Asymptotics of Tracy-Widom Distributions

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Outline

(1). Tracy-Widom distributions

(2). Undetermined constants

(3). Proof of the GUE constant

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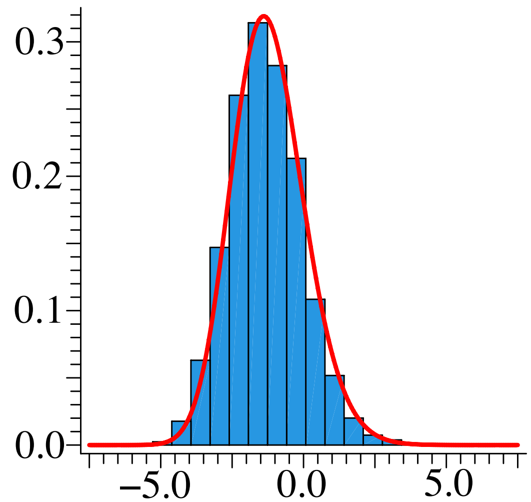
The Tracy-Widom Distributions

The Tracy-Widom distributions arise in a variety of processes, including:

- The largest eigenvalue of the GOE, GUE, GSE random matrix ensembles
- The length of the longest increasing subsequence of a random permutation
- The last passage time of a last passage percolation model
- The height of a random growth model



The Tracy-Widom GOE distribution and a histogram of largest eigenvalues (shifted and scaled) of 7500 realizations of the GOE ensemble of 100×100 matrices.

Formulas for The Tracy-Widom Distributions

Let $q(x)$ be the Hastings-McLeod solution to the homogenous Painlevé II equation

$$q'' = 2q^3 + xq,$$

satisfying the boundary condition

$$q(x) \sim \text{Ai}(x), \qquad x \to +\infty.$$

Then the GOE, GUE, and GSE Tracy-Widom distributions are

$$F_1(x) = F(x)E(x), \qquad F_2(x) = F(x)^2, \qquad F_4(x) = \frac{1}{2}\left\{E(x) + \frac{1}{E(x)}\right\}F(x),$$

where, after setting $R(x) := \int_x^\infty (q(s))^2 ds$, we have

$$F(x) = \exp\left(-\frac{1}{2}\int_x^\infty R(s)ds\right), \qquad E(x) = \exp\left(-\frac{1}{2}\int_x^\infty q(s)ds\right).$$

Asymptotics as $x \to +\infty$

From the known asymptotic expansion of the Hastings-McLeod solution:

$$
\begin{aligned}
F(x) &= 1 - \frac{e^{-\frac{4}{3}x^{3/2}}}{32\pi x^{3/2}}\left(1 - \frac{35}{24x^{3/2}} + O(x^{-3})\right), \\
E(x) &= 1 - \frac{e^{-\frac{2}{3}x^{3/2}}}{4\sqrt{\pi}x^{3/2}}\left(1 - \frac{41}{48x^{3/2}} + O(x^{-3})\right),
\end{aligned}
$$

and the asymptotic expansions of the Tracy-Widom distributions follow.

Asymptotics as $\mathrm{x} \to -\infty$

This limit is different. Now the asymptotics of the Hastings-McLeod solution yields

$$
\begin{aligned}
F_1(x) &= \tau_1 \frac{e^{-\frac{1}{24}|x|^3 - \frac{1}{3\sqrt{2}}|x|^{3/2}}}{|x|^{1/16}} \left(1 - \frac{1}{24\sqrt{2}|x|^{3/2}} + O(|x|^{-3})\right), \\
F_2(x) &= \tau_2 \frac{e^{-\frac{1}{12}|x|^3}}{|x|^{1/8}} \left(1 + \frac{3}{2^6|x|^3} + O(|x|^{-6})\right), \\
F_4(x) &= \tau_4 \frac{e^{-\frac{1}{24}|x|^3 + \frac{1}{3\sqrt{2}}|x|^{3/2}}}{|x|^{1/16}} \left(1 + \frac{1}{24\sqrt{2}|x|^{3/2}} + O(|x|^{-3})\right).
\end{aligned}
$$

Note the undetermined constants of integration τ_i!

Using Universality to Pin Down the Constants

As noted previously, the Tracy-Widom distributions are the limits of various objects. The strategy is to choose the appropriate approximate ensemble so the associated constant term is explicitly computable.

Deift-Its-Krasovsky (2006) started with the Laguerre Unitary Ensemble and took the appropriate limit to show

$$\tau_2 = 2^{1/24} e^{\zeta'(-1)}$$

(as conjectured by Tracy and Widom). Their proof used the fact that

$$F_2(x) = \det(1 - \mathrm{Airy}_x),$$

where Airy_x is the operator on $L^2(x, \infty)$ with kernel

$$\mathrm{Airy}(u, v) = \frac{\mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\mathrm{Ai}(v)}{u - v}.$$

An Alternate Method

Let $D_n(t)$ denote the $n \times n$ Toeplitz determinant with symbol $f(e^{i\theta}) = e^{2t\cos(\theta)}$ on the unit circle:

$$D_n(t) = \det \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2t\cos\theta} e^{i(j-k)\theta} d\theta \right)_{0 \le j,k \le n-1}$$

Then in the double scaling limit $n = [2t + xt^{1/3}]$ and $t \to \infty$:

$$e^{-t^2} D_n(t) \to F_2(x).$$

Orthogonal Polynomials

Let $p_j(z) = \kappa_j z^j + \cdots$ be the orthonormal polynomial of degree j with respect to the weight $e^{2t\cos\theta}\frac{d\theta}{2\pi}$:

$$\int_{-\pi}^{\pi} p_j(e^{i\theta})\overline{p_k(e^{i\theta})}e^{2t\cos\theta}\frac{d\theta}{2\pi} = \delta_{jk} \qquad \text{for } j,k \geq 0.$$

The leading coefficient $\kappa_j(t)$ is given by

$$\kappa_j(t) = \sqrt{\frac{D_j(t)}{D_{j+1}(t)}}.$$

Thus

$$e^{-t^2}D_n(t) = e^{-t^2}\prod_{q=1}^{n}\frac{D_q(t)}{D_{q-1}(t)} = e^{-t^2}\prod_{q=1}^{n}\frac{1}{\kappa_{q-1}^2(t)}.$$

Three Regimes

Let L and M be large constants.

$$\log(e^{-t^2} D_n) = -t^2 + \underbrace{\log(D_L)}_{\text{exact part}}$$

$$+ \underbrace{\sum_{q=L+1}^{[2t - Mt^{1/3} - 1]} \log(\kappa_{q-1}^{-2})}_{\text{Airy part}}$$

$$+ \underbrace{\sum_{q=[2t - Mt^{1/3}]}^{[2t + xt^{1/3}]} \log(\kappa_{q-1}^{-2})}_{\text{Painlevé part}} .$$

We get the desired limit from

$$\lim_{L,M\to\infty} \lim_{t\to\infty} \log(e^{-t^2} D_n(t)), \qquad n = [2t + xt^{1/3}].$$

The Exact Part

$$D_L(t) = \frac{1}{(2\pi)^L L!} \int_{[-\pi,\pi]^L} e^{2t \sum_{j=1}^L \cos\theta_j} \prod_{1 \le k < \ell \le L} |e^{i\theta_k} - e^{i\theta_\ell}|^2 \prod_{j=1}^L d\theta_j.$$

A stationary phase argument shows

$$\lim_{t \to \infty} D_L(t) \cdot \left(\frac{e^{2tL}}{(2\pi)^L} D_L^{\text{Herm}}(t) \right)^{-1} = 1,$$

where

$$D_L^{\text{Herm}}(t) = \frac{1}{L!} \int_{[-\infty,\infty]^L} e^{-t \sum_{j=1}^L \theta_j^2} \prod_{1 \le k < \ell \le n} |\theta_k - \theta_\ell|^2 \prod_{j=1}^L d\theta_j.$$

The Exact Part, Continued

The Selberg integral $D_L^{\mathrm{Herm}}(t)$ can be computed explicitly:

$$D_L^{\mathrm{Herm}}(t) = \frac{\pi^{L/2}}{2^{L(L-1)/2} t^{L^2/2}} \prod_{q=0}^{L-1} q! = \frac{\pi^{L/2}}{2^{L(L-1)/2} t^{L^2/2}} G(L+1).$$

Using the asymptotics of the Barnes G-function gives

$$\lim_{L\to\infty} \lim_{t\to\infty} \left(\log(D_L) - \left\{ 2Lt - \frac{L^2}{2}\log(2t) + \left(\frac{L^2}{2} - \frac{1}{12}\right)\log L - \frac{3}{4}L^2 + \zeta'(-1) \right\} \right) = 0.$$

The Painlevé Part

From Baik-Deift-Johansson (1999):

$$\lim_{t\to\infty} \sum_{q=[2t-Mt^{1/3}]}^{[2t+xt^{1/3}]} \log(\kappa_{q-1}^{-2}) = -\int_{-M}^{x} R(y)dy$$

$$= -\int_{-M}^{x} \left(R(y) - \frac{1}{4}y^2 + \frac{1}{8y} \right) dy + \frac{1}{12}x^3$$

$$-\frac{1}{8}\log|x| + \frac{1}{12}M^3 + \frac{1}{8}\log M.$$

Recall $R(x) = \int_x^\infty (q(s))^2 ds$.

The Airy Part

To compute

$$\sum_{q=L+1}^{[2t-Mt^{1/3}-1]} \log(\kappa_{q-1}^{-2})$$

we analyze the asymptotics of the orthogonal polynomials

$$p_j(z) = \kappa_j z^j + \cdots .$$

Riemann-Hilbert Analysis of Orthogonal Polynomials

Let the 2×2 matrix-valued function $Y(z;q)$ satisfy the Riemann-Hilbert problem

$$\begin{cases} Y(z;q) \text{ is analytic off the unit circle } \Sigma, \\ Y_+(z;q) = Y_-(z;q) \begin{pmatrix} 1 & \frac{1}{z^q} e^{t(z+z^{-1})} \\ 0 & 1 \end{pmatrix} \text{ on } \Sigma, \\ Y(z;q) = (I + O(z^{-1})) \begin{pmatrix} z^q & 0 \\ 0 & z^{-q} \end{pmatrix} \text{ as } z \to \infty. \end{cases}$$

Then

$$\kappa_{q-1}^2(t) = -Y_{21}(0;q,t).$$

Analysis of the Riemann-Hilbert Problem

We proceed with the analysis as in Baik-Deift-Johannson (1999), introducing a g-function and deforming contours.

The leading-order contribution to the solution comes from one band on the circle: $\{e^{i\theta}$ such that $-\theta_c \leq \theta \leq \theta_c\}$.

We need the contribution from the Airy parametrices to higher accuracy than is found in Baik-Deift-Johannson (1999).

The Airy Part, Continued

Analysis of the Riemann-Hilbert problem yields that

$$\lim_{L,M\to\infty}\lim_{t\to\infty}\left|\sum_{q=L+1}^{[2t-Mt^{1/3}-1]}\log(\kappa_{q-1}^{-2})-\left(t^2-2tL+\frac{1}{2}L^2\log(2t)\right.\right.$$

$$\left.\left.-(\frac{1}{2}L^2-\frac{1}{12})\log L+\frac{3}{4}L^2-\frac{1}{12}M^3-\frac{1}{8}\log M+\frac{1}{24}\log 2\right)\right|=0.$$

Result for $F(x)$

Combining the results from the exact, Painlevé, and Airy parts gives

$$\lim_{M,L\to\infty}\lim_{t\to\infty}\log(e^{-t^2}D_n(t))$$

$$= -\int_{-\infty}^{x}\left(R(y)-\frac{1}{4}y^2+\frac{1}{8y}\right)dy+\frac{1}{12}x^3-\frac{1}{8}\log|x|+\frac{1}{24}\log 2+\zeta'(-1).$$

Along with

$$F_2(x)=\lim_{M,L\to\infty}\lim_{t\to\infty}e^{-t^2}D_n(t)$$

this shows that (for $x<0$):

$$F(x)=2^{1/48}e^{\frac{1}{2}\zeta'(-1)}\frac{e^{-\frac{1}{24}|x|^3}}{|x|^{1/16}}\exp\left\{\frac{1}{2}\int_{-\infty}^{x}\left(R(y)-\frac{1}{4}y^2+\frac{1}{8y}\right)dy\right\}.$$

Consequences

Combined with the standard formula

$$F_2(x) = \exp\left(-\int_x^\infty R(s)ds\right),$$

this gives the following total integral formula for $R(x) = \int_x^\infty (q(s))^2 ds$ if $c < 0$:

$$\int_c^\infty R(y)dy + \int_{-\infty}^c \left(R(y) - \frac{1}{4}y^2 + \frac{1}{8y}\right)dy = -\frac{1}{24}\log 2 - \zeta'(-1) + \frac{1}{12}|c|^3 + \frac{1}{8}\log|c|.$$

Also, we can compute the asymptotics of $F_2(x)$ as $x \to -\infty$, yielding

$$\tau_2 = 2^{1/24} e^{\zeta'(-1)}.$$

Asymptotics of $E(x)$

Let $\pi_j(z;t) := \frac{1}{\kappa_j} p_j(z;t)$ be the monic orthogonal polynomials with respect to $\frac{1}{2\pi} e^{2t\cos\theta} d\theta$ on the unit circle. Define

$$D_\ell^{++}(t) = \det(I_{j-k}(2t) + I_{j+k+2}(2t))_{0\le j,k\le \ell-1},$$

$$D_\ell^{-+}(t) = \det(I_{j-k}(2t) + I_{j+k+1}(2t))_{0\le j,k\le \ell-1},$$

where

$$I_j(2t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2t\cos\theta} e^{ij\theta} d\theta.$$

Then from Baik-Rains (2000, 2001):

$$E(x)^2 = \lim_{t\to\infty} e^{-t} \frac{D_{L-1}^{++} D_L^{-+}}{D_{2L-1}} \prod_{j=L}^{(t+\frac{x}{2}t^{1/3}-1)} [(1+\pi_{2j}(0;t))(1-\pi_{2j+1}(0;t))].$$

$E(x)$, Continued

The end result is:

$$E(x) = \frac{1}{2^{1/4}} e^{-\frac{1}{3\sqrt{2}}|x|^{3/2}} \exp\left\{\frac{1}{2}\int_{-\infty}^{x}\left(q(y) - \sqrt{\frac{|y|}{2}}\right)dy\right\}.$$

Using the known asymptotics of the Hastings-McLeod solution as $x \to -\infty$, we can now identify the unknown constants τ_1 and τ_4 in the GOE and GSE Tracy-Widom distributions:

$$\tau_1 = 2^{-11/48} e^{\frac{1}{2}\zeta'(-1)}, \quad \tau_4 = 2^{-35/48} e^{\frac{1}{2}\zeta'(-1)}.$$

The Total Integral Formula for $q(x)$

Using the representation of $E(x)$ via Painlevé functions, this is equivalent to

$$\int_c^\infty q(y)dy + \int_{-\infty}^c \left(q(y) - \sqrt{\frac{|y|}{2}}\right)dy = \frac{1}{2}\log 2 + \frac{\sqrt{2}}{3}|c|^{3/2}.$$

We call this the total integral formula for the Hastings-McLeod solution.

More recently, Baik-B-DiFranco-Its have developed a method for this and other total integrals directly from the Riemann-Hilbert problem for Painlevé II functions.

Extension: Incomplete Spectra

Consider the eigenvalues of a realization of the $k \times k$ GUE ensemble:

$$\lambda_k \leq \lambda_{k-1} \leq \lambda_{k-2} \leq \cdots \leq \lambda_8 \leq \lambda_7 \leq \lambda_6 \leq \lambda_5 \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1$$

Now suppose that each eigenvalue is detected with probability p and not detected with probability $1-p$:

$$\lambda_k \leq \cancel{\lambda_{k-1}} \leq \lambda_{k-2} \leq \cdots \leq \lambda_8 \leq \cancel{\lambda_7} \leq \lambda_6 \leq \lambda_5 \leq \lambda_4 \leq \lambda_3 \leq \cancel{\lambda_2} \leq \cancel{\lambda_1}$$

Then the distribution of the largest (appropriately shifted and scaled) detected eigenvalue λ_1^{det} converges to the incomplete Tracy-Widom distribution:

$$F_2(x;p) = \lim_{k\to\infty} \mathbb{P}(\sqrt{2}k^{1/6}(\lambda_1^{\text{det}} - \sqrt{2k}) \leq x).$$

Representation in Terms of Ablowitz-Segur Solutions to Painlevé II

Tracy-Widom (1994):

$$F_2(x;p) = \exp\left(-\int_x^\infty (s-x)q(s;p)^2 dx\right),$$

where $q(x;p)$ satisfies the Painlevé II equation

$$q'' = xq + 2q^3$$

and the boundary condition

$$q(x;p) \sim \sqrt{p}\,\mathrm{Ai}(x) \text{ as } x \to +\infty.$$

Determinental Representations

We also have the relation

$$F_2(x;p) = \det(1 - p\,\mathrm{Airy}_x),$$

Recall Airy_x is the operator with kernel

$$\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}$$

restricted to $(x, +\infty)$.

Tracy-Widom (1994)

Determinental Representations

There is another useful representation:

$$F_2(x;p) = \lim_{\substack{n,t\to\infty \\ n=[2t+xt^{1/3}]}} (1+\sqrt{p})^{-n}\det(1-\sqrt{p}\boldsymbol{K}_n),$$

where $\boldsymbol{K}_n$ is the integral operator on $L^2(\Sigma)$ with kernel

$$k_n(z,w) = \frac{z^{-n}w^n - \phi(z)\phi(w)^{-1}}{2\pi i(z-w)}, \quad \phi(z) = e^{t(z-z^{-1})}.$$

This determinant can be analyzed via an associated Riemann-Hilbert problem.

Borodin-Okounkov-Olshanski (2000), Johansson (2001), Baik-Deift-Rains (2001).

Conclusions

- We give a new proof of the GUE constant.
- We give the first derivation and proof of the GOE and GSE constants.
- A new method for computing total integrals provides an alternate proof of the GOE and GSE constants.
- Work in progress: the constant problem for incomplete spectra.