Critical behavior in non-intersecting path ensembles

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Based on joint works with

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and

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Outline

1. Introduction
   - OP ensembles and non-intersecting path ensembles
   - Multiple orthogonal polynomials (MOPs) and MOP ensembles
   - Riemann-Hilbert problem for MOPs
2. Non-intersecting Brownian motions
   - One starting position, two ending positions
   - Two starting positions, two ending positions
   - Critical separation and Painlevé II
   - Brief outline of steepest descent analysis
3. Non-intersecting squared Bessel paths
   - MOP ensemble with modified Bessel weights
   - Global and local regimes
   - Critical time and new family of kernels

Part 1: Introduction

- The structure behind eigenvalues of unitary random matrices appears in other situations as well
  - Non-intersecting paths
  - Tiling problems, stochastic growth problems
  - Representation theory of large groups
- They are determinantal point processes: a random point process so that correlation functions have determinantal form

$$\det \left[K(x_j, x_k) \right]_{j,k=1,\ldots,m}$$

  - K is the correlation kernel

OP ensembles

- The eigenvalues from a unitary random matrix ensemble

$$\frac{1}{Z_n} \exp(-\operatorname{Tr} V(M)) dM$$

defined on $n \times n$ Hermitian matrices M follow a determinantal point process with correlation kernel

$$K(x,y) = \sqrt{e^{-V(x)}}\sqrt{e^{-V(y)}} \sum_{j=0}^{n-1} p_j(x) p_j(y)$$

  - p_j is the orthonormal polynomial of degree j with weight $e^{-V(x)}$
  - Orthogonal Polynomial ensemble

Non-intersecting path ensembles

- **Given n independent copies $X_1(t), \ldots, X_n(t)$ of a 1-D strong Markov process with continuous sample paths, conditioned to**
  - **start at time $t = 0$ at n given points $a_1 < a_2 < \cdots < a_n$,**
  - **end at time $t = 1$ at n given points $b_1 < b_2 < \cdots < b_n$,**
  - **not intersect in the full time interval $(0, 1)$.**
- **Then the positions of the paths at given time $t \in (0, 1)$ are a determinantal point process.**
  - **Consequence of Karlin-McGregor (1959) theorem.**

Correlation kernel

- **If $p_t(x,y)$ is the Markov process transition p.d.f., then by Karlin-McGregor theorem, the p.d.f. for the positions $x_1,\ldots,x_n$ of the paths at time t is**

$$\frac{1}{Z_n}\det\left[p_t(a_j,x_k)\right]_{j,k=1,\ldots,n}\cdot\det\left[p_{1-t}(x_k,b_j)\right]_{j,k=1,\ldots,n}$$

- **The correlation kernel K takes the form**

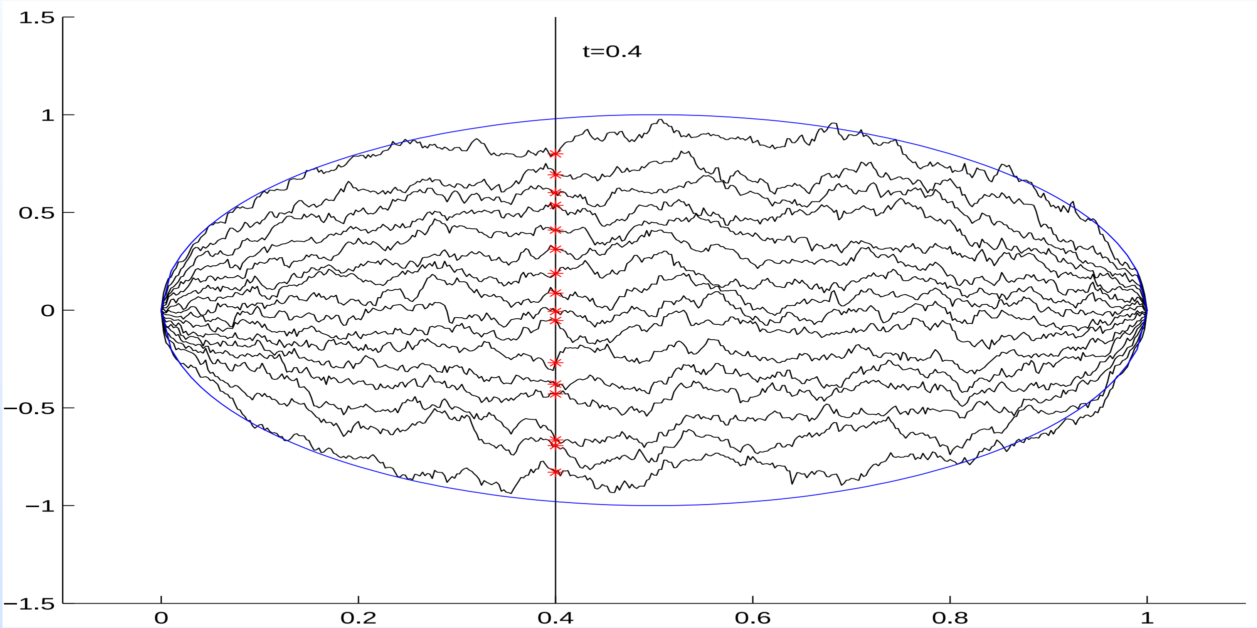
$$K(x,y)=\sum_{j=1}^{n}\phi_j(x)\psi_j(y)$$

where biorthogonal functions ϕ_j and ψ_j arise from biorthogonalizing the two sets of functions $x\mapsto p_t(a_j,x)$ **and** $x\mapsto p_{1-t}(x,b_j)$

$$\int\phi_j(x)\psi_k(x)dx=\delta_{j,k}$$

Confluent Brownian motion case as a model for GUE

- The **confluent case** $a_j \to 0$, $b_j \to 0$, for Brownian motion leads to (a variation of) Dyson's Brownian motion



Biorthogonal ensemble

- OP ensembles and non-intersecting path ensembles are examples of **biorthogonal ensembles**

$$\mathcal{P}(x_1, \ldots, x_n) = \frac{1}{Z_n} \det\left[f_j(x_k)\right]_{j,k=1,\ldots,n} \cdot \det\left[g_j(x_k)\right]_{j,k=1,\ldots,n}$$

  for given functions $f_1, \ldots, f_n$, and $g_1, \ldots, g_n$.

- A biorthogonal ensemble has a correlation kernel

$$K_n(x,y) = \sum_{j=1}^{n} \phi_j(x)\psi_j(y) \qquad \text{where}$$

  - $\phi_1, \ldots, \phi_n$ is a basis for $\mathrm{span}\{f_1, \ldots, f_n\}$
  - $\psi_1, \ldots, \psi_n$ is a basis for $\mathrm{span}\{g_1, \ldots, g_n\}$
  - $\int \phi_j(x)\psi_k(x)dx = \delta_{j,k}$ (**biorthogonality**)

Multiple Orthogonal Polynomial ensembles

- If $f_j(x) = x^{j-1}w_1(x)$ and $g_j(x) = x^{j-1}w_2(x)$, then **OP ensemble**.
- **MOP ensemble** if

$$f_j\text{'s are:} \quad x^{j-1}w_{1,k}(x), \quad \text{for } j = 1, \ldots, n_k, \quad k = 1, \ldots, p,$$

$$g_j\text{'s are:} \quad x^{j-1}w_{2,k}(x), \quad \text{for } j = 1, \ldots, m_k, \quad k = 1, \ldots, q,$$

with $\sum n_k = \sum m_k$.

- Biorthogonal functions take the form

$$\phi_j(x) = \sum_{k=1}^{p} A_{j,k}(x)w_{1,k}(x), \qquad \psi_j(x) = \sum_{k=1}^{q} B_{j,k}(x)w_{2,k}(x)$$

with $\sum_k \deg A_{j,k} = \sum_k \deg B_{j,k} = j - 1$.

- We call $A_{j,k}$ and $B_{j,k}$ **multiple orthogonal polynomials** ($p + q$ MOPs).

Riemann-Hilbert problem for $1+2$ MOPs

- **MOPs (with $1+2$ weights) satisfy a 3×3 matrix valued RH problem**

Van Assche-Geronimo-Kuijlaars (2001)

generalization of Fokas-Its-Kitaev (1992) RH problem for OPs

  - $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ **is analytic**
  - $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_{1,1}(x)w_{2,1}(x) & w_{1,1}(x)w_{2,2}(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad x \in \mathbb{R}$
  - $Y(z) = (I_3 + \mathcal{O}(1/z)) \begin{pmatrix} z^{n_1} & 0 & 0 \\ 0 & z^{-m_1} & 0 \\ 0 & 0 & z^{-m_2} \end{pmatrix}$ **as** $z \to \infty$

(recall $n_1 = m_1 + m_2$)

- **The correlation kernel for the MOP ensemble is given in terms of the solution Y of the RH problem**

$$K(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_{2,1}(y) & w_{2,2}(y) \end{pmatrix} Y_+(y)^{-1} Y_+(x) \begin{pmatrix} w_{1,1}(x) \\ 0 \\ 0 \end{pmatrix}$$

Bleher-Kuijlaars (2004), Daems-Kuijlaars (2004)

Riemann-Hilbert problem for $2+2$ MOPs

- $p+q$ **MOPs satisfy a** $(p+q)\times(p+q)$ **matrix valued RH problem stated here for the** $2+2$ **case**

  Daems-Kuijlaars (2007)

  - $Y : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{4\times 4}$ **is analytic**
  - $Y_+(x) = Y_-(x)\begin{pmatrix} I & W(x) \\ 0 & I \end{pmatrix}$, **for** $x\in\mathbb{R}$ **where**

$$W(x) = \begin{pmatrix} w_{1,1}(x) \\ w_{1,2}(x) \end{pmatrix} \begin{pmatrix} w_{2,1}(x) & w_{2,2}(x) \end{pmatrix}$$

  - $Y(z) = (I_4 + \mathcal{O}(1/z))\,\mathrm{diag}\,(z^{n_1}, z^{n_2}, z^{-m_1}, z^{-m_2})$ **as** $z\to\infty$

- **The correlation kernel for the MOP ensemble is**

$$K(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 0 & w_{2,1}(y) & w_{2,2}(y) \end{pmatrix} Y_+(y)^{-1} Y_+(x) \begin{pmatrix} w_{1,1}(x) \\ w_{1,2}(x) \\ 0 \\ 0 \end{pmatrix}$$

Part 2: Non intersecting Brownian paths

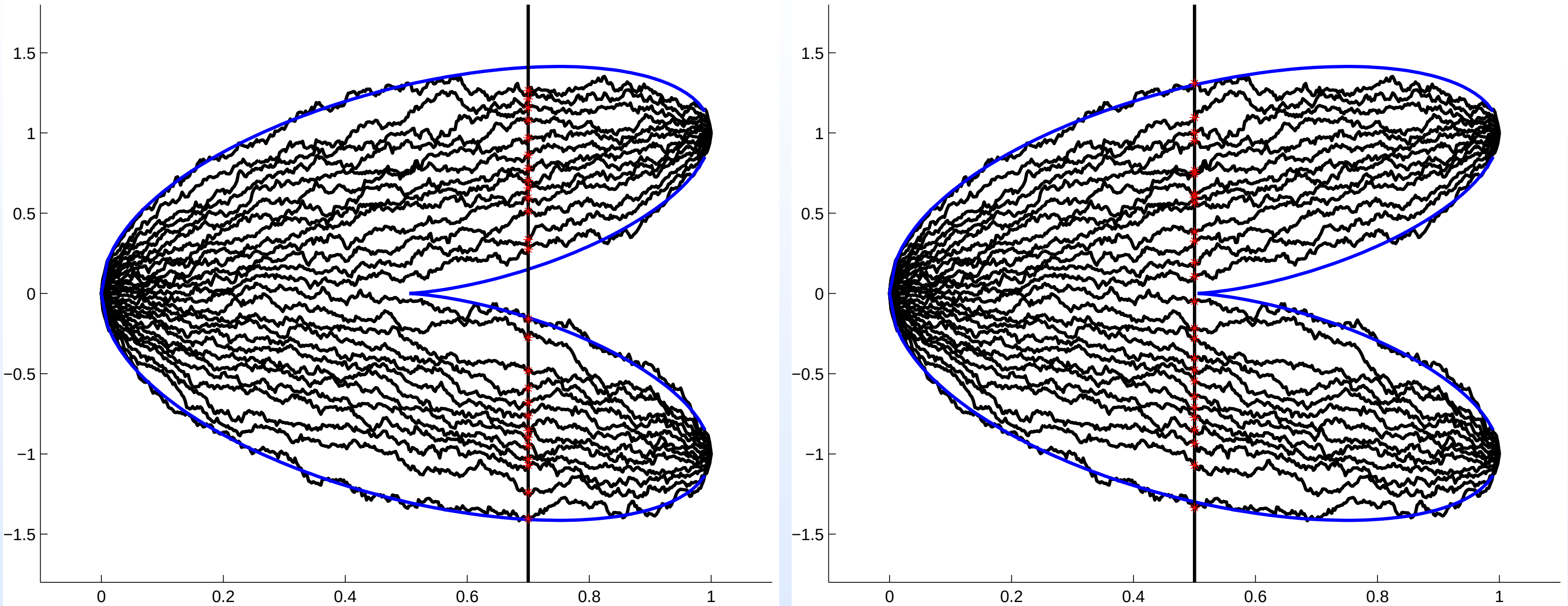
- Take n non-intersecting Brownian motions so that
  - p starting points $a_1, \ldots, a_p$ and q ending points $b_1, \ldots, b_q$
  - n_k paths start at a_k and m_k paths end at b_k
- The positions of the paths at time $t \in (0,1)$ are MOP ensemble with $p+q$ weights

$$w_{1,k}(x) = e^{-\frac{(x-a_k)^2}{2t}}, \qquad k = 1, \ldots, p,$$

$$w_{2,k}(x) = e^{-\frac{(x-b_k)^2}{2(1-t)}}, \qquad k = 1, \ldots, q.$$

- MOPs in this case are called **multiple Hermite polynomials**

One starting position, two ending positions



- Asymptotic analysis of the 3×3 matrix valued RH problem

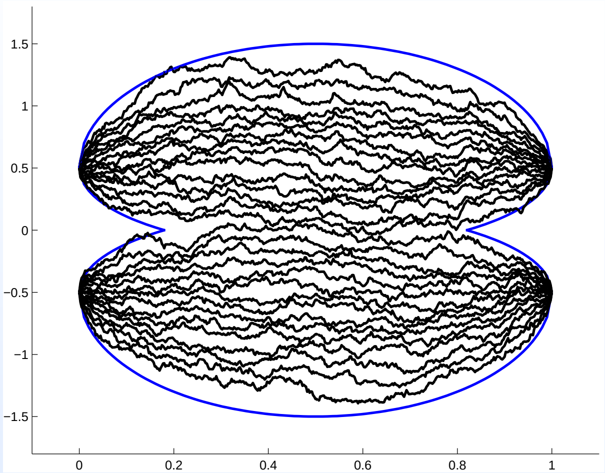
  (Aptekarev)-Bleher-Kuijlaars (2004, 2005, 2007), McLaughlin (2007), Orantin (arXiv)

  - Usual scaling limits from random matrix theory are found in the bulk (sine kernel) and at the edges (Airy kernel)
  - Double scaling limit at the cusp point: a family of Pearcey kernels

    Brézin-Hikami (1998), Tracy-Widom (2006), Okounkov-Reshetikhin (2007)

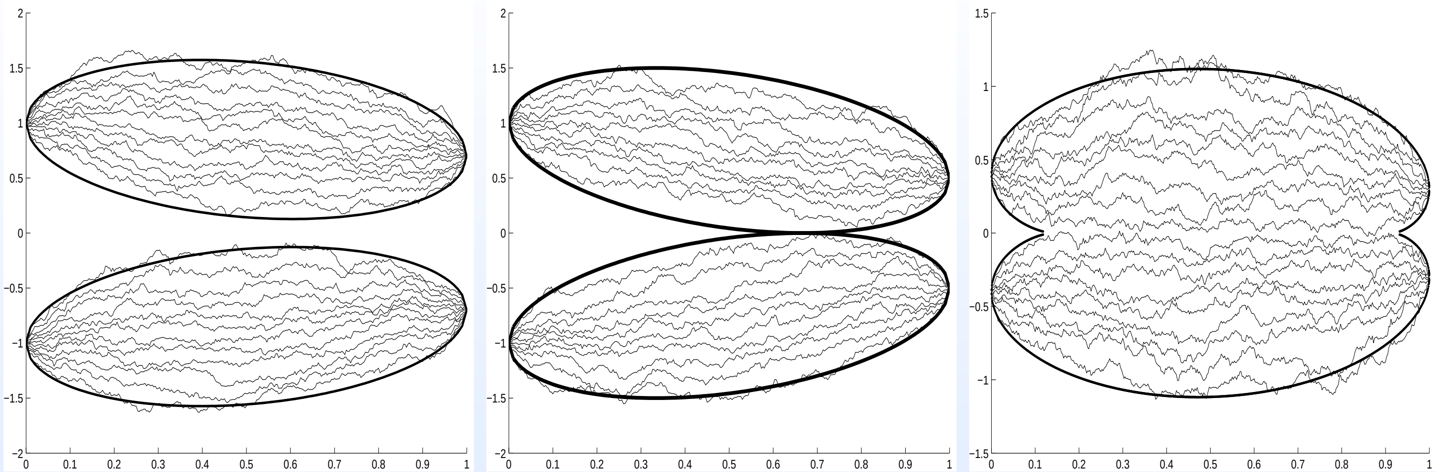
    Adler-van Moerbeke (2007), Bleher-Kuijlaars (2007)

Two starting positions, two ending positions



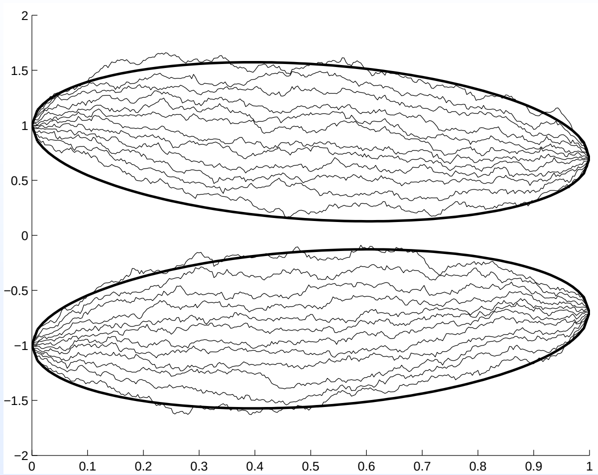
- Asymptotic analysis of 4×4 matrix valued RH problem for case $n_1 = n_2 = m_1 = m_2$ and small separation **Daems-Kuijlaars-Veys (2008)**
  - The paths fill out an algebraic curve of degree six
  - Sine kernel in the bulk and Airy kernel at the edges
  - Pearcey kernels at the cusp points is expected (we did not prove it)

Critical separation



- If $n_1 = m_1$ and $n_2 = m_2$ there is a phase transition at critical separation
  - At critical separation the paths fill out two tangent ellipses.
  - At larger separation the two ellipses are disjoint.
  - At smaller separation the paths fill out one simply connected domain bounded by an algebraic curve.

Large separation



- **Take** $n_1 = m_1$, $n_2 = m_2$,
- $\frac{n_1}{n} = p_1$, $\frac{n_2}{n} = p_2 = 1 - p_1$,
- $a_1 > a_2$, $b_1 > b_2$.
- **When separation between initial and ending positions is large enough, one expects two disjoint ellipses.**

- **Paths at time** t **are on** $[\alpha_2(t), \beta_2(t)] \cup [\alpha_1(t), \beta_1(t)]$ **where**

$$\alpha_j(t) = (1-t)a_j + tb_j - 2\sqrt{p_j t(1-t)}$$

$$\beta_j(t) = (1-t)a_j + tb_j + 2\sqrt{p_j t(1-t)}$$

- **Ellipses are disjoint if** $\beta_2(t) < \alpha_1(t)$ **for all** $t \in (0,1)$**, and this happens if and only if**

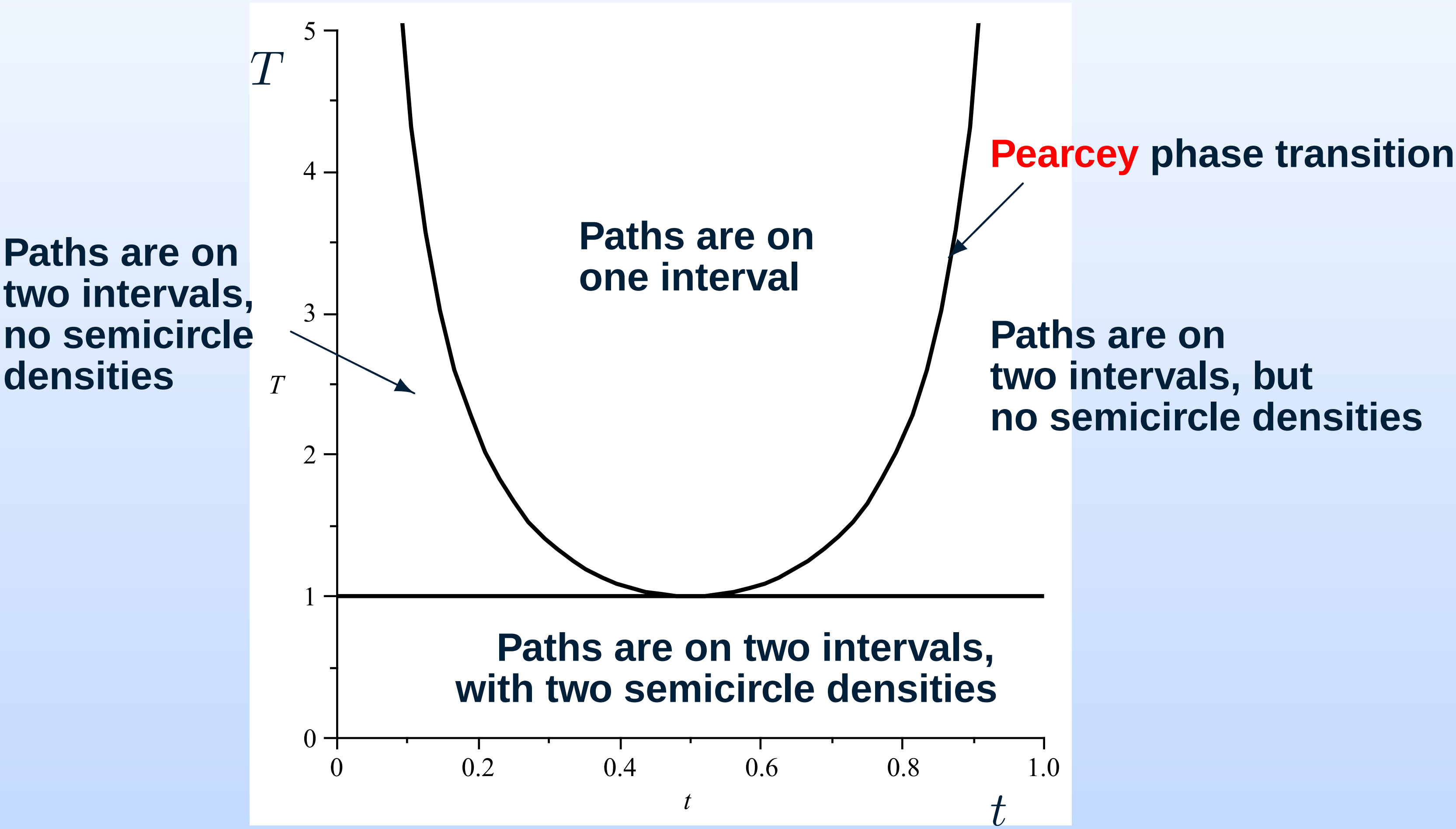
$$(a_1 - a_2)(b_1 - b_2) > (\sqrt{p_1} + \sqrt{p_2})^2$$

Phase diagram

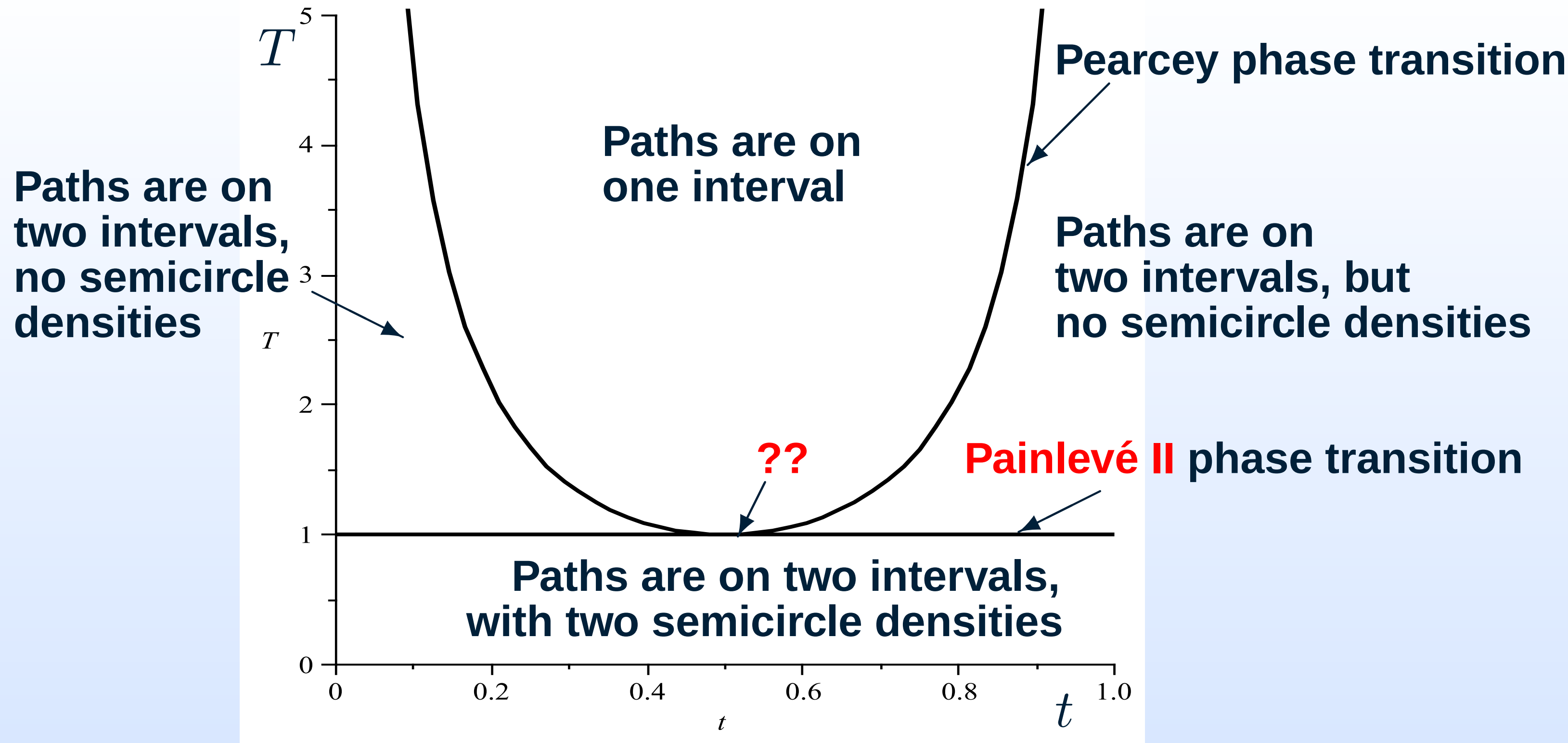
- **Brownian motion transition probability** $\sqrt{\frac{n}{2\pi t T}} \exp\left(-\frac{n}{T}\frac{(x-y)^2}{2t}\right)$
- **Take** $(a_1 - a_2)(b_1 - b_2) = (\sqrt{p_1} + \sqrt{p_2})^2$**: critical separation if** $T = 1$**.**

Phase diagram

- Brownian motion transition probability $\sqrt{\frac{n}{2\pi t T}} \exp\left(-\frac{n}{T}\frac{(x-y)^2}{2t}\right)$
- Take $(a_1 - a_2)(b_1 - b_2) = (\sqrt{p_1} + \sqrt{p_2})^2$: critical separation if $T = 1$.



Phase diagram: Painlevé II transition at $T = 1$



- **The transition at $T = 1$ at the non-critical time is governed by the Hastings-McLeod solution of the Painlevé II equation**
- **There are no new scaling limits of the correlation kernel, but the Hastings-McLeod solution is seen in the asymptotics of the recurrence coefficients of the multiple Hermite polynomials** **Delvaux-Kuijlaars**

Outline of proof

- Asymptotic analysis of 4×4 matrix valued RH problem with jump condition

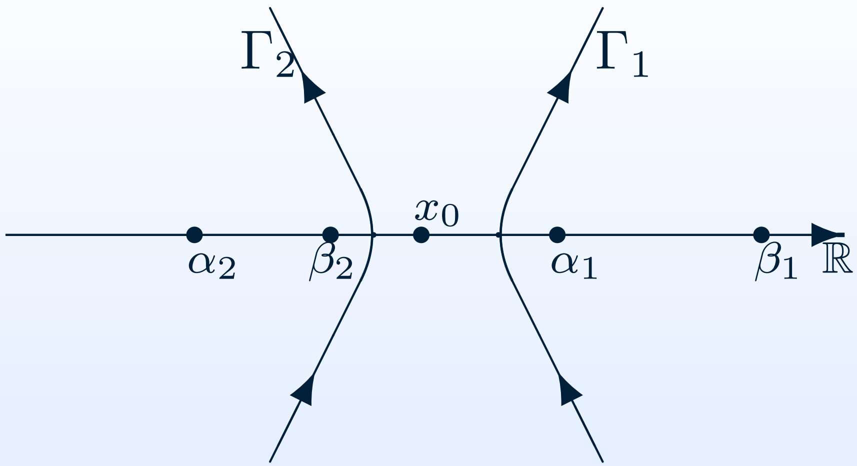
$$Y_+(x) = Y_-(x) \begin{pmatrix} I & W(x) \\ 0 & I \end{pmatrix}, \quad \text{for } x \in \mathbb{R},$$

where $W(x) = \begin{pmatrix} w_{1,1}(x) \\ w_{1,2}(x) \end{pmatrix} \begin{pmatrix} w_{2,1}(x) & w_{2,2}(x) \end{pmatrix}$ with

$$w_{1,k}(x) = \exp\left(-\frac{n}{T} \frac{(x-a_k)^2}{2t} \right), \quad w_{2,k}(x) = \exp\left(-\frac{n}{T} \frac{(x-b_k)^2}{2(1-t)} \right), \qquad k = 1, 2,$$

- Direct application of g-functions corresponding to semi-circle laws fails.
- First we do a global preliminary transformation ("global opening of lenses")

Preliminary transformation



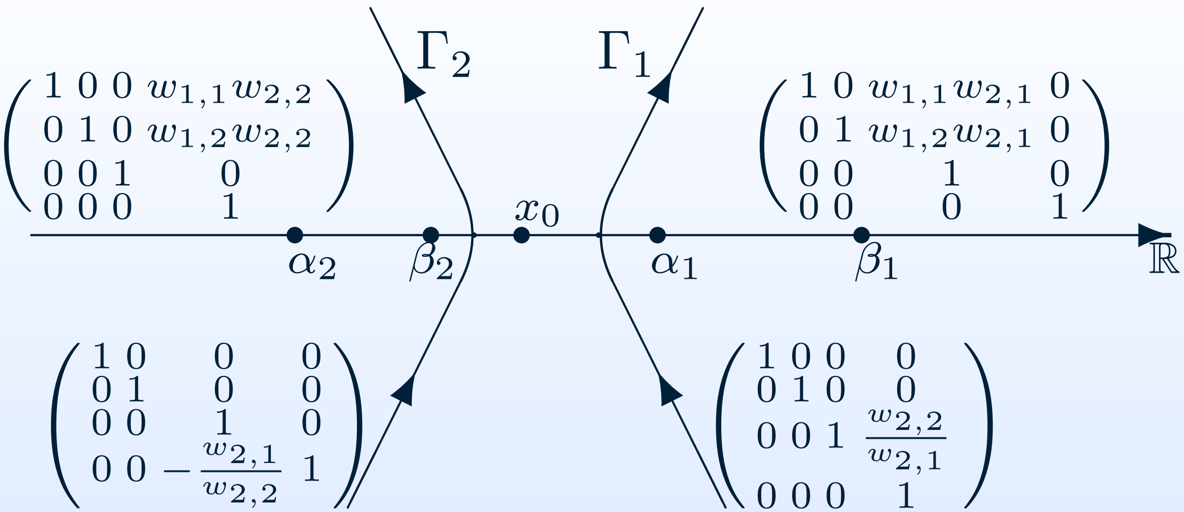
- **Assume small time $t < t_{cr}$, and define**

$$X = Y \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{w_{2,2}}{w_{2,1}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to the right of } \Gamma_1,$$

$$X = Y \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{w_{2,1}}{w_{2,2}} & 1 \end{pmatrix} \quad \text{to the left of } \Gamma_2,$$

$$X = Y \quad \text{elsewhere.}$$

Jumps in the RH problem for X



- **Now we define**

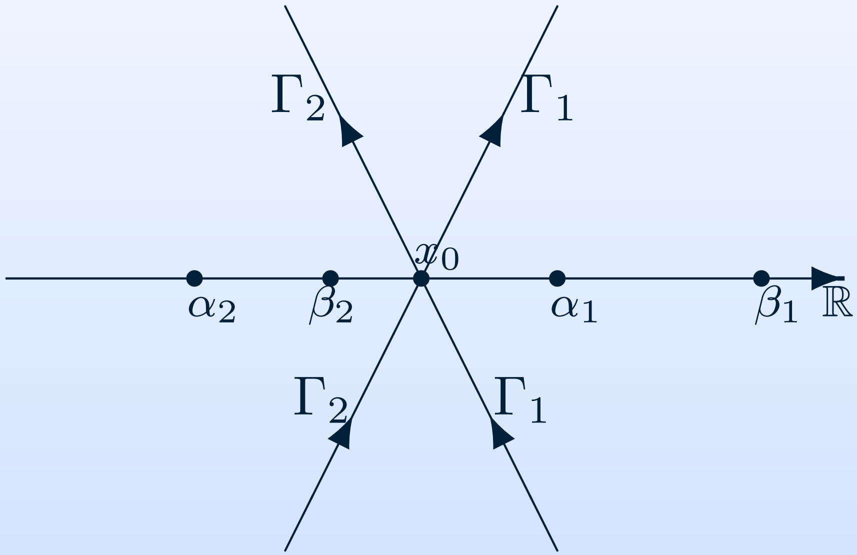
$$T = L^{-n} X G^n L^n$$

where L is a constant diagonal matrix, $G = \operatorname{diag}\left(e^{-g_1}, e^{-g_2}, e^{g_1}, e^{g_2}\right)$.

- **If we in a situation of large separation, i.e., $T < 1$, then we can take Γ_1 and Γ_2 so that jumps on these contours tend to identity matrix as $n \to \infty$ at an exponential rate.**

Critical separation

- It does not work in the case of critical separation, i.e., $T = 1$.
- Then the best we can do is to let Γ_1 and Γ_2 meet in a point $x_0 \in (\beta_2, \alpha_1)$.
- Away from x_0 the jumps are exponentially close to the identity matrix.



- A local parametrix at x_0 is constructed with the help of the RH problem for the Hastings-McLeod solution of Painlevé II $$q'' = sq + 2q^3$$

Baik-Deift-Johansson (1999), Bleher-Its (2003), Claeys-Kuijlaars (2006)

Part 3: Squared Bessel process

- **Squared Bessel process** is a strong Markov process on $[0, \infty)$ depending on a parameter $\alpha > -1$, with transition probabilities

$$p_t^\alpha(x,y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\alpha/2} e^{-(x+y)/(2t)} I_\alpha\left(\frac{\sqrt{xy}}{t}\right), \qquad x, y > 0,$$

where I_α is the **modified Bessel function** of the first kind of order α

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}$$

- If $d = 2(\alpha+1)$ is an integer, then the squared Bessel process is the square of the distance to the origin of a d-dimensional Brownian motion.
  - Non-intersecting squared Bessel processes were considered by König-O'Connell (2001), Katori-Tanemura (2007), and Desrosiers-Forrester (2008). Related work by Tracy-Widom (2007) on non-intersecting Brownian excursions.

Simulation of 50 non-intersecting paths

- **Confluent case:** all $a_j \to a > 0$, all $b_j \to 0$.

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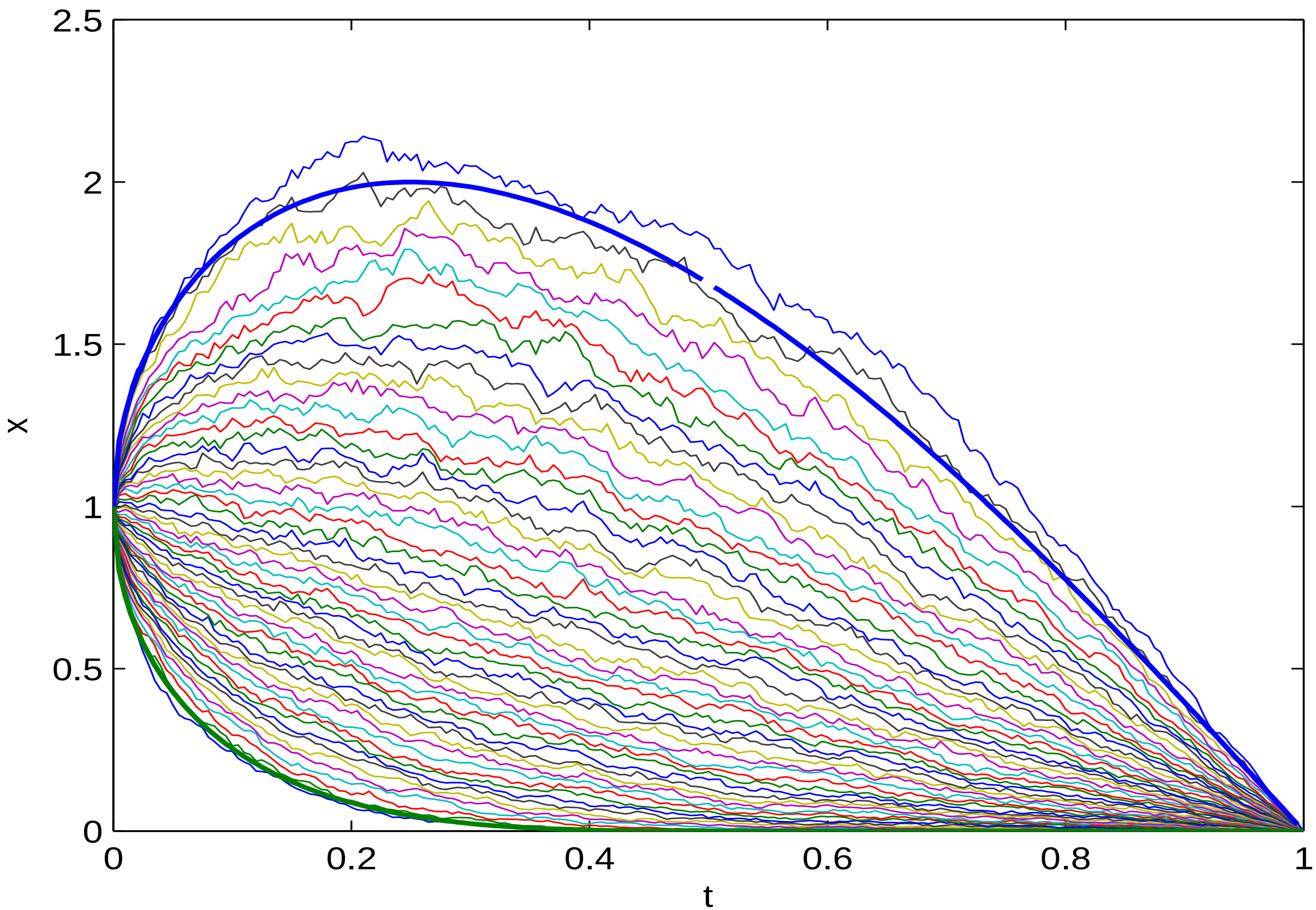

Non-intersecting squared Bessel paths

- In confluent limit $a_j \to a$, $b_j \to 0$, the positions of n non-intersecting squared Bessel paths at time $t \in (0,1)$ are a **MOP ensemble** with $1+2$ weight functions

$$w_{1,1}(x) = e^{-\frac{x}{2(1-t)}}, \qquad n_1 = n,$$

$$w_{2,1}(x) = x^{\alpha/2} e^{-\frac{x}{2t}} I_\alpha\left(\frac{\sqrt{ax}}{t}\right), \qquad m_1 := \lceil n/2 \rceil,$$

$$w_{2,2}(x) = x^{(\alpha+1)/2} e^{-\frac{x}{2t}} I_{\alpha+1}\left(\frac{\sqrt{ax}}{t}\right), \qquad m_2 := \lfloor n/2 \rfloor.$$

  - Proof depends on differential equations for modified Bessel functions

- Corollary: correlation kernel is expressible in terms of solution of RH problem with jump condition for $x > 0$,

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & x^{\alpha/2} e^{-\frac{x}{t(1-t)}} I_\alpha(\frac{\sqrt{ax}}{t}) & x^{(\alpha+1)/2} e^{-\frac{x}{t(1-t)}} I_{\alpha+1}(\frac{\sqrt{ax}}{t}) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Rescaling and ODE

- To obtain non-trivial limits as $n \to \infty$, we rescale $a \mapsto na$, $x \mapsto nx$.
- Rescaled weights

$$w_{1,1}(x)w_{2,1}(x) = x^{\alpha/2} e^{-\frac{nx}{t(1-t)}} I_\alpha\left(n\frac{\sqrt{ax}}{t}\right),$$

$$w_{1,1}(x)w_{2,2}(x) = x^{(\alpha+1)/2} e^{-\frac{nx}{t(1-t)}} I_{\alpha+1}\left(n\frac{\sqrt{ax}}{t}\right).$$

- MOPs with modified Bessel weights were studied in two papers by Coussement-Van Assche (2003). They found a third order ODE for them, which in the rescaled variables is

$$xy'''(x) + \left((2+\alpha) - \frac{2nx}{t(1-t)}\right) y''(x)$$
$$+ \left(\frac{n^2 x}{t^2(1-t)^2} + \frac{n(n-\alpha-2)}{t(1-t)} - \frac{an^2}{t^2}\right) y'(x) - \frac{n^3}{t^2(1-t)^2} y(x) = 0$$

Asymptotics

- Formal WKB analysis of the ODE (as $n \to \infty$) leads to algebraic equation

$$x\zeta^3 - \frac{2x}{t(1-t)}\zeta^2 + \left(\frac{x}{t^2(1-t)^2} + \frac{1}{t(1-t)} - \frac{a}{t^2}\right)\zeta - \frac{1}{t^2(1-t)^2} = 0$$

- The equation has branch points $x_-(t) < x_+(t)$ and 0.
- There is a **critical time** $t_{cr} = \frac{a}{a+1}$ so that
  - For $0 < t < t_{cr}$ we have $0 < x_-(t) < x_+(t)$
  - For $t_{cr} < t < 1$ we have $x_-(t) < 0 < x_+(t)$
- Asymptotic analysis of RH problem confirms that paths accumulate on $[\max(0, x_-(t)), x_+(t)]$ with limiting mean density

$$\frac{d\mu(x)}{dx} = \frac{1}{\pi} \left|\operatorname{Im} \zeta(x)\right|$$

- For $t \neq t_{cr}$ we also find usual scaling limits: **sine kernel**, **Airy kernel** and **Bessel kernel** Kuijlaars-Martinez Finkelshtein-Wielonsky (arxiv 2007, to appear in CMP)

Local regime at critical time

- **Double scaling limit at $x^* = 0$ and $t = t_{cr}$:**

$$\lim_{n\to\infty} \frac{1}{cn^{3/2}} K_n\left(\frac{x}{cn^{3/2}}, \frac{y}{cn^{3/2}}; t = t_{cr} + \frac{\tau}{c'n^3}\right) = ??$$

leads to new family of kernels depending on parameter τ

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leads to new family of kernels depending on parameter τ

  - **Limiting kernels involve a special solution of third order ODE**

$$x f''' + (2+\alpha) f'' - \tau f' - f = 0$$

which is the double scaling limit of the ODE for MOPs, and its adjoint

$$-(yg)''' + (\alpha+2) g'' + \tau g' - g = 0$$

Local regime at critical time

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  - **Limiting kernels involve a special solution of third order ODE**

$$xf''' + (2+\alpha)f'' - \tau f' - f = 0$$

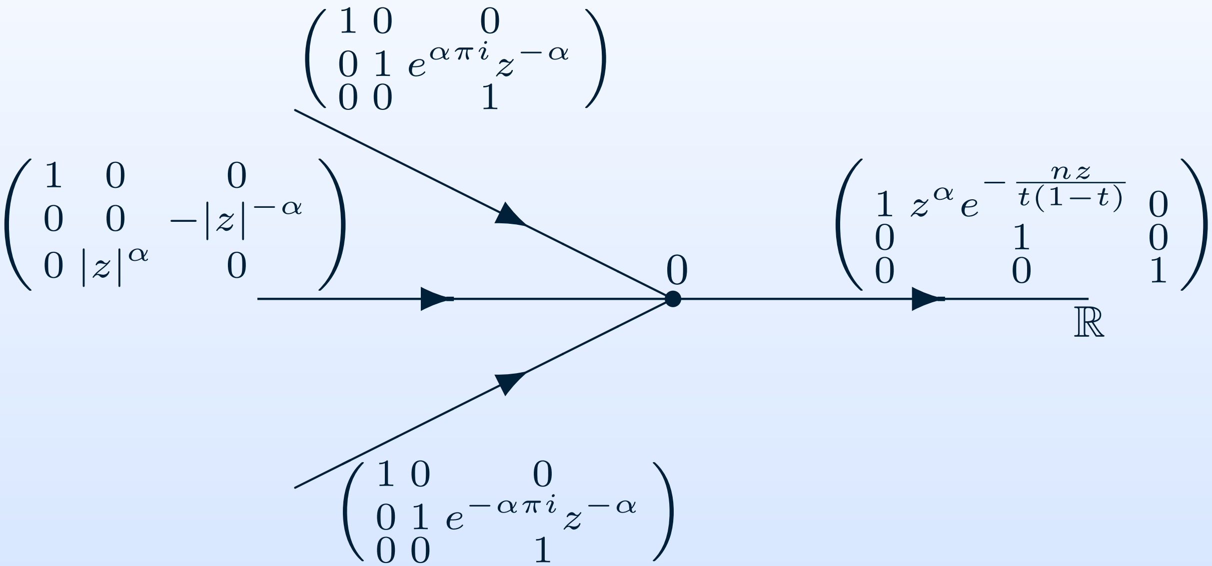
which is the double scaling limit of the ODE for MOPs, and its adjoint

$$-(yg)''' + (\alpha+2)g'' + \tau g' - g = 0$$

$$\frac{1}{\pi(x-y)}\left[f(x)(yg(y))'' - f'(x)(2g(y) + yg'(y))\right.$$
$$\left. + f''(x)yg(y) - (\alpha+2)f(x)g'(y) - \tau f(x)g(y)\right]$$

Steps in steepest descent analysis

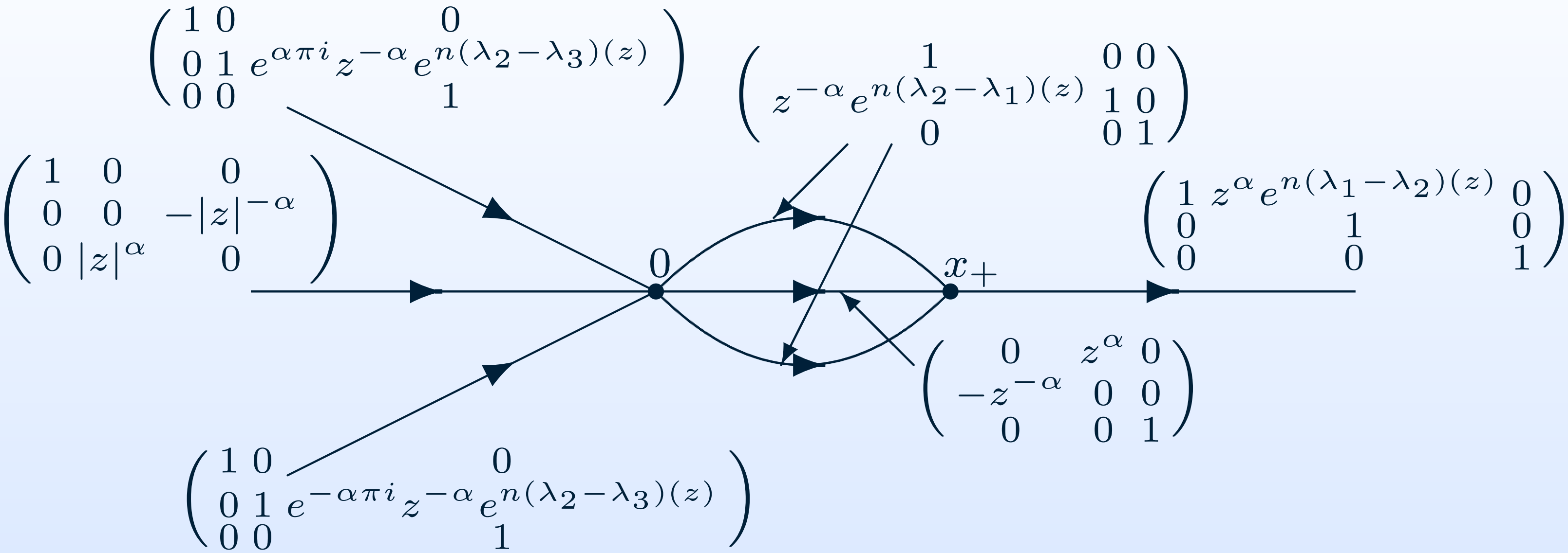
- Initial RH problem for Y has jump on $[0,\infty)$.
- First preliminary step $Y \mapsto X$ introduces jump on negative real line



$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & y^{\alpha} e^{-\frac{ny}{t(1-t)}} & 0 \end{pmatrix} X_+^{-1}(y) X_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Next steps in steepest descent analysis

- Application of g-functions $X \mapsto U$ and opening of lenses $U \mapsto T$



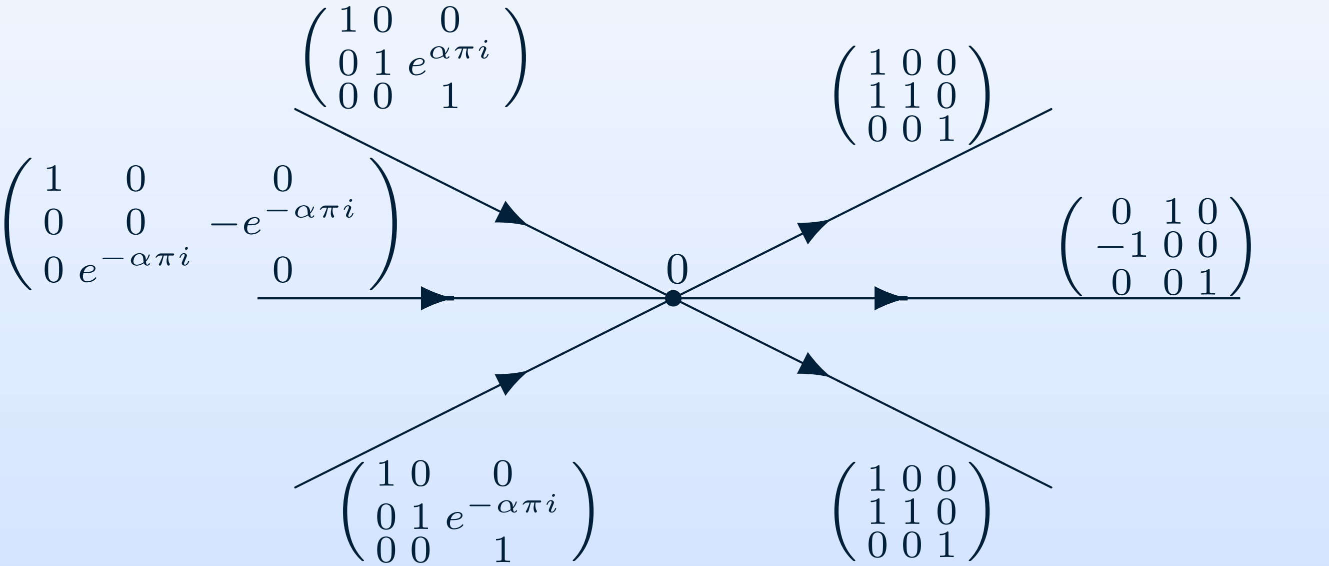
- The $\lambda_j(z)$'s are anti-derivatives of the solutions $\zeta_j(z)$ of the algebraic equation

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} -e^{-n\lambda_{1,+}(y)} & y^{\alpha} e^{-n\lambda_{2,+}(y)} & 0 \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} e^{n\lambda_{1,+}(x)} \\ x^{-\alpha} e^{n\lambda_{2,+}(x)} \\ 0 \end{pmatrix}$$

Reduction to constant jumps

- **Reduction to constant jumps near 0 (magnified picture)**

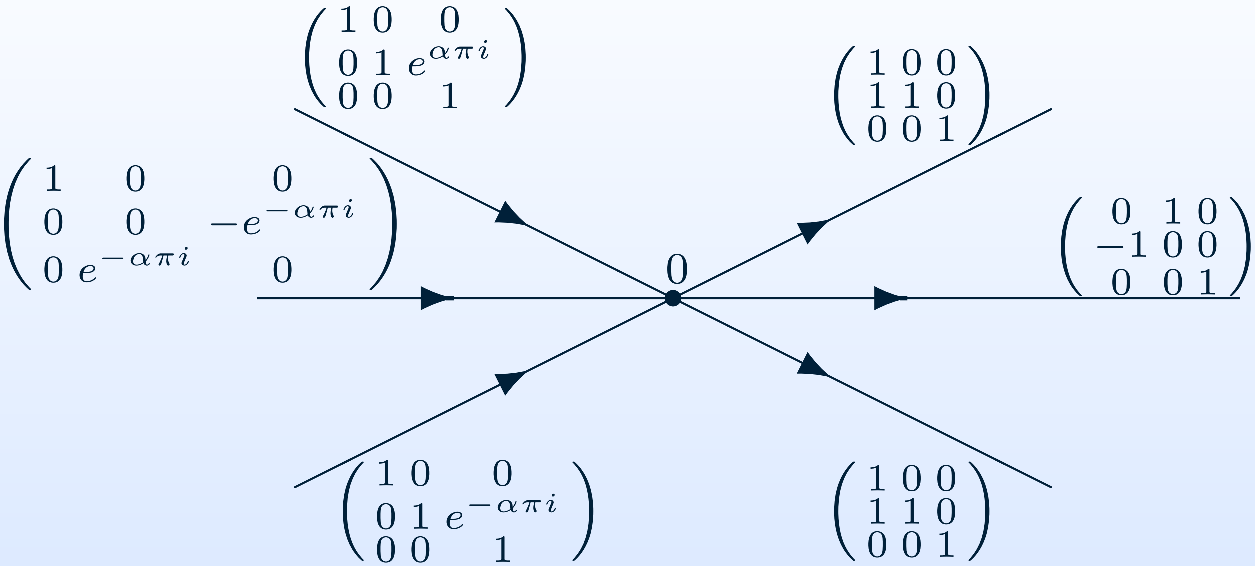
$$S(z) = T(z)\,\mathrm{diag}\begin{pmatrix} e^{n\lambda_1(z)} & z^{-\alpha}e^{n\lambda_2(z)} & e^{n\lambda_3(z)} \end{pmatrix}$$



$$K_n(x,y) = \frac{1}{2\pi i(x-y)}\begin{pmatrix} -1 & 1 & 0 \end{pmatrix} S_+^{-1}(y)S_+(x)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Parametrix at 0

- **We look for model solution Ψ with jumps on infinite rays**



- **Limiting kernel is**

$$\frac{1}{2\pi i(x-y)} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \Psi_+^{-1}(y)\Psi_+(x) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Solution Ψ

- **We construct Ψ from solutions of** $xf''' + (\alpha+2)f'' - \tau f' - f = 0$.
- **There are solutions of the form**

$$f(x) = \int_\Gamma t^{\alpha-1} e^{\tau/t} e^{1/(2t^2)} e^{xt} dt$$

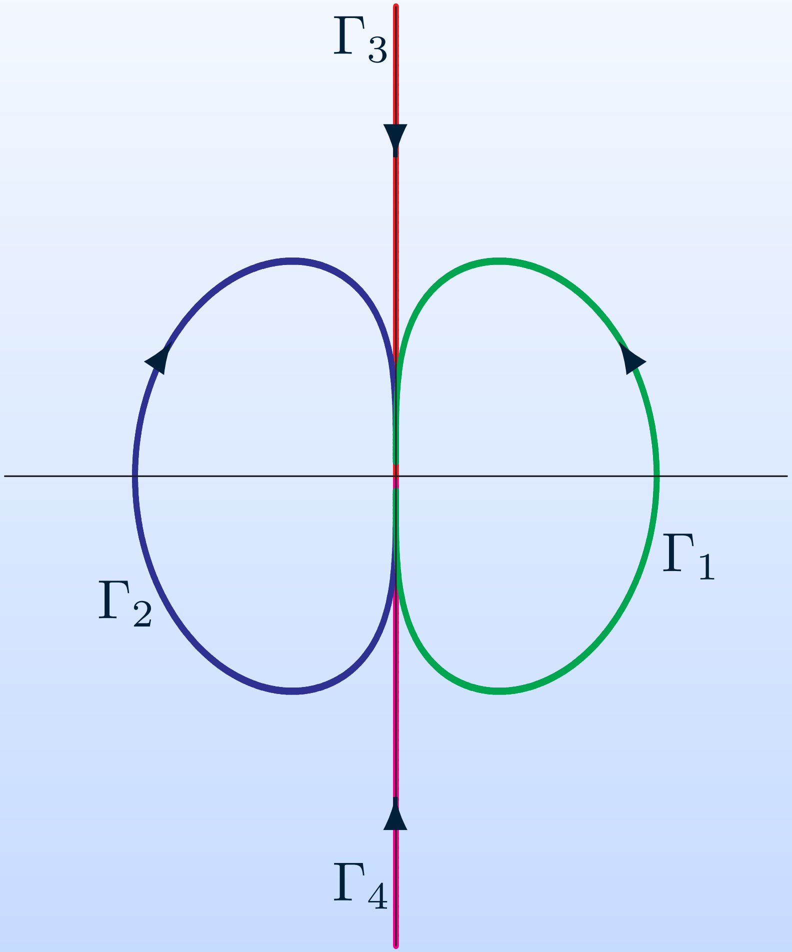
where Γ connects $0+i0$, $0-i0$, ∞.

$$f_1(x) = \frac{1}{2\pi i}\int_{\Gamma_1} \cdots$$

$$f_2(x) = \frac{e^{-\alpha\pi i}}{2\pi i}\int_{\Gamma_2} \cdots$$

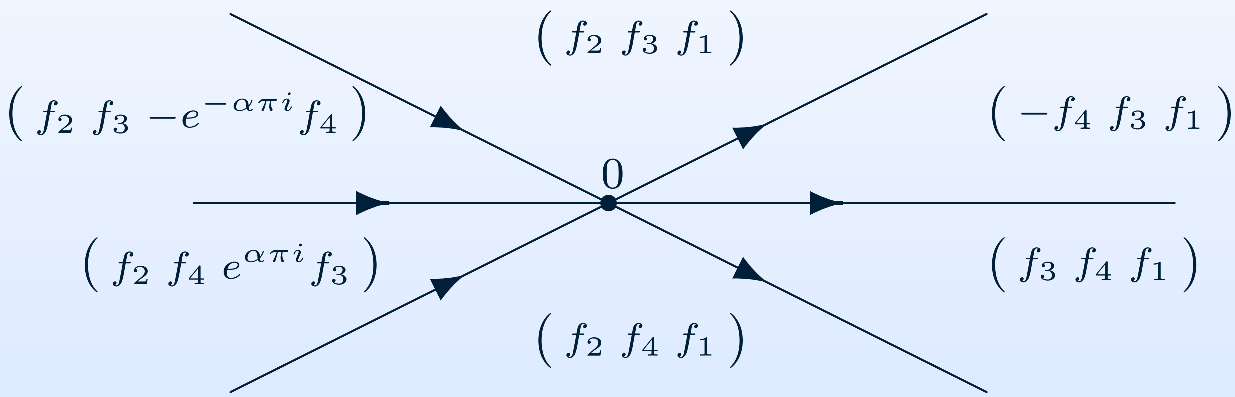
$$f_3(x) = \frac{e^{-\alpha\pi i}}{2\pi i}\int_{\Gamma_3} \cdots$$

$$f_4(x) = \frac{e^{\alpha\pi i}}{2\pi i}\int_{\Gamma_4} \cdots$$



Jumps are satisfied

- **Choose right combination of solutions in each sector so that the jump conditions are satisfied (+ an asymptotic condition ...)**



- **Other rows of Ψ contain derivatives.**
- **The special solution that appears in the limiting kernel is**

$$-f_4(x) + f_3(x) = f_2(x) = \frac{e^{-\alpha\pi i}}{2\pi i} \int_{\Gamma_2} t^{\alpha-1} e^{\tau/t} e^{1/(2t^2)} e^{xt} dt$$

- **The inverse Ψ^{-1} contains solutions of the adjoint equation**