Higher order analogues of the Tracy-Widom distribution and the Painlevé II hierarchy

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Outline

1. Unitary random matrix ensembles and the Tracy-Widom distribution
2. Largest eigenvalue distribution for critical ensembles
3. The Painlevé II hierarchy
4. Generalizations of the Tracy-Widom distribution
5. Large gap asymptotics

Unitary random matrix ensembles

- space of $n \times n$ Hermitian matrices with probability measure

$$\frac{1}{Z_n} \exp(-n \mathrm{Tr}\, V(M)) dM,$$

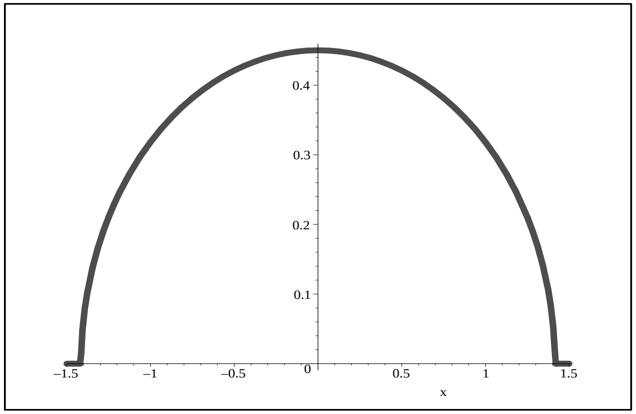
  where

  - V is a scalar real analytic function with sufficient growth at $\pm\infty$,
  - $dM = \prod_{j=1}^{n} dM_{jj} \prod_{i<j} d\mathrm{Re}\, M_{ij} d\mathrm{Im}\, M_{ij}$

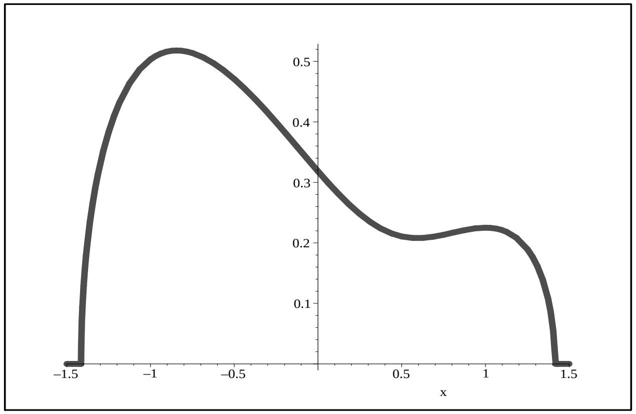
- Simplest case $V(M) = M^2/2 \longrightarrow$ GUE
  - independent Gaussian distributed matrix entries
- invariant under unitary conjugation
- behavior of eigenvalues when n is large?

Limiting mean eigenvalue density

- limiting mean density of eigenvalues
  - for GUE:
    Wigner semi-circle law



  - in general: limiting mean eigenvalue density depends on V



$\to$ smooth density on finite union of intervals

$\to$ typically square root behavior near endpoints

Eigenvalue correlation kernel

- eigenvalues in a unitary ensemble have a correlation kernel

$$K_n(x,y) = e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \sum_{k=0}^{n-1} p_k(x) p_k(y)$$

  - p_k orthonormal on $\mathbb{R}$ w.r.t. weight $e^{-nV(x)}$
  - information about eigenvalues is contained in this kernel

- local scaling limits

$$\lim_{n\to\infty} \frac{1}{cn^\gamma} K_n(x^* + \frac{u}{cn^\gamma}, x^* + \frac{v}{cn^\gamma})$$

turn out to be universal - only depending on the macroscopic nature of x^*

Airy kernel

- local scaling limits near the edge are given by the Airy kernel, *(Forrester '93, Tracy-Widom '94, Deift-Kriecherbauer-McLaughlin-Venakides-Zhou '99, Deift-Gioev '07)*

$$\lim_{n\to\infty} \frac{1}{cn^{2/3}} K_n\left(b + \frac{u}{cn^{2/3}}, b + \frac{v}{cn^{2/3}}\right) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u)}{\pi(u-v)}$$

  - universal whenever LMED has square root behavior near b

- how can one describe the fluctuations of the largest eigenvalue around the right endpoint b for large n?

Limiting distribution of largest eigenvalue

- limiting distribution is given by a Fredholm determinant

$$\lim_{n\to\infty} \mathrm{Prob}\left(\mathrm{cn}^{2/3}(\lambda_\mathrm{n} - \mathrm{b}) < \mathrm{s}\right) = \det(\mathrm{I} - \mathrm{K_s})$$

  - K_s is integrable operator acting on $L^2(s, +\infty)$ with kernel

$$K_s(u, v) = \frac{1}{u - v}\left(\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u)\right)$$

  - this Fredholm determinant can be expressed explicitly in terms of a Painlevé II solution

Tracy-Widom distribution

- Tracy-Widom formula ('94)

$$\det(I - K_s) = \exp\left(\int_s^{+\infty} (s-x)u^2(x)dx\right)$$

- u is the Hastings-McLeod solution to the Painlevé II equation $u_{ss} = su + 2u^3$
  - $u(s) \sim \text{Ai}(s)$ as $s \to +\infty$,
  - $u(s) \sim \sqrt{\frac{-s}{2}}$ as $s \to -\infty$

Large gap asymptotics

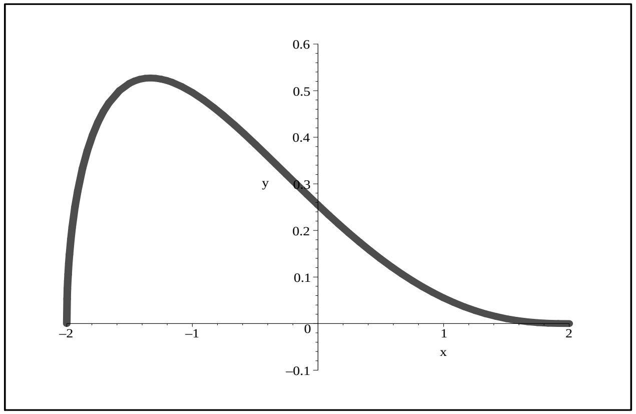
- Large gap asymptotics as $s \to -\infty$ *(Tracy-Widom, Deift-Its-Krasovsky, Baik-Buckingham-DiFranco)*

  $$\ln\det(I-K_s) = -\frac{|s|^3}{12} - \frac{1}{8}\ln|s| + \chi + \mathcal{O}(|s|^{-3/2}),$$

  with $\chi = \frac{1}{24}\ln 2 + \zeta'(-1)$.
- We want to generalize the above for critical ensembles where LMED vanishes faster than a square root
  - Generalization of Tracy-Widom formula?
  - Which function will play the role of the Hastings-McLeod solution?
  - Large gap asymptotics?

Critical ensembles

- Limiting mean eigenvalue density near the right edge of the spectrum
  - $\psi_V(x) \sim c(x-b)^{\frac{4k+1}{2}}$
  - $k=0$ is the regular case
  - The case $k=1$ is realized for $V(x)=\frac{1}{20}x^4-\frac{4}{15}x^3+\frac{1}{5}x^2+\frac{8}{5}x$
  - $k>1$ can occur for polynomials of degree at least $2k+2$



Critical ensembles

- In double scaling limits,

$$\lim_{n\to\infty} \frac{1}{cn^{\frac{2}{4k+3}}} K_n\left(b+\frac{u}{cn^{\frac{2}{4k+3}}}, b+\frac{v}{cn^{\frac{2}{4k+3}}}\right) = K^{(k)}(u,v;t_0,\ldots,t_{2k-1})$$

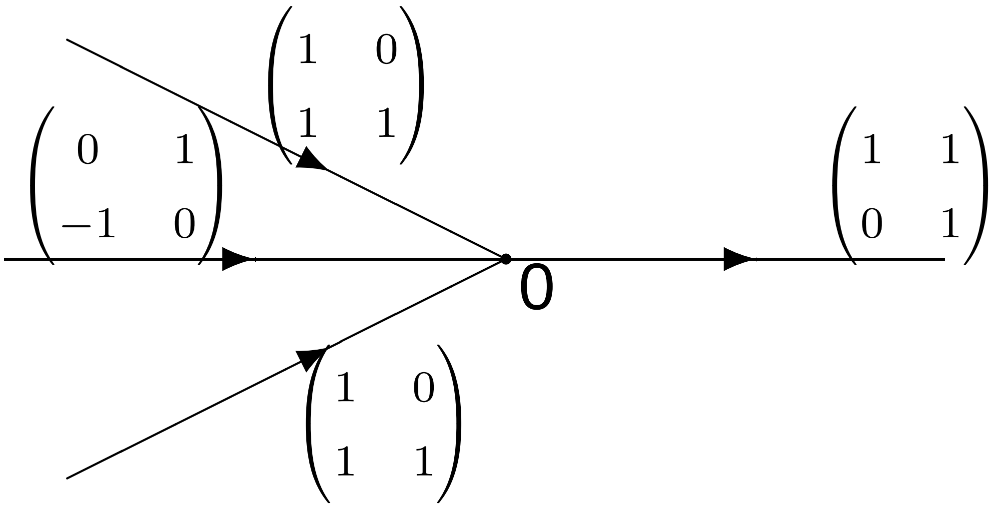
- Airy kernel is replaced by a kernel of the form

$$K^{(k)}(u,v;t_0,\ldots,t_{2k-1}) = \frac{1}{2\pi i(u-v)}(\Phi_1(u)\Phi_2(v)-\Phi_1(v)\Phi_2(u)),$$

where Φ_1 and Φ_2 are related to the Painlevé I hierarchy *(Brézin-Marinari-Parisi ’90, Bowick-Brézin ’91, TC-Vanlessen ’07)*

Limiting kernel for $k \geq 1$

- Φ_1 and Φ_2 are canonical solutions to the Lax pair associated to the $2k$-th member in the Painlevé I hierarchy (a differential equation of order $2k+2$)
- Riemann-Hilbert problem



(a) $\Phi : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$ is analytic

(b) Φ satisfies the indicated jump relations

(c) Φ has the following behavior as $\zeta \to \infty$,

$$\Phi(\zeta) = \frac{1}{\sqrt{2}} \zeta^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{1}{4}\pi i \sigma_3} \left(I + \mathcal{O}(\zeta^{-1/2})\right) e^{-\theta(\zeta)\sigma_3},$$

where

$$\theta(\zeta; t_0, \ldots, t_{2k-1}) = \frac{2}{4k+3} \zeta^{\frac{4k+3}{2}} - 2 \sum_{j=0}^{2k-1} \frac{(-1)^j t_j}{2j+1} \zeta^{\frac{2j+1}{2}}.$$

Fredholm determinant for $k \geq 1$

- largest eigenvalue distribution described by $\det(I - K_s^{(k)})$, where $K_s^{(k)}$ is integrable operator on $L^2(s, +\infty)$ with kernel $K^{(k)}$
- Three questions about this Fredholm determinant
  - Can we give an explicit expression for $\det(I - K_s^{(k)})$? (Tracy-Widom formula)
  - What is the analogue of the Hastings-McLeod solution to PII?
  - Large gap asymptotics for $\det(I - K_s^{(k)})$ as $s \to -\infty$?

Painlevé II hierarchy

- PII hierarchy

$$\left(\frac{d}{dx}+2q\right)\mathcal{L}_n[q_x-q^2]+\sum_{\ell=1}^{n-1}\tau_\ell\left(\frac{d}{dx}+2q\right)\mathcal{L}_\ell[q_x-q^2]=xq-\alpha,$$

$$\frac{d}{dx}\mathcal{L}_{n+1}f=\left(\frac{d^3}{dx^3}+4f\frac{d}{dx}+2f_x\right)\mathcal{L}_nf,\qquad \mathcal{L}_0f=\frac{1}{2}.$$

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$$\begin{aligned}
\mathrm{P}_{\mathrm{II}}^{(1)}:&\quad q_{xx}-2q^3=xq-\alpha,\\
\mathrm{P}_{\mathrm{II}}^{(2)}:&\quad (q_x^{(4)}-10qq_x^2-10q^2q_{xx}+6q^5)+\tau_1(q_{xx}-2q^3)=xq-\alpha,\\
\mathrm{P}_{\mathrm{II}}^{(3)}:&\quad (q_x^{(6)}-14q^2q_x^{(4)}-56qq_xq_x^{(3)}-70(q_x)^2q_{xx}-42q(q_{xx})^2\\
&\qquad +70q^4q_{xx}+140q^3q_x^2-20q^7)+\tau_2(q_x^{(4)}-10qq_x^2-10q^2q_{xx}+6q^5)\\
&\qquad +\tau_1(q_{xx}-2q^3)=xq-\alpha.
\end{aligned}$$

Painlevé II hierarchy

Theorem:

For n odd, there exists a real solution $q = q(x; \tau_1, \ldots, \tau_{n-1})$ to the $\mathrm{P}_{\mathrm{II}}^{(n)}$ equation with $\alpha = \frac{1}{2}$, which has no poles for real values of x, and which has the following asymptotic behavior as $x \to \pm\infty$:

$$q(x) = \frac{1}{2x} + \mathcal{O}(x^{-\frac{4n+1}{2n}}), \qquad \text{as } x \to +\infty,$$

$$q(x) = q(x) = \left(\frac{n!^2}{(2n)!}|x|\right)^{\frac{1}{2n}} + \mathcal{O}(|x|^{-1}, \qquad \text{as } x \to -\infty.$$

Painlevé II hierarchy

- for $n = 1$, this is not the usual Hastings-McLeod solution (corresponding to $\alpha = 0$), but it corresponds to $\alpha = \frac{1}{2}$
- for $n = 1$, q and u are related by a Backlund transformation *(Gambier* $\pm$ *1910)*

$$2^{-4/3}(x + 2q^2(x) + 2q_x(x)) = u(-2^{-1/3}x)^2.$$

- solutions with this local behavior at $\pm\infty$ were known to exist *(Joshi-Mazzocco '03)*
- is q uniquely determined by its asymptotics?
- Next goal is to express $\det(I - K_s^{(k)})$ in terms of q

Analogue of Tracy-Widom formula for $k \geq 1$

Theorem:

$$\det(I - K_{s_0}^{(k)}) = \exp\left[-\int_{s_0}^{+\infty} Q[x(s); \tau_1(s), \ldots, \tau_{2k}(s)] ds \right],$$

with

$$Q(x; \tau_1, \ldots, \tau_{2k}) = \int_{-\infty}^{x} u(\xi; \tau_1, \ldots, \tau_{2k})^2 d\xi,$$

$$u(x) = 2^{-\frac{4k+1}{4k+3}} \exp\left\{ -\int_2^{+\infty} \left(q(\xi) - \frac{1}{2\xi} \right) d\xi \right\} \cdot \exp\left\{ \int_2^x q(\xi) d\xi \right\},$$

- q is the special solution to the $\mathrm{P}_{\mathrm{II}}^{(2k+1)}$ equation
- $x(s)$ (of degree $2k+1$), $\tau_j(s)$ (of degree j) are explicit polynomials in s

Analogue of Tracy-Widom formula for $k \geq 1$

- A few remarks
  - for $k = 0$, using the Backlund transformation, one verifies that our formula reproduces the Tracy-Widom formula
  - formula does not get complicated if k increases, dependence on k lies only in q and $x(s), \tau_j(s)$
  - Backlund transformation between $\alpha = 0$ and $\alpha = 1/2$ for higher members of the hierarchy?

Large gap asymptotics - constant problem

- Large gap asymptotics for $\det(I - K_s^{(k)})$

$$\frac{d}{ds}\ln\det(I - K_s^{(k)}) = \frac{1}{4}a_0^2(s)|s|^{4k+2} + \frac{3a_1(s)}{16a_0(s)|s|} + \mathcal{O}(|s|^{-\frac{4k+5}{2}}),$$

where

$$a_j(s) = \frac{1}{\Gamma(j+\frac{3}{2})}\left(\frac{\Gamma(2k+\frac{3}{2})}{\Gamma(2k+2-j)}\right. \\ \left. + \sum_{m=2}^{2k+1-j} t_{2k+1-m}\frac{\Gamma(2k+\frac{3}{2}-m)}{\Gamma(2k+2-j-m)}|s|^{-m}\right),$$

- Constant of integration?

Large gap asymptotics - constant problem

- For $k = 0$, as mentioned before, $\chi = \frac{1}{24}\ln 2 + \zeta'(-1)$ *(Deift-Its-Krasovsky, Baik-Buckingham-DiFranco)*
  - proof relies on an expression for $\chi^{(0)}$ in terms of Selberg integral

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_{1\le i<j\le n} (t_i - t_j)^2 \prod_{j=1}^n e^{-nt_j} \prod_{j=1}^n dt_j,$$

for which asymptotics are known

Large gap asymptotics - constant problem

- For $k = 1$, constant is given by

$$\chi^{(1)} = \lim_{s\to-\infty}\left(\frac{5^2}{2^8\cdot 7}|s|^7 + \frac{3}{8}\ln|s| - \int_s^{+\infty} Q\left[-2^{5/7-3}5t^3; 2^{1/7-2}15t^2, 2^{-3/7}5t\right] dt\right),$$

  - formula in terms of PII solution q

- one can also obtain a closed formula in terms of

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \prod_{1\leq i<j\leq n}(t_i-t_j)^2\prod_{j=1}^n e^{-nV(t_j)}\prod_{j=1}^n dt_j,$$

  with V of degree 3 or higher

  - asymptotics for this integral are not known

Ideas for the proofs

Proofs are based on the Riemann-Hilbert approach

- Result about Painlevé II hierarchy
  - Analysis of a RH problem characterizing the special higher order Painlevé II transcendent
  - Smoothness $\leftrightarrow$ solvability of the RH problem for all real values of parameters
  - Asymptotics $\rightarrow$ Deift/Zhou steepest descent analysis of the RH problem

Ideas for the proofs

- Results about Fredholm determinant
  - RH representation for Fredholm determinant *(Its-Izergin-Korepin-Slavnov, Deift-Its-Zhou, Borodin-Deift)*
  - Relevant RH problem can be identified with the one for the Painlevé II hierarchy $\longrightarrow$ higher order Tracy-Widom formula
  - Deift/Zhou steepest descent analysis of RH problem $\longrightarrow$ large gap asymptotics