Gap Probabilities in Two-Matrix Models

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work with:

+ P. Damgaard: arXiv:0803.1171 (JHEP) & in preparation

+ J. Osborn + K. Splittorff: hep-th/0609059v2 (Nucl.Phys.B)

Outline

1. **Two Matrix Model** (2MM)
2. **Physics** Motivation: **QCD**
3. **Objects:**

   densities, gaps, individual eigenvalues
4. **Solution for densities**
5. Expansion of gaps & eigenvalues in densities
6. **Exact results for gaps & eigenvalues**
7. **Large-N & Examples**
8. Remarks and open problems

Chiral Two-Matrix Model

$$\mathcal{Z} = \int d\Phi d\Psi \, \Pi_{f=1}^{N_1} \det[i\not{D}_1 + m_f] \, \Pi_{g=1}^{N_2} \det[i\not{D}_2 + n_g] \, \mathrm{e}^{-\mathrm{Tr}(\Phi\Phi^\dagger + \Psi^\dagger\Psi)}$$

- we are interested in real non-neg. eigenvalues of
  $$\not{D}_{1,2} \equiv \begin{pmatrix} 0 & \Phi + \mu_{1,2}\Psi \\ \Phi^\dagger + \mu_{1,2}\Psi^\dagger & 0 \end{pmatrix}$$
  $\Phi, \ \Psi$ are matrices of sizes $N \times (N+\nu) \in \ \mathbb{C}$
- change of variables $\Phi_j \equiv \Phi + \mu_j \Psi, \ j = 1, 2$ couples them:

$$\mathcal{Z} = \int d\Phi_1 d\Phi_2 \, \Pi_{f=1}^{N_1} \det[\not{D}_1 + m_f] \, \Pi_{g=1}^{N_2} \det[\not{D}_2 + n_g] \, \mathrm{e}^{-\mathrm{Tr}(c_1\Phi_1\Phi_1^\dagger + c_2\Phi_2\Phi_2^\dagger - d(\Phi_1\Phi_2^\dagger + \Phi_1^\dagger\Phi_2))}$$

- Unitary dof $\Phi_1 = U_1 X V_1$ can be eliminated using a group integral (HCIZ for non-chiral)
- Φ_i Hermitean $N \times N$ for non-chiral [Ercolani,McLaughlin01]

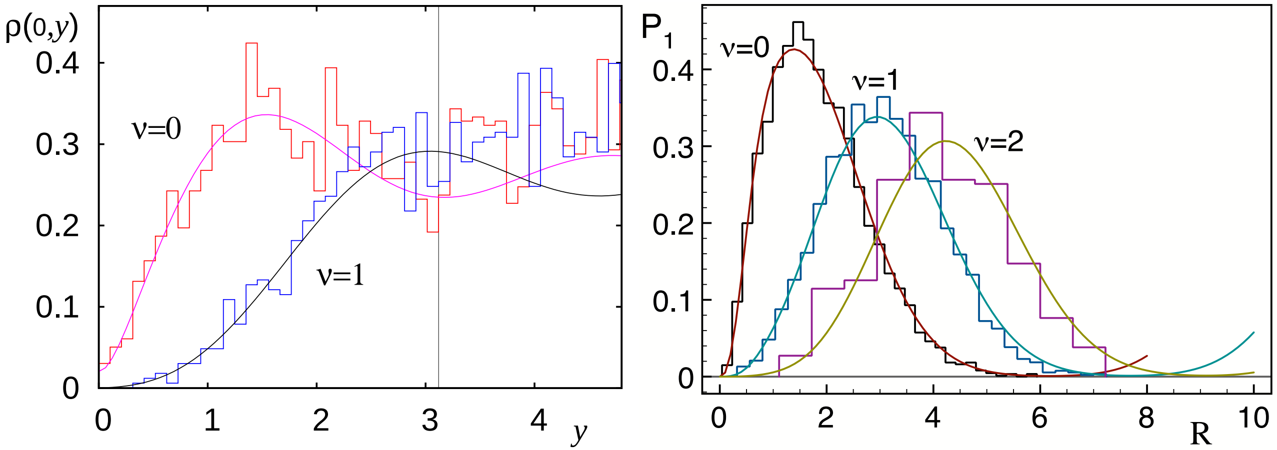
Physics Motivation

- low energy **QCD** $\longrightarrow$ chiral Perturbation Theory (χPT) $\longrightarrow$ const. Pion fields = Matrix Model ($N \to \infty$)
  [Shuryak, Verbaarschot 93;...]

$$\mathcal{Z}_{\varepsilon\chi PT} = \int_{U(N_f)} dU_0 \det[U_0]^{\nu} \mathrm{e}^{\frac{1}{4}VF^2\mu^2 \mathrm{Tr}[\Gamma,U_0][\Gamma,U_0^{\dagger}] + \frac{1}{2}\Sigma V \mathrm{Tr}\left(M(U_0+U_0^{\dagger})\right)}$$

- effective χPT has 2 LEC's: F Pion decay constant & Σ chiral condensate
- determine by comparing *analytical* MM predictions to Lattice **QCD** $\longrightarrow$ need source μ to couple to F
- easiest: imaginary isospin chemical potential $i\mu$ $\longrightarrow$ keeps $D\!\!\!/(\pm i\mu)$ spectra real!
  [Damgaard, Heller, Splittorff, Svetitzky, Toublan 05/06]
- Individual eigenvalues: num. easiest [G.A.,Damgaard 08]

Illustration Density vs. '1st' Eigenvalue on $\mathbb{C}$



- cut through density (left) vs. Lattice data [Bloch,Wettig'06]
- 1st eigenvalue (right) radially integrated **vs same data** [+ G.A., Shifrin 07]

Eigenvalue Representation

- **two sets of real eigenvalues:** $\{x\}$, $\{y\}$

  diagonalise $\Phi_1 = U_1 X V_1$, $\Phi_2 = U_2 Y V_2$ + use integral:

  [Guhr,Wettig; Jackson,Sener,Verbaarschot 96]

$$\int dU dV \exp\left[N d \Re e \mathrm{Tr}(VXUY)\right] = \frac{\det_{1\le i,j\le N}[I_\nu(Ndx_iy_j)]}{\prod_{i=1}^N (x_iy_i)^\nu \Delta_N(\{x^2\})\Delta_N(\{y^2\})}$$

- **chiral 2MM:**

$$\mathcal{Z} = \prod_{i=1}^N \int_0^\infty dx_i dy_i (x_iy_i)^{\nu+1} \prod_{f=1}^{N_1}(x_i^2+m_f^2) \prod_{g=1}^{N_2}(y_i^2+n_g^2)$$

$$\times \Delta_N(x^2)\Delta_N(y^2) \det_{k,l}\left[I_\nu(2dNx_ky_l)\right] \mathrm{e}^{-N\sum_i^N (c_1x_i^2+c_2y_i^2)}$$

- non-Gaussian!! (non-chiral: $I_\nu \to \mathrm{e}$)
- inserted $\not{D}$ operators = ave characteristic polynomials

Objects of Study

- Define jpdf: $\mathcal{Z} \equiv \prod_{i=1}^{N} \int_0^\infty dx_i dy_i \; \mathcal{P}_{jpdf}$
- **densities:**

$$\boxed{R_{k,l}(x_1, \ldots, x_k; y_1, \ldots, y_l) \sim \prod_{i=k+1}^{N} \int_0^\infty dx_i \prod_{j=l+1}^{N} \int_0^\infty dy_j \; \mathcal{P}_{jpdf}}$$

- **gap probabilities:**

$$\boxed{E_{k,l}(s,t) \sim \int_0^s dx_1 \ldots dx_k \int_s^\infty dx_{k+1} \ldots dx_N \int_0^t dx_1 \ldots dx_l \int_t^\infty dy_{l+1} \ldots dy_N \; \mathcal{P}_{jpdf}}$$

- **individual eigenvalues:** $\underline{x_k = s \;\&\; y_l = t}$

$$p_{k,l}(s,t) \sim \int_0^s dx_1 \ldots dx_{k-1} \int_s^\infty dx_{k+1} \ldots dx_N$$

$$\times \int_0^t dy_1 \ldots dy_{l-1} \int_t^\infty dy_{l+1} \ldots dy_N \mathcal{P}_{jpdf}(\ldots, x_k = s, \ldots, y_l = t, \ldots)$$

- not independent: e.g. $\partial_s \partial_t E_{0,0}(s,t) = p_{1,1}(s,t)$ for smallest

Solution for all Densities

- **bi-orthogonal polynomials** $\boxed{\delta_{nk} = \int_0^\infty dxdy\, w^{(N_1,N_2)}(x,y) P_n(x^2) Q_k(y^2)}$

weight $w^{(N_1,N_2)}(x,y) = (xy)^{\nu+1} \prod_{f=1}^{N_1}(x^2+m_f^2) \prod_{g}^{N_2}(y^2+m_g^2) I_\nu(2dNxy)\, \mathrm{e}^{-N(c_1x^2+c_2y^2)}$

- in terms of **4 kernels**: using [Eynard, Mehta 98]

$$\boxed{K_N = \sum_k^{N-1} P_k Q_k,\ \ H_N = \sum_k^{N-1} \hat{\chi}_k P_k,\ \ \hat{H}_N = \sum_k^{N-1} \chi_k Q_k,\ \ M_N = \sum_k^{N-1} \hat{\chi}_k \chi_k}$$

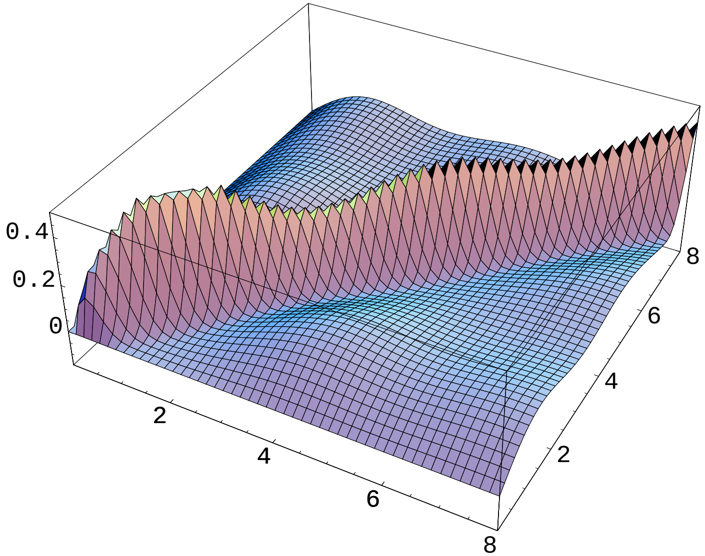
containing **integral trafo** $\chi_k(x) = \int_0^\infty dy\, w(x,y) P_k(y)$ & ditto for Q_k

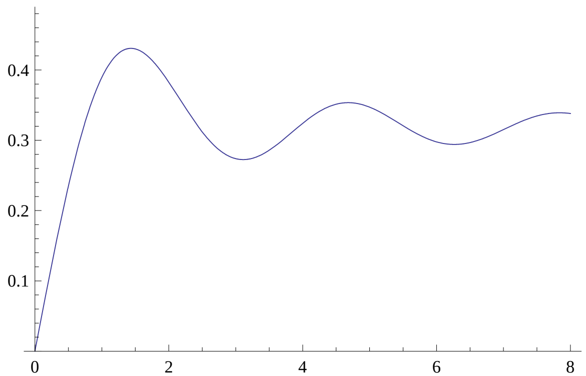
⇒ **all density correlation functions** known explicitly

$$R_{n,k}(\{x\}_n; \{y\}_k) = \det_{1\le i_1,i_2\le n;\ 1\le j_1,j_2\le k} \begin{bmatrix} H_N(x_{i_1},x_{i_2}) & M_N(x_{i_1},y_{j_2}) - w^{(N_1,N_2)}(x_{i_1},y_{j_2}) \\ K_N(y_{j_1},x_{i_2}) & \hat{H}_N(y_{j_1},y_{j_2}) \end{bmatrix}$$

Example Density: 2MM vs. 1MM

example quenched $(N_1 = N_2 = 0)$: $P_k, Q_k \sim L_k^\nu$ and $\chi_k, \hat{\chi}_k \sim \mathrm{e}^{-x^2} L_k^\nu$ **Laguerre**





- **1MM** $N \to \infty$: **Bessel** $\rho_1(x) = \frac{x}{2}(J_\nu(x)^2 - J_{\nu+1}(x) J_{\nu-1}(x))$
- **2MM** $N \to \infty$: **generalised Bessel** $\alpha^2 = 2N\delta\mu^2$

$$\rho_{1,1}(x,y) = \rho_1(x)\rho_1(y) - xy\, \mathcal{K}^+(x,y) \left(\mathcal{K}^-(x,y) - \mathbf{e}^{-(x^2+y^2)/\alpha^2} I_\nu(xy/\alpha^2) \right)$$

$$\boxed{\mathcal{K}^\pm(x,y) \equiv \int_0^1 dt\ \mathbf{e}^{\pm t^2\alpha^2} J_\nu(tx) J_\nu(ty)}$$

Densities – Gaps – Eigenvalues: Expansion

- all densities $\longrightarrow$ all gap prob's $\longrightarrow$ all indiv. ev.

$$E_{k,l}(s,t) = \sum_{i=0}^{N-k} \sum_{j=0}^{N-l} \frac{(-)^{i+j}}{i!j!} \int_0^s dx_1 \dots dx_{k+i} \int_0^t dy_1 \dots dy_{l+j} R_{k+i,l+j}(\{x\}_{k+i}, \{y\}_{l+j})$$

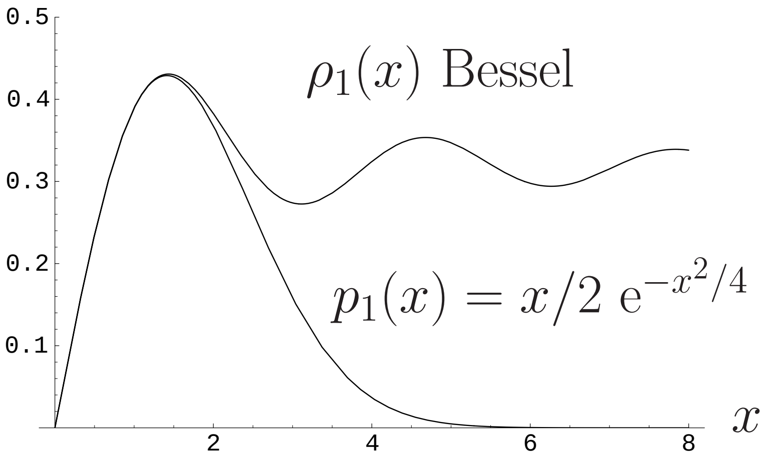
- example $p_{1,1} = \partial\partial E_{0,0}$:

$$p_{1,1}(s,t) \; = \; R_{1,1}(s,t) - \int_0^s dx \, R_{2,1}(x,s,t) - \int_0^t dy \, R_{1,2}(s,t,y) + \dots$$

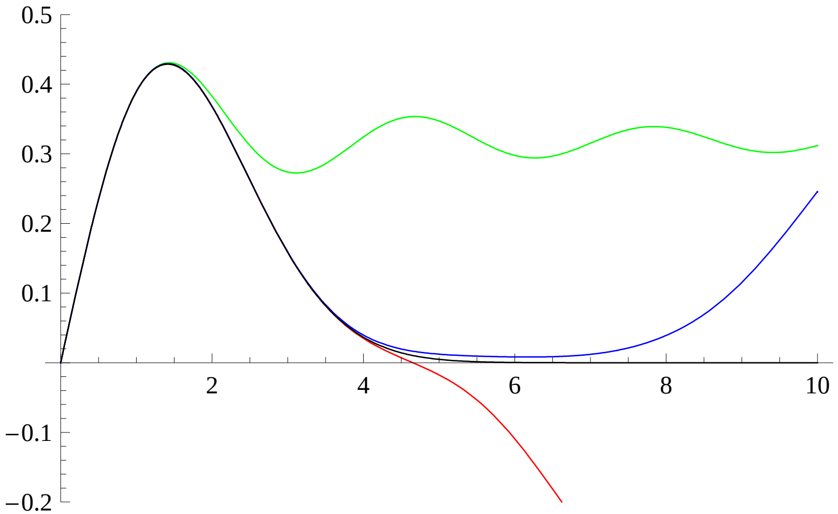
- same expansion for 1MM:
  1st eigenvalue = density – $\int$ **2-pt density +...**

Expansion in 1MM

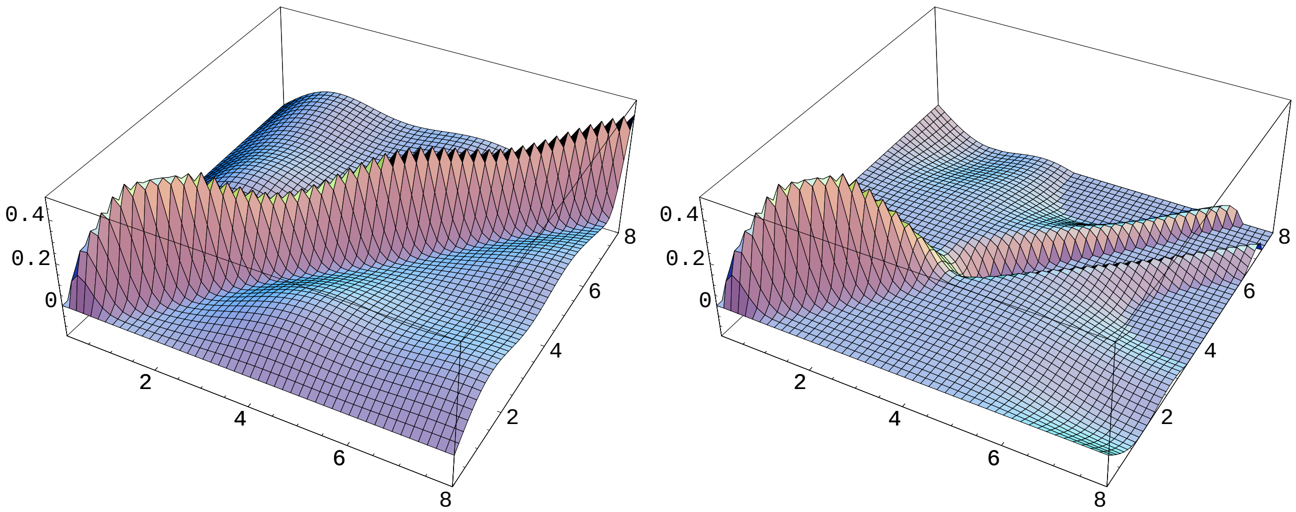
- exact result:



- expansion to 3rd order



Expansion in 2MM



$\rho_{1,1}$ generalised Bessel vs. $p_{1,1}$ expanded to 2nd order

- Can we find an exact result as in the 1MM?

Reminder: Gaps in 1MM

$$E_0(s) = \frac{1}{\mathcal{Z}} \prod_{i=1}^{N} \int_{s^2}^{\infty} dx_i \; x_i^{\nu} \prod_{f=1}^{N_1} (x_i + m_f^2) \; \mathrm{e}^{-Nc_1 x_i} \; \Delta_N(x)^2$$

- after changing variables $x^2 \longrightarrow x$
- Vandermonde $\prod_{i>j}(x_i - x_j)$ **invariant under shift**

$\Rightarrow$ $\boxed{E_0(s) \sim \mathrm{e}^{-s^2} \times \mathbf{Z}}$ **Gap = new Z** with

- shifted masses $m_f^2 \longrightarrow m_f^2 + s^2$,
- ν extra masses s^2
- "effective $\nu = 0$" [Damgaard, Nishigaki,Wettig 98]

Result $\boxed{E_0(s) \sim \mathbf{e}^{-s^2} \det_{f,k}[L_{N+k}(m_f^2 + s^2)]/\Delta(\{m^2 + s^2\})}$
$(\nu = 0)$

- same strategy for p_k: shift jpdf of k-th eigenvalue

[Damgaard, Nishigaki 01]

Exact Results for 2MM

- partial results for objects **totally symmetric in** $\{y\}$
  $\longrightarrow$ can replace $\det\left[I_\nu(x_i y_j)\right]$ by diagonal $\prod_i I_\nu(x_i y_i)$

$\Rightarrow$ **Gaps of x only, k-th individual x-eigenvalues nontrivial** due to presence of N_2 $\not{D}$'s of y's

- **example 1st gap:**

$$E_{0,0}(s, t=0) \sim \prod_{i=1}^{N} \int_s^\infty dx_i \int_0^\infty dy_i \, \mathcal{P}_{jpdf}$$

- Shift easy as for 1MM?

How to do 1st Gap in 2MM

$$E_{0,0}(s,t=0) \sim \int_s^\infty dx\ \ \exp \mathrm{mass1}\ \Delta(x) \int_0^\infty dy\ \ \exp \mathrm{mass2}\ \Delta(y)\, \det I_\nu(xy)$$

1.) $\int_0^\infty dy$ symmetric: **replace** $\det I_\nu(xy) \to \prod I_\nu(xy)$

2.) replace:

$I_\nu(xy) \to \sum_j^\infty a_j L_j(x) L_j(y)$ **and** $\mathrm{mass2}\ \Delta(y) \to \det \begin{bmatrix} L_j(n) \\ L_j(y) \end{bmatrix}$

orthogonality: $\Rightarrow\ \ E_{0,0}(s,t=0) \sim \prod_{i=1}^N \int_s^\infty dx\ \ \exp \mathrm{mass1}\ \Delta(x) \det \begin{bmatrix} L_j(n) \\ a_j L_j(x) \end{bmatrix}$

3.) det identity & shift & 2b.):

$$E_{0,0}(s,t=0) \sim \int_0^\infty dx\ \ \exp\ \ \det_{N+N_1} \begin{bmatrix} L_j(m') \\ L_j(x) \end{bmatrix} \det_{N+N_2} \begin{bmatrix} q_j(n) \\ L_j(x) \end{bmatrix}$$

- compute like average of characteristic polynomials

First Gap in 2MM

$$E_{0,0}(s,0) \sim \frac{\det}{\Delta(m')\Delta(n)} \begin{bmatrix} K(m'_1,n_1)\dots & K(m'_1,n_{N_2}) & L_{N+N_2}(m'^2_1)\dots & L_{N+N_1-1}(m'^2_1) \\ \vdots & \vdots & \vdots & \vdots \\ K(m'_{N_1},n_1)\dots & K(m'_{N_1},n_{N_2}) & L_{N+N_2}(m'^2_{N_1})\dots & L_{N+N_1-1}(m'^2_{N_1}) \end{bmatrix}$$

for $N_1 \geq N_2$

- **shifted mass** $m'^{\,2} = m^2 + s^2$
- **Kernel** $K(m',n) = \sum_{j=0}^{N+N_2-1} L_j(m^2+s^2)q_j(n)$
- **generalised shifted Laguerre** $\boxed{q_j(n) = \sum_{l=0}^{j} \frac{1}{(1-\tau)^l} L_l^{\nu}(n^2) L_{j-l}^{-\nu-1}(s^2)}$

1MM $\lim_{\mu_1\to\mu_2} \tau(\mu_1,\mu_2) = 0$: $\qquad q_j(n) = L_j(n^2+s^2)$

- for $N_2 > N_1$ replace $L_j \longrightarrow q_j$ in right half of det

Averages of Characteristic Polynomials

$$\left\langle \prod_{f=1}^{N_1} \det[i\not{D}_1 + m_f] \prod_{g=1}^{N_2} \det[i\not{D}_2 + n_g] \right\rangle$$ [G.A., Vernizzi 03]

$$= \frac{1}{\mathcal{Z}} \prod_{i=1}^{N} \int dx_i dy_i w(x_i, y_i) \prod_{f=1}^{N_1} (x_i^2 + m_f^2) \prod_{g=1}^{N_2} (y_i^2 + n_g^2) \, \Delta_N(x^2) \Delta_N(y^2)$$

$$= \frac{const.}{\Delta_{N_1}(m^2)\Delta_{N_2}(n^2)} \det \begin{bmatrix} K(m_1, n_1) \ldots & K(m_1, n_{N_2}) & P_{N+N_2}(m_1^2) \ldots & P_{N+N_1-1}(m_1^2) \\ \vdots & \vdots & \vdots & \vdots \\ K(m_{N_1}, n_1) \ldots & K(m_{N_1}, n_{N_2}) & P_{N+N_2}(m_{N_1}^2) \ldots & P_{N+N_1-1}(m_{N_1}^2) \end{bmatrix}$$

for $N_1 \geq N_2$

The k-th x-eigenvalue in 2MM

- definition

$$p_k(x) = \int_0^x dx_1 \int_{x_1}^x dx_2 \ldots \int_{x_{k-2}}^x dx_{k-1}\, \Omega_{jpdf}(x_1, \ldots, x_k)$$

- partial jpdf of the k-th ev

$$\Omega_{jpdf}(x_1, \ldots, x_k) \sim \prod_{i=k+1}^{N} \int_{x_k}^{\infty} dx_i \prod_{j=1}^{N} \int_0^{\infty} dy_j\, \mathcal{P}_{jpdf}$$

- **Result:**

$$\Omega_{jpdf} \sim\sim \prod_{i=k+1}^{N} \int_0^{\infty} dx_i \;\exp\; \det \begin{bmatrix} L_j^1(m') \\ L_j^1(x) \end{bmatrix} \det \begin{bmatrix} q_j^1(n) \\ L_j^1(x) \end{bmatrix}$$

The Large-N Limit: Hard Edge

- rescale Nx_j, Ny_j, Nm, Nn,
  and $N\delta\mu^2$ 'weak non-Hermitecity'
- **building blocks:**

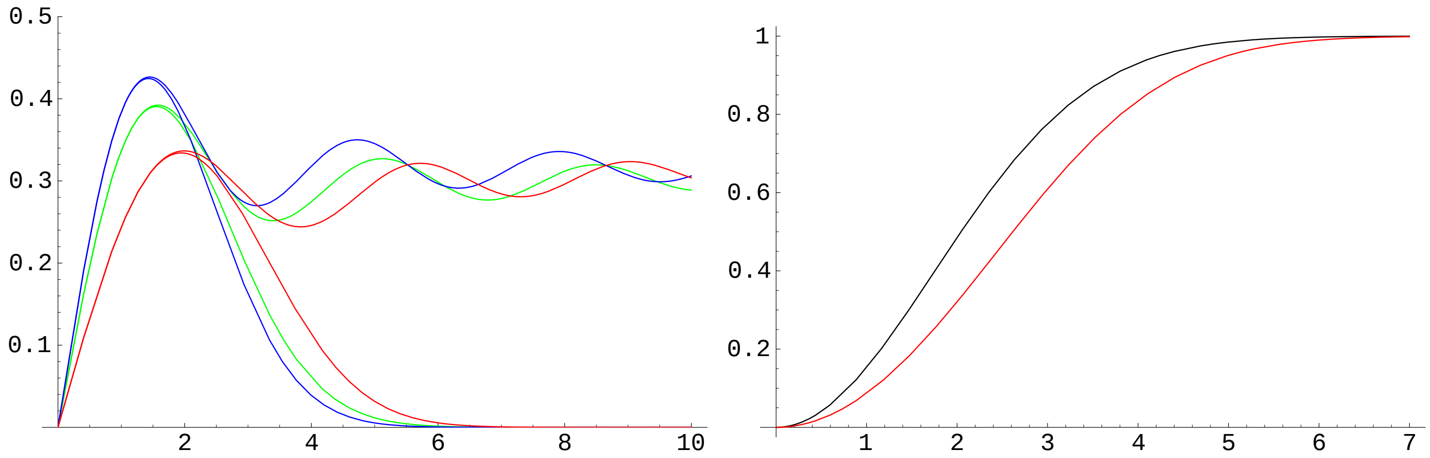
  * $L_N(N(m^2+s^2)) \longrightarrow I_0(\sqrt{\hat{m}^2+\hat{s}^2})$ Laguerre

  * $$q_N(n) \;=\; \sum_{l=0}^{N} \frac{1}{(1-\tau)^l} L_l(n^2) L_{N-l}^{-1}(s^2) \quad (\nu=0)$$
    $$\longrightarrow \frac{1}{2}\int_0^1 dr\ \mathrm{e}^{\frac{1}{2}r\hat{\delta}^2} I_0(\sqrt{r}\ \hat{n}) \frac{1}{\sqrt{1-r}}\ \hat{s} I_1(\sqrt{(1-r)}\ \hat{s}) + \mathrm{e}^{\frac{1}{2}\hat{\delta}^2} I_0(\hat{n})$$

- tricky: $\sum \to \int$: $\quad L_0^{-1}=1$
  (check: $\lim\ \hat{\delta}\to 0$: $\longrightarrow\ I_0(\sqrt{\hat{n}^2+\hat{s}^2})$ Sonine identity)

$\Rightarrow$ Kernel asymptotic $=\sum_j L_j q_j$ & Gap

Examples



- **left:** $\rho_1(x)$ and $p_1(x) \sim \partial q(\hat{n})$ for $N_1 = 0,\ N_2 = 1$
  ($\hat{\delta} \longrightarrow 0$: 1MM with 1 mass, $\hat{\delta} \longrightarrow \infty$: 1MM quenched)
- **right:** gap $1 - E_{0,0}(s, 0)$ comparing
  $N_1 = 0,\ N_2 = 2$ and $N_1 = 1 = N_2$ at fixed mass and $\hat{\delta}$

Remarks

- for $N_2 = 0$ integration over all $\{y\}$:
  reduces 2MM to 1MM

- **Factorisation into 1MM densities & indiv ev.:**
  for $N_1 = N_2 = 0$ and **parameter** μ_j **NOT scaled**
  with N one of the kernels converges to the weight:

$$M(x,y) \to \sum_{j=0}^{N=\infty} \chi_j(x)\hat{\chi}_j(y) = w(x,y)$$

$$\sum_{j=0}^{\infty}(1-\tau)^j L_j(x^2)L_j(y^2) \sim I_0(xy)$$

= **Mehler formula for Hermite or Laguerre**

(compare: for scaled $N\mu^2$: $\boxed{\int_0^1 dt\ \mathrm{e}^{-t^2\alpha^2} J_\nu(tx)J_\nu(ty)}$)

Open Problems

- asymptotic large-N rigorous and **universality proof**

- complete the solution of 2MM:
  mixed individual eigenvalues of x- and y-type

- **2MM with complex eigenvalues:**
  exact individual eigenvalues, partial quenching?