Matrix biorthogonal polynomials on the unit circle and non-abelian Ablowitz-Ladik hierarchy.

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Plan of the talk.

Review of the connection between biorthogonal polynomials on the unit circle and the Ablowitz-Ladik hierarchy (M. Adler and P. van Moerbeke '01, B. Simon and N. I. Nenciu '05):

Recursion coefficients for BOPUC satisfy semidiscrete zero-curvature equations for the Ablowitz-Ladik hierarchy.

Extension to the case of matrix biorthogonal polynomials on the unit circle:

Recursion coefficients for MBOPUC satisfy semidiscrete zero-curvature equations for the non-abelian Ablowitz-Ladik hierarchy.
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Ablowitz-Ladik hierarchy; a discretization for AKNS (Ablowitz-Kaup-Newell-Segur).
Ablowitz-Ladik hierarchy; a discretization for AKNS (Ablowitz-Kaup-Newell-Segur).

Zakharov-Shabat spectral problem for AKNS:

\[
\begin{cases}
\frac{\partial x}{\partial x} \Psi = L \Psi \\
\frac{\partial \tau}{\partial \tau} \Psi = M \Psi 
\end{cases}
\]

\[
L := \begin{pmatrix} z & q \\ r & -z \end{pmatrix}
\]

\[
\implies \frac{\partial \tau}{\partial \tau} L := [M, L] + \frac{\partial x}{\partial x} M
\]
Ablowitz-Ladik hierarchy; a discretization for AKNS (Ablowitz-Kaup-Newell-Segur).

- **Zakharov-Shabat spectral problem for AKNS:**

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  \]

- **Ablowitz-Ladik discretization:**

  \[
  \begin{align*}
  \Psi_{k+1} &= L_k \Psi_k \\
  \partial_\tau \Psi_k &= M_k \Psi_k
  \end{align*}
  \]

  \[
  L_k := \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix}
  \]

  \[
  \Rightarrow \partial_\tau L_k = M_{k+1} L_k - L_k M_k
  \]
An example: NLS and its discretization.
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- Complexified non-linear Schrödinger:

\[
L := \begin{pmatrix} z & q \\ r & -z \end{pmatrix} \quad M := \begin{pmatrix} 2z^2 - qr & 2zq + q_x \\ 2zr - r_x & -2z^2 + qr \end{pmatrix}
\]

\[
\implies \begin{cases} 
\partial_\tau q = q_{xx} - 2q^2 r \\
\partial_\tau r = -r_{xx} + 2qr^2.
\end{cases}
\]
An example: NLS and its discretization.

- **Complexified non-linear Schrödinger:**
  
  \[ L := \begin{pmatrix} z & q \\ r & -z \end{pmatrix}, \quad M := \begin{pmatrix} 2z^2 - qr & 2zq + qx \\ 2zr - rx & -2z^2 + qr \end{pmatrix} \]

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- **Discrete complexified non-linear Schrödinger:**
  
  \[ L_k := \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix}, \quad M_k := \begin{pmatrix} z^2 - 1 - x_k y_{k-1} & zx_k - z^{-1}x_{k-1} \\ zy_{k-1} - z^{-1}y_k & -z^{-2} + 1 + x_{k-1}y_k \end{pmatrix} \]

  \[ \Rightarrow \begin{cases} \partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \partial_\tau y_k = -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}). \end{cases} \]
Our goal, our strategy.

1. **The scalar case.**
2. **The block case.**

---

**Our goal, our strategy.**

- **The goal:** Expressing the connection between Ablowitz-Ladik hierarchy and the theory of biorthogonal polynomials on the unit circle (M. Adler, P. van Moerbeke '01, I. Nenciu '05).

- **The strategy:** We slightly change the discrete Lax operator (P. Miller, N. Ercolani, I. Krichever, C. D. Levermore '95.)

\[
L_k = (z x_k y_k z_k - 1) \rightarrow L_k = (z x_k y_k 1)
\]

We write eigenvalues \( \Phi_k \) such that \( \Phi_k + 1 = L_k \Phi_k \) in terms of biorthogonal polynomials on the unit circle.

We make them evolve using 2D-Toda flow.

---

M. Cafasso

MBOPUC and non-abelian AL hierarchy.
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L_k = \begin{pmatrix}
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in terms of biorthogonal polynomials on the unit circle.

- We make them evolve using 2D-Toda flow.
2D-Toda I.

Definitions:

\[
\Lambda := (\delta_{i,j-1})_{i,j \geq 0}, \quad \Lambda^{-1} := \Lambda^T \quad \begin{cases}
L_1 := \Lambda + \sum_{i \leq 0} a_i^{(1)} \Lambda^i \\
L_2 := a_{-1}^{(2)} \Lambda^{-1} + \sum_{i \geq 0} a_i^{(2)} \Lambda^i
\end{cases}
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Zakharov-Shabat Spectral problem:

\[
\begin{align*}
L_1 \Psi_1 &= z \Psi_1 \\
L_2^T \Psi_2^* &= z^{-1} \Psi_2^* \\
\partial_t \Psi_1 &= (L_1^n)\Psi_1 \\
\partial_t \Psi_2^* &= -(L_1^n)^T \Psi_2^* \\
\partial_s \Psi_1 &= (L_2^n)\Psi_1 \\
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L_2^n T \Psi_2^* &= -(L_1^n)^T \Psi_2^* \\
\partial_t \Psi_1 &= (L_1^n)_+ \Psi_1 \\
\partial_t \Psi_2^* &= -(L_1^n)^T \Psi_2^* \\
\partial_s \Psi_1 &= (L_2^n)_- \Psi_1 \\
\partial_s \Psi_2^* &= -(L_2^n)^T \Psi_2^*
\end{align*}
\]

Lax equations:

\[
\begin{align*}
\partial_{t_n} L_i &= [(L_1^n)_+, L_i] \\
\partial_{s_n} L_i &= [(L_2^n)_-, L_i]
\end{align*}
\]
2D-Toda II (linearization).

Theorem (Takasaki ’84)

Given any semi-infinite matrix $M := S^{-1}_{1}S_{2}$ with

\[ S_{1} = I + \sum_{i \geq 1} s^{(1)}_{i} \Lambda^{-i} \quad S_{2} = s^{(2)}_{0} + \sum_{i \geq 1} s^{(2)}_{i} \Lambda^{i} \]

define its time evolution through

\[ M(t; s) := \exp(\xi(t, \Lambda))M \exp(-\xi(s, \Lambda^{-1})) = S_{1}^{-1}(t; s)S_{2}(t; s). \]

with $\xi(t, \Lambda) = \sum_{i \geq 1} t_{i} \Lambda^{i}$. 
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define its time evolution through

$$M(t; s) := \exp(\xi(t, \Lambda))M \exp(-\xi(s, \Lambda^{-1})) = S_1^{-1}(t; s)S_2(t; s).$$

with $\xi(t, \Lambda) = \sum_{i \geq 1} t_i \Lambda^i$. Then

$$L_1 := S_1 \Lambda S_1^{-1} \quad L_2 := S_2 \Lambda^{-1} S_2^{-1}$$

$$\Psi_1 := \exp(\xi(t, z))S_1 \chi(z) \quad \Psi_2^* := \exp(-\xi(s, z^{-1}))(S_2^{-1})^T \chi(z^{-1})$$

with $\chi(z) := (1, z, z^2, \ldots)^T$ solve 2D-Toda.
Generalized biorthogonal polynomials

Proposition

Given a matrix $M = S_1^{-1} S_2$ define a bilinear pairing on the space of polynomials in $z$ imposing $< z^i, z^j >_M = M_{i,j}$. Then the polynomials

$$q^{(1)} := (q^{(1)}_0, q^{(1)}_1, q^{(1)}_2, \ldots)^T = S_1 \chi(z)$$

$$q^{(2)} := (q^{(2)}_0, q^{(2)}_1, q^{(2)}_2, \ldots)^T = (S_2^{-1})^T \chi(z)$$

are biorthonormal, i.e. $< q^{(1)}_i, q^{(2)}_j >_M = \delta_{i,j}$
The Toeplitz lattice I (Adler-van Moerbeke ’01).

Suppose that your initial datum $M$ is a Toeplitz matrix; i.e. 
\[
(M_{ij})_{i,j \geq 0} = (M_{j-i})_{i,j \geq 0}.
\] Then:

Toeplitz symmetry is conserved along 2D-Toda flow.

The pairing $\langle ., . \rangle_M(t,s)$ can be written as
\[
\langle P(z), Q(z) \rangle_M = \oint P(z) Q(z-1) \exp(\xi(t,z)) \gamma(z) \exp(-\xi(s,z-1)) dz
\]
with
\[
\gamma(z) = \sum_k M_k z^k
\]
initial value.

Monic biorthogonal polynomials \{\textit{p}^1_i, \textit{p}^2_j\} such that
\[
\langle \textit{p}^1_i(z), \textit{p}^2_j(z) \rangle_M = \delta_{i,j}
\]
satisfy the recursion relation
\[
(\textit{p}^1_{n+1}(z) \sim \textit{p}^2_n(z)) = L_n(\textit{p}^1_n(z) \sim \textit{p}^2_n(z)) = (z x^n z y^n) \langle \textit{p}^1_n(z), \textit{p}^2_n(z) \rangle
\] with
\[
\sim \textit{p}^2_n(z) := z^n \textit{p}^2_n(z-1)
\]
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$$< P(z), Q(z) >_M = \oint P(z)Q(z^{-1}) \exp(\xi(t,z))\gamma(z) \exp(-\xi(s,z^{-1})) \frac{dz}{2\pi i z}$$

with $\gamma(z) = \sum_k M_k z^k$ initial value.
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  \]
  
  with \( \gamma(z) = \sum_k M_k z^k \) initial value.
- Monic biorthogonal polynomials \( \{ p_1^{(1)}, p_2^{(2)} \} \) such that
  
  \[
  < p_1^{(1)}(z), p_2^{(2)}(z) >_{M} = h_i \delta_{i,j}
  \]
  
  satisfy the recursion relation

\[
\begin{pmatrix}
  p_{n+1}^{(1)}(z) \\
  \tilde{p}_{n+1}^{(2)}(z)
\end{pmatrix}
= \mathcal{L}_n
\begin{pmatrix}
  p_n^{(1)}(z) \\
  \tilde{p}_n^{(2)}(z)
\end{pmatrix}
= 
\begin{pmatrix}
  z & x_{n+1} \\
  z y_{n+1} & 1
\end{pmatrix}
\begin{pmatrix}
  p_n^{(1)}(z) \\
  \tilde{p}_n^{(2)}(z)
\end{pmatrix}
\]

with \( \tilde{p}_n^{(2)}(z) := z^n p_n^{(2)}(z^{-1}) \).
The Toeplitz lattice II (Adler-van Moerbeke ’01).

**Theorem**

Lax operators of the Toeplitz lattice are of the following form:

\[ h^{-1}L_1h = \begin{pmatrix} -x_1y_0 & 1 - x_1y_1 & 0 & \ldots & \ldots \\ -x_2y_0 & -x_2y_1 & 1 - x_2y_2 & 0 & \ldots \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1 - x_3y_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \]

\[ L_2(x, y) = (h^{-1}L_1h)^T(y, x). \]
The scalar case.

The block case.

The Toeplitz lattice II (Adler-van Moerbeke ’01).

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Lax operators of the Toeplitz lattice are of the following form:

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-x_3 y_0 & -x_3 y_1 & -x_3 y_2 & 1 - x_3 y_3 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

\[
L_2(x, y) = (h^{-1}L_1 h)^T(y, x).
\]

Hence the Toeplitz lattice gives time evolution for recursion coefficients for BOPUC.
Theorem

Consider time-dependent BOPUC \( \{ p_n^{(1)}(z), p_n^{(2)}(z) \} \) with respect to the pairing

\[
\langle P(z), Q(z) \rangle := \int P(z)Q(z^{-1})\gamma(t, s; z) \frac{dz}{2\pi i z}
\]

with \( \gamma(t, s; z) := \exp(\xi(t, z))\gamma(z) \exp(-\xi(s, z^{-1})) \). Then the related recursion operators \( \mathcal{L}_k \) evolve according to semidiscrete zero-curvature equations for the Ablowitz-Ladik hierarchy

\[
\partial_{t_i/s_i} \mathcal{L}_k = \mathcal{M}_{t_i/s_i,k+1} \mathcal{L}_k - \mathcal{L}_k \mathcal{M}_{t_i/s_i,k}.
\]
Sketch of the proof.

\[ \Phi_k := ( p_1^k(z), \tilde{p}_2^k(z) ) \] satisfy equation
\[ \Phi_k + 1 = L_k \Phi_k. \]

We have just to find \( M_{t|i/s}k \) such that
\[ \partial_t i/s\Phi_k = M_{t|i/s}k \Phi_k. \]

Then semidiscrete zero-curvature equations will be given by compatibility conditions.

Using Zakharov-Shabat equations, for every \( k \), \( \partial_t i p_1^k \) is a linear combination of \( \{ p_1^k, p_1^k+1, p_1^k+2, \ldots \} \) with coefficients in \( C[x_j, y_j] \).

In the same way, for every \( k \), \( \partial_t i \tilde{p}_2^k \) is a linear combination of \( \{ \tilde{p}_2^k, \tilde{p}_2^k-1, \tilde{p}_2^k-2, \ldots \} \) with coefficients in \( C[x_j, y_j] \).

Using recursion relations you arrive to write \( M_{t|i/s}k \) depending on \( \{ x_j, y_j, z \} \).
Sketch of the proof.

\[ \Phi_k := (p^{(1)}_k(z), \tilde{p}^{(2)}_k(z))^T \] satisfy equation \( \Phi_{k+1} = \mathcal{L}_k \Phi_k \).

We have just to find \( \mathcal{M}_{t_i/s_i,k} \) such that \( \partial_{t_i/s_i} \Phi_k = \mathcal{M}_{t_i/s_i,k} \Phi_k \).

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Sketch of the proof.

- $\Phi_k := (p_k^{(1)}(z), \tilde{p}_k^{(2)}(z))^T$ satisfy equation $\Phi_{k+1} = L_k \Phi_k$.

We have just to find $M_{t_i/s_i,k}$ such that $\partial_{t_i/s_i} \Phi_k = M_{t_i/s_i,k} \Phi_k$.

Then semidiscrete zero-curvature equations will be given by compatibility conditions.

- Using Zakharov-Shabat equations, for every $k$, $\partial_{t_i} p_k^{(1)}$ is a linear combination of $\{p_{k+1}^{(1)}, p_k^{(1)}, p_{k+2}^{(1)}, \ldots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$.

In the same way, for every $k$, $\partial_{t_i} \tilde{p}_k^{(2)}$ is a linear combination of $\{\tilde{p}_k^{(2)}, \tilde{p}_{k-1}^{(2)}, \tilde{p}_{k-2}^{(2)}, \ldots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$. 
Sketch of the proof.

- $\Phi_k := (p_k^{(1)}(z), \tilde{p}_k^{(2)}(z))^T$ satisfy equation $\Phi_{k+1} = \mathcal{L}_k \Phi_k$.
- We have just to find $M_{t_i/s_i,k}$ such that $\partial_{t_i/s_i} \Phi_k = M_{t_i/s_i,k} \Phi_k$.
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- In the same way, for every $k$, $\partial_{t_i} \tilde{p}_k^{(2)}$ is a linear combination of $\{\tilde{p}_k^{(2)}, \tilde{p}_{k-1}^{(2)}, \tilde{p}_{k-2}^{(2)}, \ldots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$.

- Using recursion relations you arrive to write $M_{t_i,k}$ depending on $\{x_j, y_j, z\}$. 
Example:

\[
\begin{align*}
\partial_t p(1)_k &= -zp(1)_k - x_k + 1y_k p(1)_k + p(1)_k + 1 = -x_k + 1y_k p(1)_k - z \tilde{p}(2)_k \\
\partial_t \tilde{p}(2)_k &= -zh_k + 1h_k \tilde{p}(2)_k - 1 = zy_k p(1)_k - z \tilde{p}(2)_k \\
\Rightarrow \partial_t (p(1)_k \tilde{p}(2)_k) &= M_t_1, k (p(1)_k \tilde{p}(2)_k)
\end{align*}
\]
Example:

\[
\begin{align*}
\partial_{t_1} p_k^{(1)} &= -zp_k^{(1)} - x_{k+1}ykp_k^{(1)} + p_{k+1}^{(1)} = -x_{k+1}ykp_k^{(1)} + x_k+1\tilde{p}_k^{(2)} \\
\partial_{t_1} \tilde{p}_k^{(2)} &= -z\frac{h_{k+1}}{h_k}\tilde{p}_{k-1}^{(2)} = zykp_k^{(1)} - z\tilde{p}_k^{(2)}
\end{align*}
\]
Example:

\[
\partial_t p_k^{(1)} = -zp_k^{(1)} - x_{k+1}yp_k^{(1)} + p_{k+1}^{(1)} = -x_{k+1}yp_k^{(1)} + x_{k+1}\tilde{p}_k^{(2)}
\]

\[
\partial_t \tilde{p}_k^{(2)} = -z\frac{h_{k+1}}{h_k}\tilde{p}_{k-1}^{(2)} = zykp_k^{(1)} - z\tilde{p}_k^{(2)}
\]

\[
\implies \partial_t \begin{pmatrix} p_k^{(1)} \\ \tilde{p}_k^{(2)} \end{pmatrix} = M_{t_1,k} \begin{pmatrix} p_k^{(1)} \\ \tilde{p}_k^{(2)} \end{pmatrix} = \begin{pmatrix} -x_{k+1}y & x_{k+1} \\ zy & -z \end{pmatrix} \begin{pmatrix} p_k^{(1)} \\ \tilde{p}_k^{(2)} \end{pmatrix}
\]
A natural question.

Time evolution for orthogonal polynomials on the real line leads to the Toda hierarchy.

Time evolution for biorthogonal polynomials on the unit circle leads to the Ablowitz-Ladik hierarchy.

Time evolution for matrix orthogonal polynomials on the real line leads to the non-abelian Toda hierarchy.

What about time evolution for matrix biorthogonal polynomials on the unit circle?
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- Time evolution for matrix orthogonal polynomials on the real line leads to the non-abelian Toda hierarchy.

What about time evolution for matrix biorthogonal polynomials on the unit circle?
The setting I.

- Introduce a matrix-valued symbol \( \gamma(z) = \sum_k M_k z^k \) with \( \{M_k\} \) \((n \times n)\) matrices.
The setting I.

- Introduce a matrix-valued symbol $\gamma(z) = \sum_k M_k z^k$ with \{\(M_k\)\} \((n \times n)\) matrices.

- Consider the two matrix-valued pairings

\[
< P(z), Q(z) >_L = \oint P(z) \gamma(z) Q^*(z) \frac{dz}{2\pi i z}
\]

\[
< P(z), Q(z) >_R = \oint P^*(z) \gamma(z) Q(z) \frac{dz}{2\pi i z}
\]

and corresponding matrix-valued biorthogonal polynomials

\[
\{P_n^{(1)R}(z), P_n^{(1)L}(z), P_n^{(2)R}(z), P_n^{(2)L}(z)\}
\]

such that

\[
< P_k^{(2)R}, P_j^{(1)R} >_R = \delta_{kj} h_k^R \quad < P_k^{(1)L}, P_j^{(2)R} >_L = \delta_{kj} h_k^L.
\]
The following block recursion relations can be obtained:

\[
\begin{pmatrix}
P_{N+1}^{(1)L} \\
\tilde{P}_{N+1}^{(2)R}
\end{pmatrix}
= \mathcal{L}_N^L
\begin{pmatrix}
P_{N}^{(1)L} \\
\tilde{P}_{N}^{(2)R}
\end{pmatrix}
= \begin{pmatrix}
zI & x_{N+1}^L \\
zy_{N+1}^R & I
\end{pmatrix}
\begin{pmatrix}
P_{N}^{(1)L} \\
\tilde{P}_{N}^{(2)R}
\end{pmatrix}
\]

\[
\begin{pmatrix}
P_{N+1}^{(1)R} \\
\tilde{P}_{N+1}^{(2)L}
\end{pmatrix}
= \begin{pmatrix}
P_{N}^{(1)R} & \tilde{P}_{N}^{(2)L}
\end{pmatrix}
\mathcal{L}_N^R
= \begin{pmatrix}
P_{N}^{(1)R} & \tilde{P}_{N}^{(2)L}
\end{pmatrix}
\begin{pmatrix}
zI & zy_{N+1}^L \\
x_{N+1}^R & I
\end{pmatrix}.
\]
The following block recursion relations can be obtained:

\[
\begin{pmatrix}
P^{(1)\!L}_{N+1} \\
\tilde{P}^{(2)\!R}_{N+1}
\end{pmatrix} = \mathcal{L}^L_N \begin{pmatrix}
P^{(1)\!L}_N \\
\tilde{P}^{(2)\!R}_N
\end{pmatrix} = \begin{pmatrix}
zI & x_{N+1}^L \\
y_{N+1}^R & I
\end{pmatrix} \begin{pmatrix}
P^{(1)\!L}_N \\
\tilde{P}^{(2)\!R}_N
\end{pmatrix}
\]

\[
\begin{pmatrix}
P^{(1)\!R}_{N+1} & \tilde{P}^{(2)\!L}_{N+1}
\end{pmatrix} = \begin{pmatrix}
P^{(1)\!R}_N & \tilde{P}^{(2)\!L}_N
\end{pmatrix} \mathcal{L}^R_N = \begin{pmatrix}
P^{(1)\!R}_N & \tilde{P}^{(2)\!L}_N
\end{pmatrix} \begin{pmatrix}
zI & y_{N+1}^L \\
x_{N+1}^R & I
\end{pmatrix}.
\]

Now introduce time evolution

\[
\gamma(t, s; z) := \exp(\xi(t, zI))\gamma(z) \exp(-\xi(s, z^{-1}I))
\]

and study time evolution for corresponding matrix biorthogonal polynomials.
The setting III.

Look for some matrices $\mathcal{M}_{t_i/s_i,n}^L$, $\mathcal{M}_{t_i/s_i,n}^R$ such that

$$
\partial_{t_i/s_i} \begin{pmatrix} P_N^{(1)L}(z) \\ \tilde{P}_N^{(2)R}(z) \end{pmatrix} = \mathcal{M}_{t_i/s_i,n}^L \begin{pmatrix} P_N^{(1)L}(z) \\ \tilde{P}_N^{(2)R}(z) \end{pmatrix}
$$

$$
\partial_{t_i/s_i} \begin{pmatrix} P_N^{(1)R}(z) \\ \tilde{P}_N^{(2)L}(z) \end{pmatrix} = \begin{pmatrix} P_N^{(1)R}(z) \\ \tilde{P}_N^{(2)L}(z) \end{pmatrix} \mathcal{M}_{t_i/s_i,n}^R
$$

Compatibility conditions will give non-abelian semidiscrete zero-curvature equations.
Look for some matrices $\mathcal{M}_{t_i/s_i,n}^L, \mathcal{M}_{t_i/s_i,n}^R$ such that

$$
\partial_{t_i/s_i} \begin{pmatrix} P_N^{(1)L}(z) \\ \tilde{P}_N^{(2)R}(z) \end{pmatrix} = \mathcal{M}_{t_i/s_i,n}^L \begin{pmatrix} P_N^{(1)L}(z) \\ \tilde{P}_N^{(2)R}(z) \end{pmatrix}
$$

$$
\partial_{t_i/s_i} \begin{pmatrix} P_N^{(1)R}(z) \\ \tilde{P}_N^{(2)L}(z) \end{pmatrix} = \begin{pmatrix} P_N^{(1)R}(z) \\ \tilde{P}_N^{(2)L}(z) \end{pmatrix} \mathcal{M}_{t_i/s_i,n}^R
$$

Compatibility conditions will give non-abelian semidiscrete zero-curvature equations.

$$
\partial_\tau \mathcal{L}_k^L = \mathcal{M}_{k+1}^L \mathcal{L}_k^L - \mathcal{L}_n^L \mathcal{M}_k^L
$$

$$
\partial_\tau \mathcal{L}_k^R = \mathcal{L}_k^R \mathcal{M}_{k+1}^R - \mathcal{M}_k^R \mathcal{L}_k^R.
$$
**Non-abelian AL and MBOPUC**

**Theorem (M.C.)**

Consider time-dependent matrix BOPUC \( \{P_n^{(1)R}(z), P_n^{(1)L}(z)\} \) and \( \{P_n^{(2)R}(z), P_n^{(2)L}(z)\} \) with respect to the pairings

\[
< P, Q >_R := \oint P^*(z)\gamma(t, s; z)Q(z) \frac{dz}{2\pi iz}
\]

\[
< P, Q >_L := \oint P(z)\gamma(t, s; z)Q^*(z) \frac{dz}{2\pi iz}
\]

with \( \gamma(t, s; z) := \exp(\xi(t, zI))\gamma(z)\exp(-\xi(s, z^{-1}I)) \). Then the related block recursion operators \( \mathcal{L}^R \) and \( \mathcal{L}^L \) evolves according to the non-abelian AL equations:

\[
\partial_{t_i / s_i} \mathcal{L}_k^L = \mathcal{M}_{t_i / s_i, k+1}^L \mathcal{L}_k^L - \mathcal{L}_n^L \mathcal{M}_{t_i / s_i, k}^L
\]

\[
\partial_{t_i / s_i} \mathcal{L}_k^R = \mathcal{L}_k^R \mathcal{M}_{t_i / s_i, k+1}^R - \mathcal{M}_{t_i / s_i, k}^R \mathcal{L}_k^R.
\]
Discrete NLS and its non abelian version.

Discrete NLS:

\[
\begin{align*}
\partial_{\tau} x_k &= x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\
\partial_{\tau} y_k &= -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}).
\end{align*}
\]
Discrete NLS and its non abelian version.

- **Discrete NLS:**
  \[
  \begin{align*}
  \partial_{\tau} x_k &= x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\
  \partial_{\tau} y_k &= -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}).
  \end{align*}
  \]

- **Non-abelian discrete NLS:**
  \[
  \begin{align*}
  \partial_{\tau} x^L_k &= x^L_{k+1} - 2x^L_k + x^L_{k-1} - x^L_{k+1} y^R_k x^L_k - x^L_k y^R_k x^L_{k-1} \\
  \partial_{\tau} y^R_k &= -y^R_{k+1} + 2y^R_k - y^R_{k-1} + y^R_{k+1} x^L_k y^R_k + y^R_k x^L_k y^R_{k-1} \\
  \partial_{\tau} x^R_k &= x^R_{k+1} - 2x^R_k + x^R_{k-1} - x^R_{k-1} y^L_k x^R_k - x^R_k y^L_k x^R_{k+1} \\
  \partial_{\tau} y^L_k &= -y^L_{k+1} + 2y^L_k - y^L_{k-1} + y^L_{k-1} x^R_k y^L_k + y^L_k x^R_k y^L_{k+1}
  \end{align*}
  \]
Sketch of the proof I.

Consider right and left block-Toeplitz matrices

\[ T_L(t, s) := (M_{j-i})_{i,j \geq 0} \quad T_R(t, s) := (M_{i-j})_{i,j \geq 0} \]

(they are moment matrices for \(< ., . >_L\) and \(< ., . >_R\).)
Consider right and left block-Toeplitz matrices

\[ T^L(t, s) := (M_{j-i})_{i,j \geq 0} \quad T^R(t, s) := (M_{i-j})_{i,j \geq 0} \]

(they are moment matrices for \(< ., . >_L\) and \(< ., . >_R\).)

Consider the two factorizations

\[ T^L(t, s) = S_1(t, s)^{-1} S_2(t, s) \quad T^R(t, s) = Z_2(t, s) Z_1^{-1}(t, s) \]
Sketch of the proof I.

- Consider right and left block-Toeplitz matrices

\[ T^L(t, s) := (M_{j-i})_{i,j \geq 0} \quad T^R(t, s) := (M_{i-j})_{i,j \geq 0} \]

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- Consider the two factorizations

\[ T^L(t, s) = S_1(t, s)^{-1}S_2(t, s) \quad T^R(t, s) = Z_2(t, s)Z_1^{-1}(t, s) \]

- Define the following block Lax matrices and block wave vectors:

\[ L_1 := S_1 \Lambda S_1^{-1} \quad L_2 := S_2 \Lambda^{-1}S_2^{-1} \]

\[ R_1 := Z_1^{-1} \Lambda^{-1}Z_1 \quad R_2 := Z_2^{-1} \Lambda Z_2 \]

\[ \Psi_1(z) := \exp(\xi(t, zI))S_1 \chi(z) \quad \Psi_2^*(z) := \exp(-\xi(s, z^{-1}I))(S_2^{-1})^T \chi(z^{-1}) \]

\[ \Phi_1(z) := \exp(\xi(t, zI)) \left[ \chi(z) \right]^T Z_1 \quad \Phi_2^*(z) := \exp(-\xi(s, z^{-1}I)) \chi(z^{-1})^T (Z_2^{-1})^T. \]
Sketch of the proof II.

**Theorem**

The following equations hold true:

\[
L_1 \Psi_1(z) = z \Psi_1(z) \quad \Phi_1(z) R_1 = z \Phi_1(z)
\]

\[
L_2^T \Psi_2^*(z) = z^{-1} \Psi_2^*(z) \quad \Phi_2^*(z) R_2^T = z^{-1} \Phi_2^*(z)
\]

\[
\partial_{t_n} \Psi_1 = (L_{1n})^+ \Psi_1 \quad \partial_{t_n} \Phi_1 = \Phi_1(R_{1n})^-
\]

\[
\partial_{s_n} \Psi_1 = (L_{2n})^- \Psi_1 \quad \partial_{s_n} \Phi_1 = \Phi_1(R_{2n})^+
\]

\[
\partial_{t_n} \Psi_2^* = -(L_{1n}^+)^T \Psi_2^* \quad \partial_{t_n} \Phi_2^* = -\Phi_2^*(R_{1n}^-)^T
\]

\[
\partial_{s_n} \Psi_2^* = -(L_{2n}^-)^T \Psi_2^* \quad \partial_{s_n} \Phi_2^* = -\Phi_2^*(R_{2n}^+)^T.
\]
Theorem (A. Sinap-W. van Assche ’96, M.C.)

Lax operators are given in terms of recursion coefficients by

\[
(L_1)_{N,M+1} = -x_{N+1}^l \left( \prod_{j=N+2}^{M} (I - y_j^r x_j^l) \right) y_{M+1}^r \quad \forall N > M \geq -1
\]

\[
(R_2)_{N,M+1} = -y_{N+1}^r \left( \prod_{j=N+2}^{M} (I - x_j^l y_j^r) \right) x_{M+1}^l \quad \forall N > M \geq -1
\]

\[
(L_2)_{M+1,N} = -h_{M+1}^{-l} x_{M+1}^r \left( \prod_{j=N+2}^{M+1} (I - y_j^r x_j^l) \right) y_{N+1}^l h_{N}^l \quad \forall N > M \geq -1
\]

\[
(R_1)_{M+1,N} = -h_{M+1}^{-r} y_{M+1}^l \left( \prod_{j=N+2}^{M+1} (I - x_j^r y_j^l) \right) x_{N+1}^r h_{N}^r \quad \forall N > M \geq -1
\]

\[
(L_1)_{N,N+1} = (R_2)_{N,N+1} = I
\]

\[
(L_2)_{N+1,N} = h_{N+1}^l h_{N}^{-l} \quad (R_1)_{N+1,N} = h_{N+1}^r h_{N}^{-r}.
\]
Conclusions and Perspectives

The link between BOPUC and AL hierarchy is extended to the matrix case considering the non-abelian version of AL. In the scalar case AL equations combined with string equations coming from the unitary matrix models lead to Painlevé and discrete Painlevé equations. Is it possible to generalize these results to the matrix case? Any applications in combinatorics for the matrix case? (Gessel, Baik, Deift, Johansson, Its, Widom, Tracy, Borodin, Okounkov, Adler, van Moerbeke, Hisakado...)

M. Cafasso
MBOPUC and non-abelian AL hierarchy.
Conclusions and Perspectives

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Conclusions and Perspectives

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MBOPUC and non-abelian AL hierarchy.