

Outpost Colonization in Hermitian Matrix Model (aka Birth of a Cut)

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Broad Goal:

Given orthogonal (monic) polynomials:

$$\int_{\mathbb{R}} p_n(x)p_m(x)d\mu(x) = \delta_{nm}h_n,$$

find the asymptotic behavior of the orthogonal polynomial (or, of its zeros):

$$\lim_{n \rightarrow \infty} p_n(x) = ?$$

“**For convenience,**” the integral measure is often given by

$$d\mu(x) = e^{-\frac{n}{T}V(x)} dx,$$

with a fixed potential $V(x)$ and a parameter T .

(In this way, [equilibrium measure](#) description is natural.)

Broad Goal 2:

Given an ensemble of $n \times n$ Hermitian matrices weighted by

$$e^{-\frac{n}{T} \text{tr} V(M)} dM,$$

find the statistics of eigenvalues.

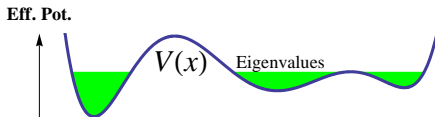
Equivalently, given n coulomb charges (= eigenvalues) on \mathbb{R} , weighted by

$$e^{-\frac{1}{T} \sum_{j=1}^n V(\lambda_j) + 2 \sum_{i>j} \log |\lambda_i - \lambda_j|} \prod_{j=1}^n d\lambda_j,$$

find the distribution of the charges.

g -function and effective potential

The eigenvalues are described by **2 dimensional electric charges** confined in **real line \mathbb{R}** and subject to **potential $V(x)$** .



Effective (electrostatic) potential is same on every support of eigenvalues.

$$\varphi(x) \equiv \frac{1}{2}V(x) - g(x) + \frac{1}{2}\ell,$$

$$g(x) = \int \log(x-s)\rho(s)ds = \log x + \mathcal{O}(1/x),$$

where $\rho(s)$ is the normalized density of eigenvalues.

Colonization of Outpost: A critical transition

As an example, take

$$V(x) = x^4 - x^2 + tx,$$

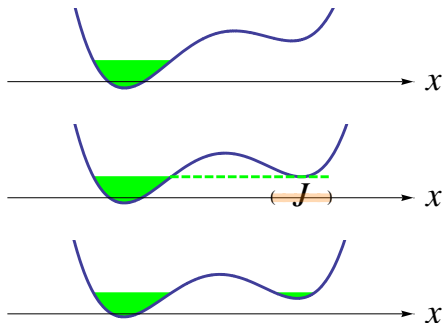
and change t .

There is a critical point when **a new cut begins to form**.

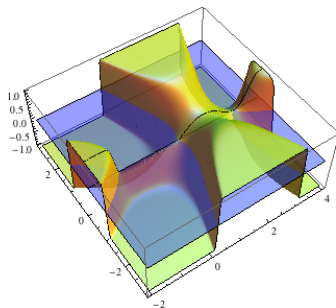
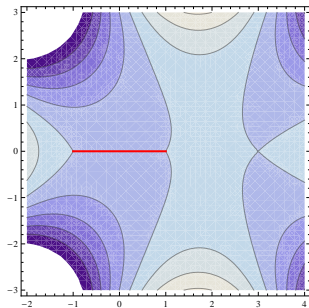
Double scaling limit:

Scaling of T or/and $V(x)$ as $N \rightarrow \infty$, to stay at (or move slowly away from) the critical point.

How to Keep only a **finite eigenvalues** at outpost as $N \rightarrow \infty$?



Landscape



Double Scaling Limit: Recipe

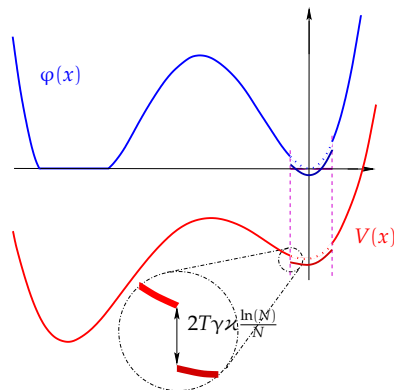
To keep only **finitely fixed** number of zeros at the output:

- $d\mu_N := \begin{cases} e^{-\frac{N}{T}V(x)}dx, & x \in \mathbb{R} \setminus J, \\ N^{2\gamma\kappa} e^{-\frac{N}{T}V(x)}dx, & x \in J. \end{cases}$
($\kappa \in \mathbb{R}$),
- Or, equivalently, define potential to be

$$V(x) - 2\gamma T\kappa \frac{\log N}{N} \chi_J$$

where χ_J is characteristic function of J .

- (# of zeros at the outpost)
= (Nearest non-negative integer to κ).



Some results ..

- 1 The size of the colony increases at every half-integer κ .
- 2 Locally, the eigenvalues are governed by the microscopic matrix model of size $K \times K$ with weight: K is the nearest integer to κ .

$$e^{\text{tr}(M_{\text{micro}}^{2\kappa+2})} dM_{\text{micro}} .$$

Define local (zooming) coordinates ζ by

$$\zeta^{2\kappa+2} = \frac{N}{T} \varphi(x) \approx \frac{N}{T} C_0 x^{2\kappa+2} .$$

More detailed result, Subleading behavior, and Movie:

$P_m(\zeta) \equiv$ (m th degree OP with respect to the measure $e^{-\zeta^{2\nu+2}} d\zeta$).

The zeros at the outpost are described by

$$P_K(\zeta) + \# N^{-2\gamma\delta} P_{K-1}(\zeta), \quad \delta \equiv \kappa - K.$$

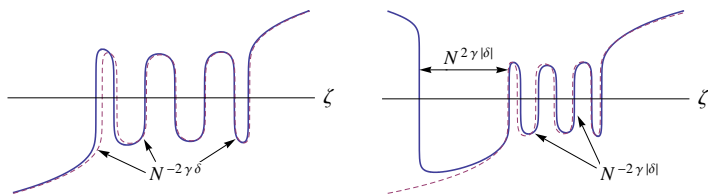


Figure: For $\delta > 0$ (left) and for $\delta < 0$ (right), a schematic plot of polynomial at the outpost. The dashed line is the plot of the leading term.

- The deviation gets smallest at half-integer κ .
- **Stray zero:** For $\delta < 0$ one zero is found away from the rest (in ζ -coordinate).
- At $\delta = -1/2$, this stray zero is globally away.

Statistics of outpost eigenvalues:

- When $|\delta| \neq 1/2$, the kernel at the outpost is given by

$$K_N(x(\zeta), y(\zeta')) = \frac{C_0^\gamma e^{\frac{N}{T}(g(x)+g(y)-\ell)} }{N^{\gamma(2\alpha-1)}} \mathbb{K}_K(\zeta, \zeta') \left(1 + \mathcal{O}(N^{2\gamma|\delta|-\gamma}) \right),$$

where \mathbb{K} stands for the kernel from M_{micro} .

✓ The location of zeros is given asymptotically by $P_{[\alpha]}(\zeta)$.

- When $\delta = \pm 1/2$, the kernel at the outpost is given by the replacement:

$$\mathbb{K}_K(\zeta, \zeta') \Rightarrow \mathbb{K}_K(\zeta, \zeta') + \frac{\alpha_\pm}{\eta_K} P_{K-\frac{1}{2} \pm \frac{1}{2}}(\zeta) P_{K-\frac{1}{2} \pm \frac{1}{2}}(\zeta'),$$

where η_K is the squared norm of P_K , and $\pm\alpha$ is some (non-universal) number.

⇒ No matrix model description.

Comparing with traditional scaling

Traditionally, we do not scale $V(x)$ by a discontinuous jump.

The same result can be reached by a traditional scaling of parameter T .

It is a certain **continuous approximation to our discontinuous potential**.

For instance, one needs to scale T such that:

$$\gamma\kappa \frac{\log N}{N} = (T - 1) \int_b^{x_0} \frac{A(x - x_0)^{2\nu}}{\sqrt{(x - a)(x - b)}} dx, \quad (1)$$

and the end points of the main cut move with N .

By using the discontinuous jump, we can simplify all the side steps and focus on the outpost.

More results:

- Instead of simple $V_{\text{micro}}(\zeta) = \zeta^{2\nu+2}$ we can have an **arbitrary local potential** of the same degree, $V_{\text{micro}}(\zeta) = \zeta^{2\nu+2} + f(\zeta)$, by

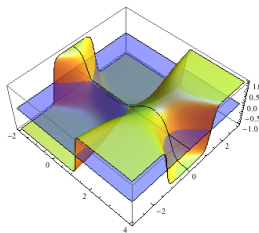
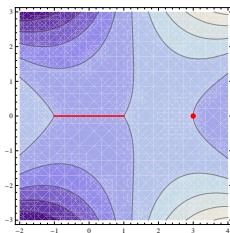
$$V(x) + \frac{T}{N} \left(f(\zeta) - 2\gamma\kappa \log N \right) \chi_I .$$

- **Port Colonization:** For positive definite hermitian matrix model with weight

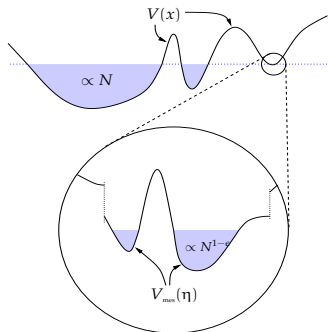
$$(\det M)^\alpha e^{\frac{N}{T} \text{tr} V(M)} dM ,$$

we have an emerging hard-edge. The port is locally governed by the microscopic measure:

$$(\det M_{\text{micro}})^\alpha e^{\text{tr} V_{\text{micro}}(M_{\text{micro}})} dM_{\text{micro}} .$$



Mesoscopic colonization



- We can have **mesoscopic** colony of size $\propto N^{1-\epsilon}$ where $0 < \epsilon < 1$ (Work in progress with M. Bertola, M.Y. Mo).

Relevant Literature

- Eynard B. Universal distribution of random matrix eigenvalues near the "birth of a cut" transition. Journal of Statistical Mechanics (2006) 7. P07005
- Tom Claeys, Birth of a Cut in Unitary Random Matrix Ensembles. Int Math Res Notices Vol. 2008 (submitted to arXiv on 16th Nov.)
- M. Y. Mo, The Riemann-Hilbert approach to double scaling limit of random matrix eigenvalues near the "birth of a cut" transition (submitted to arXiv on 21st Nov.)
- M. Bertola, SYL, First Colonization of a Spectral Outpost in Random Matrix Theory (submitted to arXiv on 23rd Nov.)
- M. Bertola, SYL, First Colonization of a Hard-edge in Random Matrix Theory (2008)

Orthogonality \Leftrightarrow Riemann-Hilbert problem (Its, Kitaev, Fokas)

From a series of polynomials,

$$Y(x) = \begin{bmatrix} p_n(x) & \mathcal{C}[p_n](x) \\ \frac{-2\pi i}{h_{n-1}} p_{n-1}(x) & \frac{-2\pi i}{h_{n-1}} \mathcal{C}[p_{n-1}](x) \end{bmatrix}, \quad \mathcal{C}[p](x) := \frac{1}{2\pi i} \int \frac{p(s)}{s-x} d\mu(s).$$

The orthogonality is equivalent to the following conditions.

- (1) $Y(x) = \left(\mathbb{I} + \mathcal{O}\left(\frac{1}{x}\right) \right) \begin{bmatrix} x^n & 0 \\ 0 & x^{-n} \end{bmatrix},$
- (2) $Y_+(x) = Y_-(x) \begin{bmatrix} 1 & \mu'(x) \\ 0 & 1 \end{bmatrix}, \quad x \in \mathbb{R}.$

The solution of the matrix Riemann-Hilbert problem is **unique**.

From now, let us write

$$d\mu(x) = \omega e^{-NV(x)} dx,$$

where ω includes all the non-conventional part. In our case,

$$\omega = N^{\kappa \times J}.$$

Steepest Descent Method: (DKMVZ '99)

(Overly simplified) Three steps to solving the RHP: $\Rightarrow g(x), \Psi^G, \mathbb{E}$.

(1) Find g -function (or Boutroux curve). It amounts to finding the limit:

$$\lim_{n \rightarrow \infty} p_n(x)^{1/n} = e^{g(x)}.$$

(2) Find the global parametrix $\Psi^G(x)$ using the **asymptotic RHP**. It amounts to

$$Y(x) \approx \Psi^G(x) e^{Ng(x)\sigma_3}, \quad \sigma_3 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

(3) **Local parametrix** near the outpost.

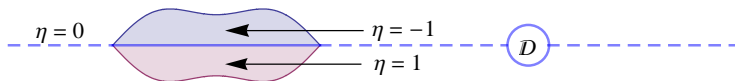
$$Y(x) \approx \Psi^G(x) \mathbb{E} e^{Ng(x)\sigma_3}.$$

Step (1): See the landscape

Step (2) Lens opening: (or Lip augmentation)

We define a matrix from $Y(x)$.

- $\Psi(x) := Y(x) e^{-Ng(x)\sigma_3} \begin{bmatrix} 1 & 0 \\ \eta \frac{e^{2N\varphi(x)}}{\omega} & 1 \end{bmatrix}.$



New **small** jump along the lip lines.

Approximate model RHP:

- $\Psi_+^G = \Psi_-^G \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ on cut.

In the large N limit,

$$\Psi \longrightarrow \Psi^G.$$

Ψ^G : (Spinor part)

For one cut, solution can be written explicitly as

$$\Psi^G = \frac{2^{-r\sigma_3}}{\sqrt{2x'(t)}} \begin{bmatrix} t^r & it^{-r-1} \\ -it^{r-1} & t^{-r} \end{bmatrix} \left(\frac{t-t_0}{t_0t-1} \right)^{K\sigma_3},$$

with a uniformizing paratrization of $x = \frac{1}{2} \left(t + \frac{1}{t} \right)$.

Note the **freedom of adding any number (K) of singularities.**

Step (3): Local parametrix

To keep K zeros at the outpost (at $x = 0$), Ψ^G must have K zeros there.

$$\Psi^G = A(x) N^{-\frac{1}{2}\delta\sigma_3} \tilde{x}^{K\sigma_3}.$$

$A(x)$ is analytic at $x = 0$. δ will be determined soon.

✓ **Remove the singular (blue) part by changing** $\Psi^G \rightarrow \Psi^G \mathbb{E}$ **inside** \mathbb{D} .

Substitute the singular (blue) part by a smooth function $R(\zeta)$.

$$\begin{aligned}\Psi^G \mathbb{E} &= A(x) R(\zeta) \\ &= \Psi^G \cdot \boxed{N^{\frac{1}{2}\delta\sigma_3} \tilde{x}^{-K\sigma_3} R(\zeta)}.\end{aligned}$$

We have introduced a new jump (= \mathbb{E}).

⇒ Want \mathbb{E} to be close to 1 on $\partial\mathbb{D}$.

Solve for $R(\zeta)$

Demand the **exact** jump condition on $\Psi^G \mathbb{E}$.

$$R_+ = R_- \begin{bmatrix} 1 & N^\kappa e^{-\zeta^2} \\ 0 & 1 \end{bmatrix}.$$

Then the solution to the jump matrix is written by Freud polynomials:

$$\begin{aligned} R &= \begin{bmatrix} P_K(\zeta) & \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{P_K(s) e^{-s^2} ds}{s-\zeta} \\ \frac{-2\pi i}{\eta_{K-1}} P_{K-1}(\zeta) & \frac{-1}{\eta_{K-1}} \int_{\mathbb{R}} \frac{P_{K-1}(s) e^{-s^2} ds}{s-\zeta} \end{bmatrix} N^{-\frac{1}{2}\kappa\sigma_3} \\ &= \left(\mathbb{I} + \mathcal{O}\left(\frac{1}{\zeta}\right) \right) \zeta^{K\sigma_3} N^{-\frac{1}{2}\kappa\sigma_3} \\ &= \left(\mathbb{I} + \mathcal{O}\left(\frac{1}{\zeta}\right) \right) \tilde{x}^{K\sigma_3} N^{-\frac{1}{2}(\kappa-K)\sigma_3} \end{aligned}$$

This is a solution when $\delta := \kappa - K$.

Error:

Now let us estimate the error on $\partial\mathbb{D}$.

$$\begin{aligned}\mathbb{E} &= \boxed{N^{\frac{1}{2}\delta\sigma_3} \tilde{x}^{-K\sigma_3} R(x)} \\ &= N^{\frac{1}{2}\delta\sigma_3} \tilde{x}^{-K\sigma_3} \left(\mathbb{I} + \mathcal{O}\left(\frac{1}{\zeta}\right) \right) \tilde{x}^{K\sigma_3} N^{-\frac{1}{2}(\kappa-K)\sigma_3} \\ &= \mathbb{I} + \mathcal{O}\left(N^{|\delta|-\frac{1}{2}}\right).\end{aligned}$$

- When κ is a half integer, we need to improve the error bound!

(Partial) Schlesinger transform

The global parametrix Ψ^G with K zeros and $K + 1$ zeros at the outpost are related by [Schlesinger transform](#).

$$\Psi_{K+1}^G = \left(\mathbf{I} + \frac{S}{x} \right) \Psi_K^G.$$

Proof) $\Psi_{K+1}^G (\Psi_K^G)^{-1}$ has no jump, converges to the identity at the infinity, and has at most a simple pole at the outpost.

“Partial Schlesinger transform”

Idea is to find F_κ that interpolates between K and $K + 1$ such that

$$\left(\mathbf{I} + \frac{F_\kappa}{x} \right) \Psi^G,$$

can reduce the error further for $K < \kappa < K + 1$.

All order correction: Schematically

Note that the error comes from the **red part** of $R(\zeta)$.

$$R(\zeta) = \left(\mathbb{I} + \frac{Y_1}{\zeta} + \frac{Y_2}{\zeta^2} + \dots \right) \tilde{x}^{K\sigma_3} N^{-\frac{1}{2}(x-K)\sigma_3} .$$

Absorb some of **the red part** into (new) Ψ^G so that the jump gets smaller,

$$\left(\mathbb{I} + \frac{Y_1}{\zeta} + \frac{Y_2}{\zeta^2} + \dots \right) = \left(\mathbb{I} + \frac{G_1}{\zeta} + \dots + \frac{G_m}{\zeta^m} \right) \left(\mathbb{I} + \frac{\tilde{Y}_1}{\zeta^{p+1}} + \frac{\tilde{Y}_2}{\zeta^{p+2}} + \dots \right) .$$

If **blue part** goes to Ψ^G then the error is reduced by $\mathcal{O}(N^{p/2})$.

✓ Such decomposition is possible up to an arbitrary order.

Since $\Psi_{\text{new}}^G (\Psi^G)^{-1}$ has no jump and only has a **finite order pole at the outpost**,

$$\Psi_{\text{new}}^G = \left(\mathbb{I} + \frac{F_1}{x} + \dots + \frac{F_m}{x^m} \right) \Psi^G .$$

This correction procedure involves only linear algebra.

Having $A(x) = A_0 + A_1x + A_2x^2 + \dots$,

One absorbs $\left(\mathbb{I} - \frac{N_1}{x}\right)$ by $\left(1 - \frac{A_0 N_1 (A_0)^{-1}}{x}\right)$.

One absorbs $\left(\mathbb{I} - \frac{N_k}{x^k}\right)$ by $\left(\mathbb{I} + \frac{F_1}{x} + \dots + \frac{F_k}{x^k}\right)$ where

$(A_0, \dots, A_{k-1}) \cdot N_k =$

$$= (F_1, \dots, F_k) \left(\begin{bmatrix} 0 & \dots & 0 & A_0 \\ 0 & \dots & A_0 & A_1 \\ \vdots & & & \vdots \\ A_0 & A_1 & \dots & A_{k-1} \end{bmatrix} - \begin{bmatrix} A_1 & A_2 & \dots & A_k \\ A_2 & A_3 & \dots & A_{k+1} \\ \vdots & & & \vdots \\ A_k & A_{k+1} & \dots & A_{2k-1} \end{bmatrix} \cdot N_k \right).$$

Ercolani & McLaughlin '02

Assume one obtained Ψ^G and Ψ^D .

The error is given by the jump on $\partial\mathbb{D}$:

$$(\Psi^G)^{-1}\Psi^D = \mathbb{I} + \frac{V_1}{N} + \frac{V_2}{N^2} + \dots$$

Improve both Ψ_G and Ψ^D by

$$\begin{aligned}\Psi^G &\rightarrow \Psi^G \left(\mathbb{I} + \frac{S_1^-}{N} + \frac{S_2^-}{N^2} + \dots \right) \\ \Psi^D &\rightarrow \Psi^D \left(\mathbb{I} + \frac{S_1^+}{N} + \frac{S_2^+}{N^2} + \dots \right)\end{aligned}$$

Improved system does not have any jump if

$$\left(\mathbb{I} + \frac{V_1}{N} + \frac{V_2}{N^2} + \dots \right) \left(\mathbb{I} + \frac{S_1^+}{N} + \frac{S_2^+}{N^2} + \dots \right) = \left(\mathbb{I} + \frac{S_1^-}{N} + \frac{S_2^-}{N^2} + \dots \right),$$

on $\partial\mathbb{D}$.

This gives series of additive Riemann-Hilbert problems:

$$S_1^- - S_1^+ = V_1;$$

$$S_k^- - S_k^+ = \sum_{j=1}^{k-1} S_j^+ V_{k-j}, \quad k = 2, 3, \dots$$

which are solved by

$$S_1(z) = \frac{-1}{2\pi i} \int \frac{V_1(s)}{s-z} ds,$$

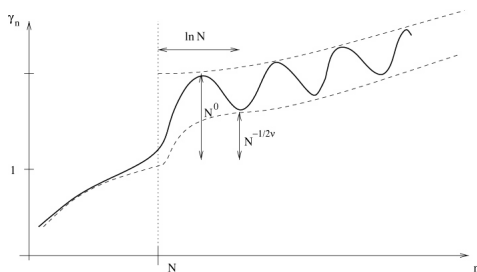
$$S_k(z) = \frac{-1}{2\pi i} \int \frac{\sum_{j=1}^{k-1} S_j^+ V_{k-j}}{s-z} ds, \quad k = 2, 3, \dots$$

Application: Gradient catastrophe (Grava, Claeys, Klein).

$$V(x) = x^4 - 2x^2 + 4tx .$$

$$y_{12}^{(3)} - 16N^2 y'_{12} - 32N^2 y_{12} (2Nt - 3y_{21} y'_{12}) = 0 ,$$

$$y_{21}^{(3)} - 16N^2 y'_{21} + 32N^2 y_{21} (2Nt + 3y_{12} y'_{21}) = 0 .$$



(from Eynard's paper)