$\zeta(s) = P_X(s) Z_X(s) + error$$

$$P_X(s) := \exp\Big(\sum_{n \le X} \frac{\Lambda(n)}{n^s \log n} \Big)$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n \text{ is prime power} \\ 0 & \text{else} \end{cases}$$

$$Z_X(s) := \exp\Big(-\sum_{\rho} U\big((s-\rho)\log X\big) \Big)$$

where U acts to limit the sum over zeros to those near s (on a scale of $1/\log X$)

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Assume independence: use arithmetic on P and random matrix theory on Z.

Look at maxima of P and Z separately.

Conjecture (Farmer, Gonek, Hughes 2007)

$$\zeta(\tfrac{1}{2}+it) = O\Big(\exp\Big((\tfrac{1}{\sqrt{2}}+\epsilon)\sqrt{\log t \log\log t}\Big)\Big)$$

for all $\epsilon > 0$ and no $\epsilon < 0$.

L-functions

- Dirichlet sum

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- Euler product over the primes

- Functional equation (normalised so that critical line is at $\mathrm{Re}(s) = 1/2$)

Generalised Riemann Hypothesis - the non-trivial zeros of L-functions lie on the critical line

Katz and Sarnak:

Averaged over a family of L-functions, zeros close to $s = 1/2$ show statistics like ONE of

$$U(N), O(N), USp(2N)$$

depending on the family

Elliptic curve L-functions:

eg.

$$E_{11}:\ y^2 = 4x^3 - 4x^2 - 40x - 79$$

L-function:

$$L_{E_{11}}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

a_n determined by E_{11}

Mystery 4: How are ranks of curves in this family distributed?

Create a family:

$$L_E(s,\chi_d) = \sum_{n=1}^{\infty} \frac{a_n \chi_d(n)}{n^s}$$

d fundamental discriminant, $\chi_d(n) = \left(\frac{d}{n}\right)$

$$E_d : dy^2 = ax^3 + bx + c$$

Family:

$\mathcal{F}_{E+} = \{L_E(s,\chi_d)$ with even functional equation$\}$

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Orthogonal symmetry - zeros near 1/2 have statistics like eigenvalues near 1 of matrices from $SO(2N)$ chosen with Haar measure.

Orthogonal $2N\times 2N$ matrices with determinant $+1$:

The eigenvalues come in complex conjugate pairs $e^{i\theta_1}$, $e^{-i\theta_1}$, $e^{i\theta_2}$, $e^{-i\theta_2}, \ldots\ e^{i\theta_N}, e^{-i\theta_N}$.

The characteristic polynomial is

$$\Lambda_A(e^{i\theta}) = \prod_{n=1}^{N}\left(1-e^{i(\theta-\theta_n)}\right)\left(1-e^{i(\theta+\theta_n)}\right),$$

so

$$\Lambda_A(1) = \prod_{n=1}^{N}\left|1-e^{i\theta_n}\right|^2.$$

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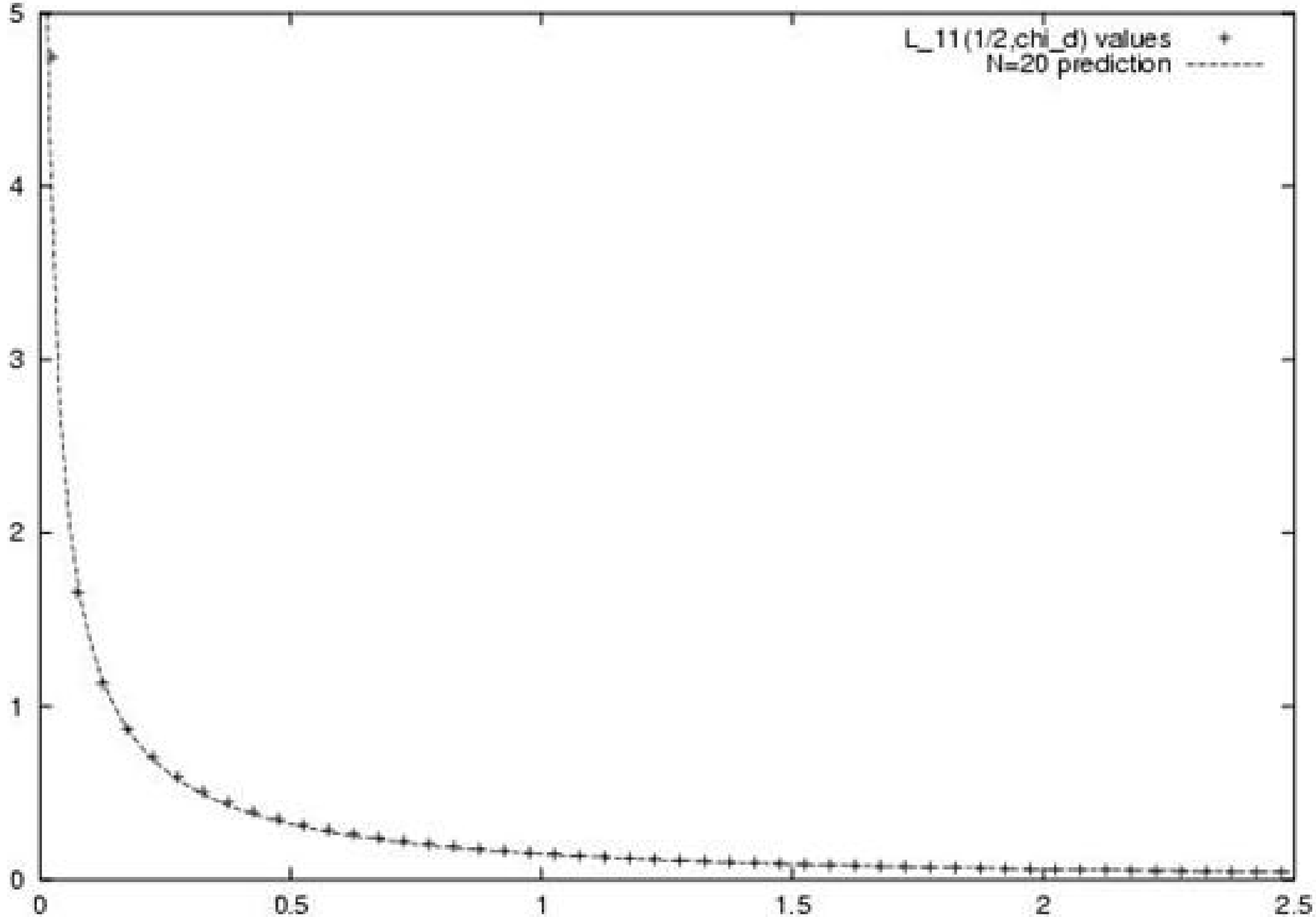

The value distribution of $L_{E_{11}}(1/2, \chi_d)$ for prime |d|, -788299808<d<0, even functional equation, compared to the value distribution of characteristic polynomials from SO(40)

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Conjecture (Conrey, Keating, Rubinstein, Snaith 2002):

Let E be an elliptic curve defined over $\mathbb{Q}$. Then there is a constant $c_E \geq 0$ such that

$$\frac{\sum_{\substack{p \leq T,\ \text{prime} \\ L_E(1/2,\chi_p)=0 \\ L_E(s,\chi_p)\in\mathcal{F}_{E^+}}} 1}{\sum_{\substack{p \leq T,\ \text{prime} \\ L_E(s,\chi_p)\in\mathcal{F}_{E^+}}} 1} \sim c_E T^{-1/4} (\log T)^{3/8}$$

Conjecture (Birch and Swinnerton-Dyer):

$L_E(1/2, \chi_d) = 0$ if and only if E_d has infinitely many rational points (ie. rank greater than zero)

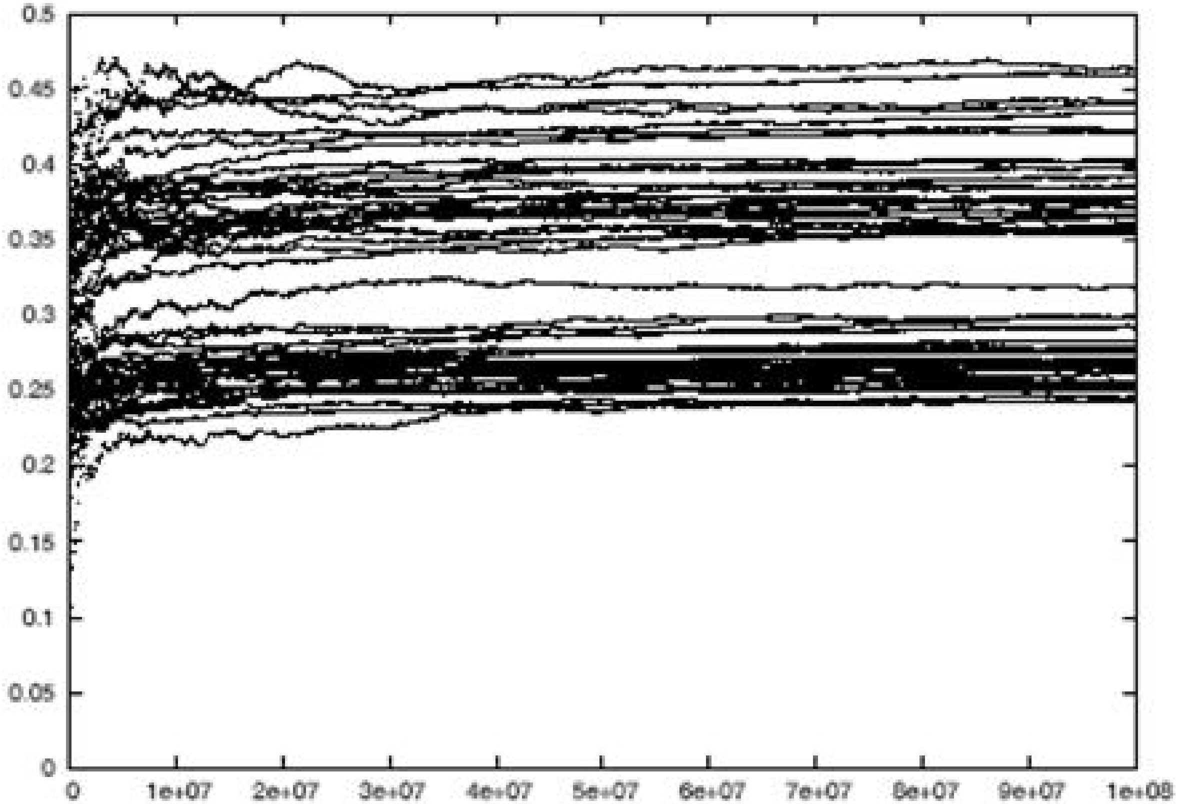

FIGURE 5. A test of Conjecture 5.1 for 55 elliptic curves in our database: we are plotting for each curve the left-hand side of (5–2) divided by $A^{\pm}\left(\frac{-1}{2}\right)\sqrt{\kappa_E^{\pm}}X^{-1/4}(\log X)^{3/8}$, $X = 100\,000, 200\,000, \ldots, 10^8$. The graphs appear relatively flat.

J.B.Conrey, J.P.Keating,M.O.Rubinstein,N.C.Snaith, "Random matrix theory and the Fourier coefficients of half-integral weight forms" Exp. Math. 15, No.1, 2006

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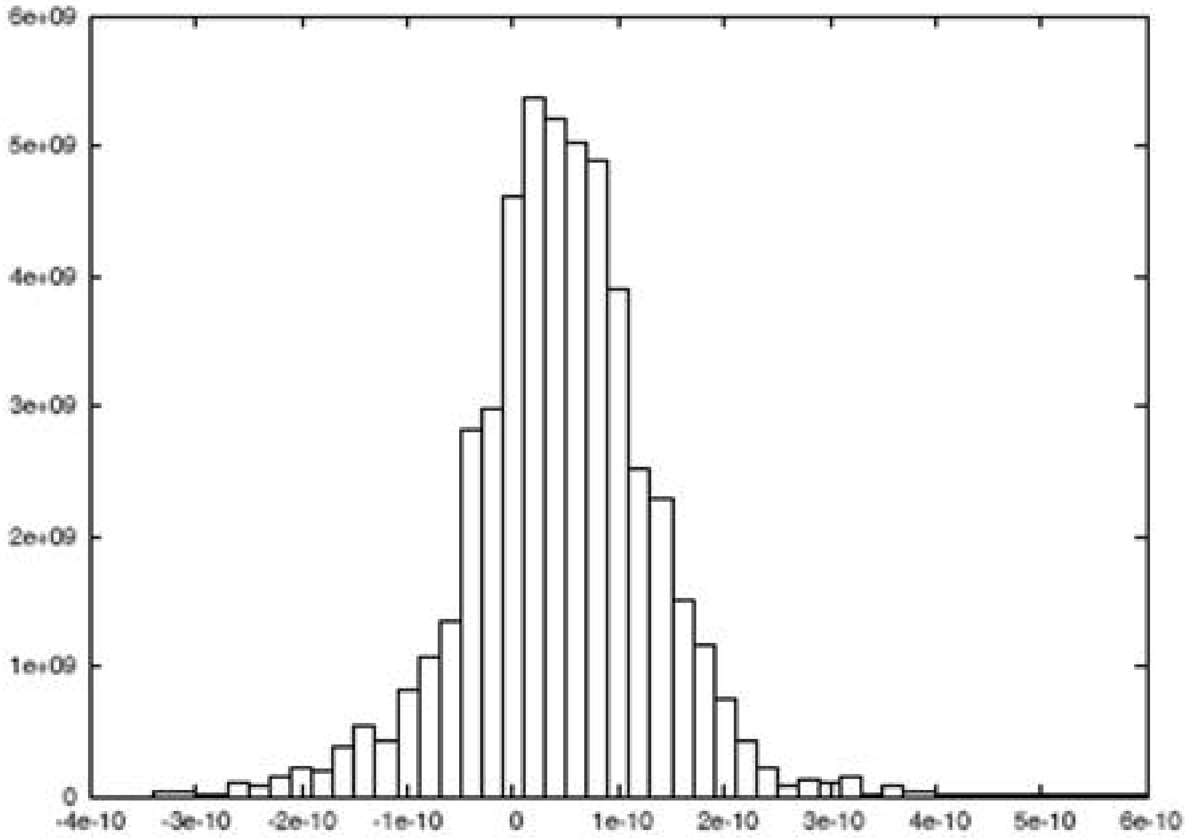

FIGURE 7. Distribution of the slopes of the graphs in Figure 6 from $X = 5 \times 10^7$ to $X = 10^8$. This tells us that the graphs are relatively flat.

J.B.Conrey, J.P.Keating,M.O.Rubinstein,N.C.Snaith, "Random matrix theory and the Fourier coefficients of half-integral weight forms" Exp. Math. 15, No.1, 2006

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