Matrix biorthogonal polynomials on the unit circle and non-abelian Ablowitz-Ladik hierarchy.

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- Zakharov-Shabat spectral problem for AKNS:

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- Ablowitz-Ladik discretization:

$$\begin{cases} \Psi_{k+1} = L_k \Psi_k \\ \partial_\tau \Psi_k = M_k \Psi_k \end{cases} \qquad L_k := \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix}$$

$$\Longrightarrow \partial_\tau L_k = M_{k+1} L_k - L_k M_k$$

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- Discrete complexified non-linear Schrödinger:

$$L_k := \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix} \quad M_k := \begin{pmatrix} z^2 - 1 - x_k y_{k-1} & z x_k - z^{-1} x_{k-1} \\ z y_{k-1} - z^{-1} y_k & -z^{-2} + 1 + x_{k-1} y_k \end{pmatrix}$$

$$\Longrightarrow \begin{cases} \partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \partial_\tau y_k = -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}). \end{cases}$$

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- We make them evolve using 2D-Toda flow.

2D-Toda I.

Definitions:

$$\Lambda := (\delta_{i,j-1})_{i,j\geq 0}, \quad \Lambda^{-1} := \Lambda^T \qquad \begin{cases} L_1 := \Lambda + \sum_{i\leq 0} a_i^{(1)} \Lambda^i \\ L_2 := a_{-1}^{(2)} \Lambda^{-1} + \sum_{i\geq 0} a_i^{(2)} \Lambda^i \end{cases}$$

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Zakharov-Shabat Spectral problem:

$$\begin{cases} L_1 \Psi_1 = z \Psi_1 \\ L_2^T \Psi_2^* = z^{-1} \Psi_2^* \end{cases} \qquad \begin{cases} \partial_{t_n} \Psi_1 = (L_1^n)_+ \Psi_1 \\ \partial_{t_n} \Psi_2^* = -(L_1^n)_+^T \Psi_2^* \end{cases} \qquad \begin{cases} \partial_{s_n} \Psi_1 = (L_2^n)_- \Psi_1 \\ \partial_{s_n} \Psi_2^* = -(L_2^n)_-^T \Psi_2^* \end{cases}$$

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Lax equations:

$$\partial_{t_n} L_i = \left[(L_1^n)_+, L_i \right] \quad \partial_{s_n} L_i = \left[(L_2^n)_-, L_i \right]$$

2D-Toda II (linearization).

Theorem (Takasaki '84)

Given any semi-infinite matrix $M := S_1^{-1} S_2$ *with*

$$S_1 = \mathrm{I} + \sum_{i \geq 1} s_i^{(1)} \Lambda^{-i} \qquad S_2 = s_0^{(2)} + \sum_{i \geq 1} s_i^{(2)} \Lambda^{i}$$

define its time evolution through

$$M(t;s) := \exp(\xi(t,\Lambda)) M \exp(-\xi(s,\Lambda^{-1})) = S_1^{-1}(t;s) S_2(t;s).$$

with $\xi(t,\Lambda) = \sum_{i \geq 1} t_i \Lambda^i$.

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with $\xi(t,\Lambda) = \sum_{i \geq 1} t_i \Lambda^i$. *Then*

$$L_1 := S_1 \Lambda S_1^{-1} \quad L_2 := S_2 \Lambda^{-1} S_2^{-1}$$

$$\Psi_1 := \exp \xi(t,z) S_1 \chi(z) \quad \Psi_2^* := \exp(-\xi(s,z^{-1}))(S_2^{-1})^T \chi(z^{-1})$$

with $\chi(z) := (1, z, z^2, \ldots)^T$ *solve 2D-Toda.*

Generalized biorthogonal polynomials

Proposition

Given a matrix $M = S_1^{-1} S_2$ *define a bilinear pairing on the space of polynomials in* z *imposing* $< z^i, z^j >_M = M_{i,j}$. *Then the polynomials*

$$q^{(1)} := (q_0^{(1)}, q_1^{(1)}, q_2^{(1)}, \dots)^T = S_1 \chi(z)$$

$$q^{(2)} := (q_0^{(2)}, q_1^{(2)}, q_2^{(2)}, \dots)^T = (S_2^{-1})^T \chi(z)$$

are biorthonormal, i.e. $< q_i^{(1)}, q_j^{(2)} >_M = \delta_{i,j}$

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- Toeplitz simmetry is conserved along 2D-Toda flow.
- The pairing $< .,. >_{M(t,s)}$ can be written as

$$< P(z), Q(z) >_M = \oint P(z)Q(z^{-1}) \exp(\xi(t,z))\gamma(z) \exp(-\xi(s,z^{-1})) \frac{dz}{2\pi i z}$$

with $\gamma(z) = \sum_k M_k z^k$ initial value.

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- Monic biorthogonal polynomials $\{p_i^{(1)}, p_j^{(2)}\}$ such that $< p_i^{(1)}(z), p_j^{(2)}(z) >_M = h_i \delta_{i,j}$ satisfy the recursion relation

$$\begin{pmatrix} p_{n+1}^{(1)}(z) \\ \tilde{p}_{n+1}^{(2)}(z) \end{pmatrix} = \mathcal{L}_n \begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} z & x_{n+1} \\ zy_{n+1} & 1 \end{pmatrix} \begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix}$$

  with $\tilde{p}_n^{(2)}(z) := z^n p_n^{(2)}(z^{-1})$

The Toeplitz lattice II (Adler-van Moerbeke '01).

Theorem

Lax operators of the Toeplitz lattice are of the following form:

$$h^{-1}L_1h = \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & \dots & \dots \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \dots \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$L_2(x,y) = (h^{-1}L_1h)^T(y,x).$$

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Hence the Toeplitz lattice gives time evolution for recursion coefficients for BOPUC.

Theorem

Consider time-dipendent BOPUC $\{p_n^{(1)}(z), p_n^{(2)}(z)\}$ *with respect to the pairing*

$$< P(z), Q(z) >:= \oint P(z)Q(z^{-1})\gamma(t,s;z)\frac{dz}{2\pi i z}$$

with $\gamma(t,s;z) := \exp(\xi(t,z))\gamma(z)\exp(-\xi(s,z^{-1}))$. *Then the related recursion operators* $\mathcal{L}_k$ *evolve according to semidiscrete zero-curvature equations for the Ablowitz-Ladik hierarchy*

$$\partial_{t_i/s_i}\mathcal{L}_k = \mathcal{M}_{t_i/s_i,k+1}\mathcal{L}_k - \mathcal{L}_k\mathcal{M}_{t_i/s_i,k}.$$

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- Using Zakharov-Shabat equations, for every k, $\partial_{t_i} p_k^{(1)}$ is a linear combination of $\{p_k^{(1)}, p_{k+1}^{(1)}, p_{k+2}^{(1)}, \ldots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$.
  In the same way, for every k, $\partial_{t_i} \tilde{p}_k^{(2)}$ is a linear combination of $\{\tilde{p}_k^{(2)}, \tilde{p}_{k-1}^{(2)}, \tilde{p}_{k-2}^{(2)}, \ldots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$.

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- Using recursion relations you arrive to write $\mathcal{M}_{t_i,k}$ depending on $\{x_j, y_j, z\}$.

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$$\partial_{t_1} \tilde{p}_k^{(2)} = -z \frac{h_{k+1}}{h_k} \tilde{p}_{k-1}^{(2)} = z y_k p_k^{(1)} - z \tilde{p}_k^{(2)}$$

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$$\Longrightarrow \partial_{t_1} \begin{pmatrix} p_k^{(1)} \\ \tilde{p}_k^{(2)} \end{pmatrix} = \mathcal{M}_{t_1,k} \begin{pmatrix} p_k^{(1)} \\ \tilde{p}_k^{(2)} \end{pmatrix} = \begin{pmatrix} -x_{k+1} y_k & x_{k+1} \\ z y_k & -z \end{pmatrix} \begin{pmatrix} p_k^{(1)} \\ \tilde{p}_k^{(2)} \end{pmatrix}$$

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What about time evolution for matrix biorthogonal polynomials on the unit circle?

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- Consider the two matrix-valued pairings

$$< P(z), Q(z) >_L = \oint P(z)\gamma(z)Q^*(z)\frac{dz}{2\pi i z}$$
$$< P(z), Q(z) >_R = \oint P^*(z)\gamma(z)Q(z)\frac{dz}{2\pi i z}$$

and corresponding matrix-valued biorthogonal polynomials

$$\{P_n^{(1)R}(z), P_n^{(1)L}(z), P_n^{(2)R}(z), P_n^{(2)L}(z)\}$$

such that

$$< P_k^{(2)R}, P_j^{(1)R} >_R = \delta_{kj} h_k^R \qquad < P_k^{(1)L}, P_j^{(2)L} >_L = \delta_{kj} h_k^L.$$

The setting II.

- The following block recursion relations can be obtained:

$$\begin{pmatrix} P_{N+1}^{(1)L} \\ \tilde{P}_{N+1}^{(2)R} \end{pmatrix} = \mathcal{L}_N^L \begin{pmatrix} P_N^{(1)L} \\ \tilde{P}_N^{(2)R} \end{pmatrix} = \begin{pmatrix} z\mathrm{I} & x_{N+1}^L \\ zy_{N+1}^R & \mathrm{I} \end{pmatrix} \begin{pmatrix} P_N^{(1)L} \\ \tilde{P}_N^{(2)R} \end{pmatrix}$$

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- Now introduce time evolution

$$\gamma(t,s;z) := \exp(\xi(t,z\mathrm{I}))\gamma(z)\exp(-\xi(s,z^{-1}\mathrm{I}))$$

and study time evolution for corresponding matrix biorthogonal polynomials.

The setting III.

- Look for some matrices $\mathcal{M}^L_{t_i/s_i,n}, \mathcal{M}^R_{t_i/s_i,n}$ such that

$$\partial_{t_i/s_i}\begin{pmatrix} P_N^{(1)L}(z) \\ \tilde{P}_N^{(2)R}(z) \end{pmatrix} = \mathcal{M}^L_{t_i/s_i,n}\begin{pmatrix} P_N^{(1)L}(z) \\ \tilde{P}_N^{(2)R}(z) \end{pmatrix}$$

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- Compatibility conditions will give non-abelian semidiscrete zero-curvature equations.

$$\partial_\tau \mathcal{L}_k^L = \mathcal{M}_{k+1}^L\mathcal{L}_k^L - \mathcal{L}_n^L\mathcal{M}_k^L$$
$$\partial_\tau \mathcal{L}_k^R = \mathcal{L}_k^R\mathcal{M}_{k+1}^R - \mathcal{M}_k^R\mathcal{L}_k^R.$$

Non-abelian AL and MBOPUC

Theorem (M.C.)

Consider time-dipendent matrix BOPUC $\{P_n^{(1)R}(z), P_n^{(1)L}(z)\}$ *and* $\{P_n^{(2)R}(z), P_n^{(2)L}(z)\}$ *with respect to the pairings*

$$< P, Q >_R := \oint P^*(z)\gamma(t,s;z)Q(z)\frac{dz}{2\pi i z}$$
$$< P, Q >_L := \oint P(z)\gamma(t,s;z)Q^*(z)\frac{dz}{2\pi i z}$$

with $\gamma(t,s;z) := \exp(\xi(t,z\mathrm{I}))\gamma(z)\exp(-\xi(s,z^{-1}\mathrm{I}))$. *Then the related block recursion operators* $\mathcal{L}^R$ *and* $\mathcal{L}^L$ *evolves according to the non-abelian AL equations:*

$$\partial_{t_i/s_i}\mathcal{L}_k^L = \mathcal{M}^L_{t_i/s_i,k+1}\mathcal{L}_k^L - \mathcal{L}_n^L\mathcal{M}^L_{t_i/s_i,k}$$
$$\partial_{t_i/s_i}\mathcal{L}_k^R = \mathcal{L}_k^R\mathcal{M}^R_{t_i/s_i,k+1} - \mathcal{M}^R_{t_i/s_i,k}\mathcal{L}_k^R.$$

Discrete NLS and its non abelian version.

- Discrete NLS:

$$\begin{cases} \partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \partial_\tau y_k = -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}). \end{cases}$$

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- Non-abelian discrete NLS:

$$\begin{cases} \partial_\tau x_k^L = x_{k+1}^L - 2x_k^L + x_{k-1}^L - x_{k+1}^L y_k^R x_k^L - x_k^L y_k^R x_{k-1}^L \\ \\ \partial_\tau y_k^R = -y_{k+1}^R + 2y_k^R - y_{k-1}^R + y_{k+1}^R x_k^L y_k^R + y_k^R x_k^L y_{k-1}^R \end{cases}$$

$$\begin{cases} \partial_\tau x_k^R = x_{k+1}^R - 2x_k^R + x_{k-1}^R - x_{k-1}^R y_k^L x_k^R - x_k^R y_k^L x_{k+1}^R \\ \\ \partial_\tau y_k^L = -y_{k+1}^L + 2y_k^L - y_{k-1}^L + y_{k-1}^L x_k^R y_k^L + y_k^L x_k^R y_{k+1}^L \end{cases}$$

Sketch of the proof I.

- Consider right and left block-Toeplitz matrices

$$T^L(t,s) := (M_{j-i})_{i,j\geq 0} \qquad T^R(t,s) := (M_{i-j})_{i,j\geq 0}$$

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- Define the following block Lax matrices and block wave vectors:

$$L_1 := S_1\Lambda S_1^{-1} \quad L_2 := S_2\Lambda^{-1}S_2^{-1}$$

$$R_1 := Z_1^{-1}\Lambda^{-1}Z_1 \quad R_2 := Z_2^{-1}\Lambda Z_2$$

$$\Psi_1(z) := \exp(\xi(t,z\mathrm{I}))S_1\chi(z) \quad \Psi_2^*(z) := \exp(-\xi(s,z^{-1}\mathrm{I}))(S_2^{-1})^T\chi(z^{-1})$$

$$\Phi_1(z) := \exp(\xi(t,z\mathrm{I}))\Big[\chi(z)\Big]^T Z_1 \quad \Phi_2^*(z) := \exp(-\xi(s,z^{-1}\mathrm{I}))\chi(z^{-1})^T(Z_2^{-1})^T.$$

Sketch of the proof II.

Theorem

The following equations hold true:

$$
\begin{aligned}
L_1\Psi_1(z) &= z\Psi_1(z) & \Phi_1(z)R_1 &= z\Phi_1(z)\\
L_2^T\Psi_2^*(z) &= z^{-1}\Psi_2^*(z) & \Phi_2^*(z)R_2^T &= z^{-1}\Phi_2^*(z)\\
\partial_{t_n}\Psi_1 &= (L_1^n)_+\Psi_1 & \partial_{t_n}\Phi_1 &= \Phi_1(R_1^n)_-\\
\partial_{s_n}\Psi_1 &= (L_2^n)_-\Psi_1 & \partial_{s_n}\Phi_1 &= \Phi_1(R_2^n)_+\\
\partial_{t_n}\Psi_2^* &= -(L_{1+}^n)^T\Psi_2^* & \partial_{t_n}\Phi_2^* &= -\Phi_2^*(R_{1-}^n)^T\\
\partial_{s_n}\Psi_2^* &= -(L_{2-}^n)^T\Psi_2^* & \partial_{s_n}\Phi_2^* &= -\Phi_2^*(R_{2+}^n)^T.
\end{aligned}
$$

Theorem (A.Sinap-W.van Assche '96,M.C.)

Lax operators are given in terms of recursion coefficients by

$$(L_1)_{N,M+1} = -x^l_{N+1}\Big(\prod_{j=N+2}^{M-}(\mathrm{I} - y^r_j x^l_j)\Big)y^r_{M+1} \quad \forall\, N > M \geq -1$$

$$(R_2)_{N,M+1} = -y^r_{N+1}\Big(\prod_{j=N+2}^{M-}(\mathrm{I} - x^l_j y^r_j)\Big)x^l_{M+1} \quad \forall\, N > M \geq -1$$

$$(L_2)_{M+1,N} = -h^{-l}_{M+1}x^r_{M+1}\Big(\prod_{j=N+2}^{M+}(\mathrm{I} - y^l_j x^r_j)\Big)y^l_{N+1}h^l_N \quad \forall\, N > M \geq -1$$

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$$(L_1)_{N,N+1} = (R_2)_{N,N+1} = \mathrm{I}$$

$$(L_2)_{N+1,N} = h^l_{N+1}h^{-l}_N \quad (R_1)_{N+1,N} = h^r_{N+1}h^{-r}_N.$$

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- The link between BOPUC and AL hierarchy is extended to the matrix case considering the non-abelian version of AL.
- In the scalar case AL equations combined with string equations coming from the unitary matrix models lead to Painleve' and discrete Painleve' equations. Is it possible to generalize these results to the matrix case?
- Any applications in combinatorics for the matrix case?

(...Gessel,Baik,Deift,Johansson,Its,Widom,Tracy,Borodin, Okounkov,Adler,van Moerbeke,Hisakado...)