The Airy Process with Past and Future Outliers

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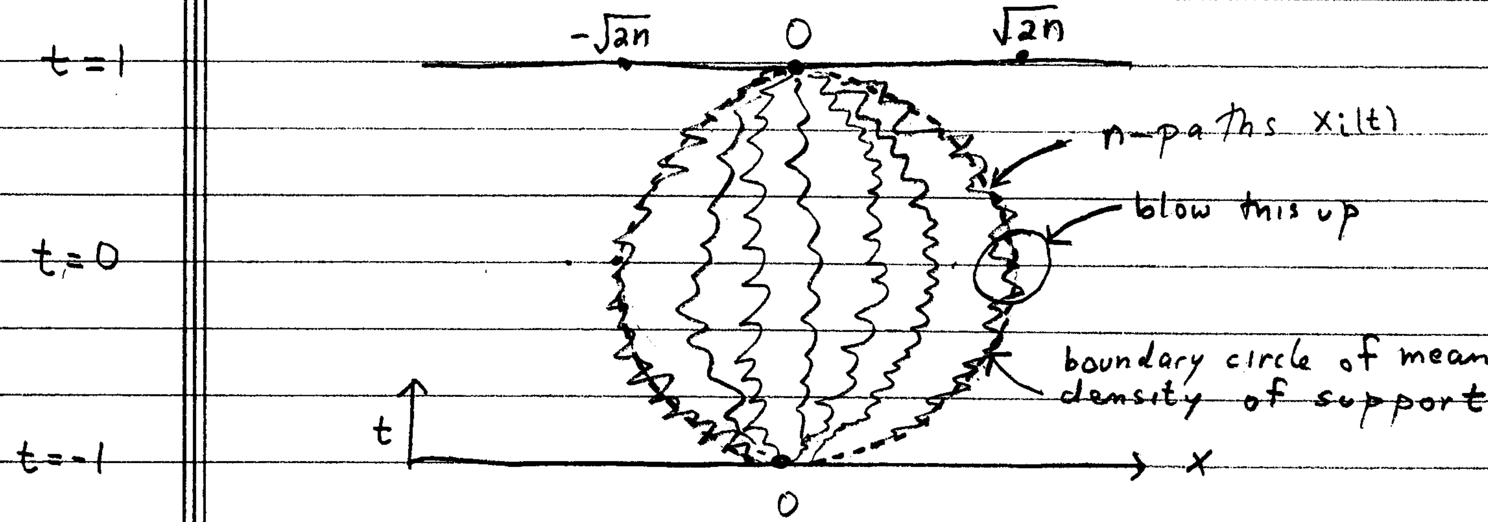
I. Introduction and Results

Airy Process



n- nonintersecting Brownian particles on $\mathbb{R}$

$$-\infty < x_1(t) < \ldots < x_n(t) < \infty$$

Brownian transition probability : $\left\{ p(x,y,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \right\}$

Airy process : $A(\tau)$ is rescaled Brownian motion viewed from right-hand edge of support with a microscope in slowed down time

Prähofer and Spohn '02

Johansson '03

⋮

$$\lim_{n\to\infty} \mathbb{P}_0^{\circ}\left(\text{all } x_i\left(\frac{\tau}{n^{1/3}}\right) \in \sqrt{2n} + \frac{E^c - \tau^2}{\sqrt{2}\, n^{1/6}} \,\middle|\, \text{all } x_i(0) = x_i(1) = 0 \right)$$

$$= \det\left(I - K^{(0)}\right)_{L^2(E)}$$

with

$$K^{(0)}(\xi,\eta) = \frac{1}{(2\pi i)^2} \int d\omega \int d\omega' \, \frac{e^{-\omega^3/3 + \xi\omega + \omega'^3/3 - \eta\omega'}}{\omega - \omega'}$$

The extended-kernel - looking at 2 windows:

$$\mathbb{P}_0^{\circ}\left(\text{all } x_i\left(\tau_j/n^{1/3}\right) \in \sqrt{2n} + \frac{E_j^c - \tau_j^2}{\sqrt{2}\, n^{1/6}},\ j=1,2 \,\middle|\, \text{all } x_i(0) = x_i(1) = 0 \right)$$

$$= \det\left(I - \left(I_{E_i} \bar{K}^{(0)} I_{E_j}\right)_{1\le i,j\le 2}\right)$$

with

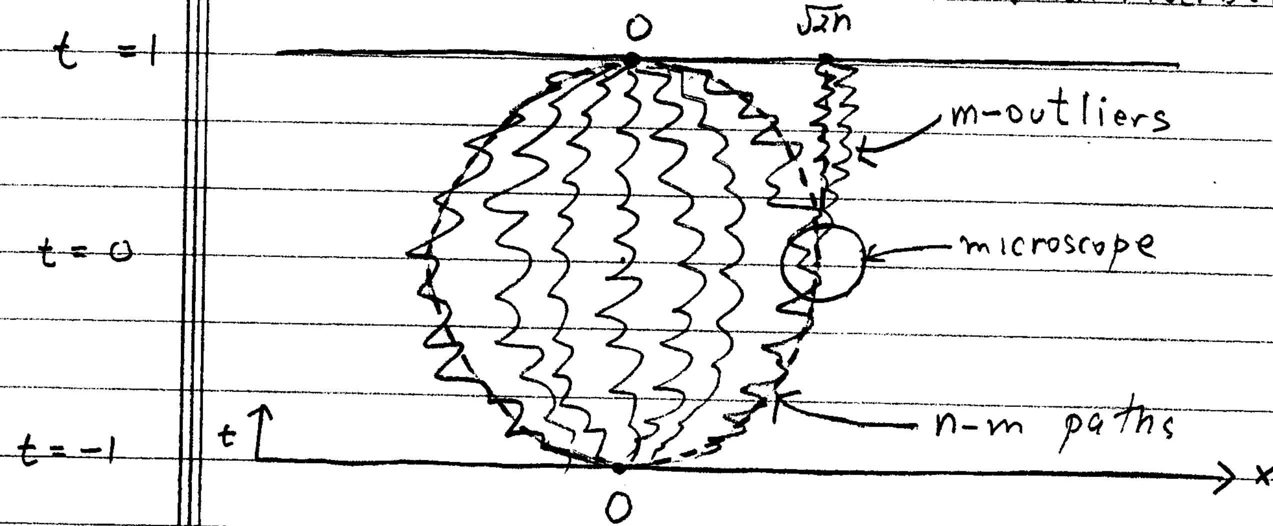
$$\bar{K}^{(0)}(\xi_1,\tau_1;\xi_2,\tau_2) = \frac{1}{(2\pi i)^2} \int d\omega \int d\omega' \, \frac{e^{-\omega^3/3 + \xi_1\omega + \omega'^3/3 - \xi_2\omega'}}{(\omega+\tau_2) - (\omega+\tau_1)}$$

$$- \frac{H(\tau_2-\tau_1)}{\sqrt{4\pi(\tau_2-\tau_1)}} \exp\left(-\frac{(\xi_2-\xi_1)^2}{4(\tau_2-\tau_1)} - \frac{(\tau_2-\tau_1)(\xi_2+\xi_1)}{2} + \frac{(\tau_2-\tau_1)^3}{12} \right)$$

H = Heavyside fcn

Airy Process with m - Outliers

(van Moerbeke + A. '05)



$$\lim_{n\to\infty} P_0^{(0,\sqrt{\frac{n}{2}})}\left(\text{all } x_i\left(\frac{\tau}{n^{1/3}}\right) \in \sqrt{2n} + \frac{E^c - \tau^2}{\sqrt{2}\, n^{1/6}} \;\middle|\; \begin{array}{l} n-m \text{ paths start + end at } 0 \\ m \text{ paths end at } \sqrt{2n} \\ \text{nonintersecting Brownian motion} \end{array} \right)$$

$$= \det\left(I - K_\tau^{(r)}\right)_{L^2(E)}$$

with

$$K_\tau^{(r)}(\xi,\eta) = \frac{1}{(2\pi i)^2} \int d\omega \int_{-\tau} d\omega' \; \frac{e^{-\frac{\omega^3}{3} + \xi\omega + \frac{\omega'^3}{3} - \eta\omega'}}{\omega - \omega'} \left(\frac{\omega + \tau}{\omega' + \tau}\right)^m$$

Suppose m-outliers slight separate $O(n^{1/6})$:

$$\sqrt{2n}\left(1 - \frac{b_i}{n^{1/3}}\right), \quad 1 \le i \le m$$

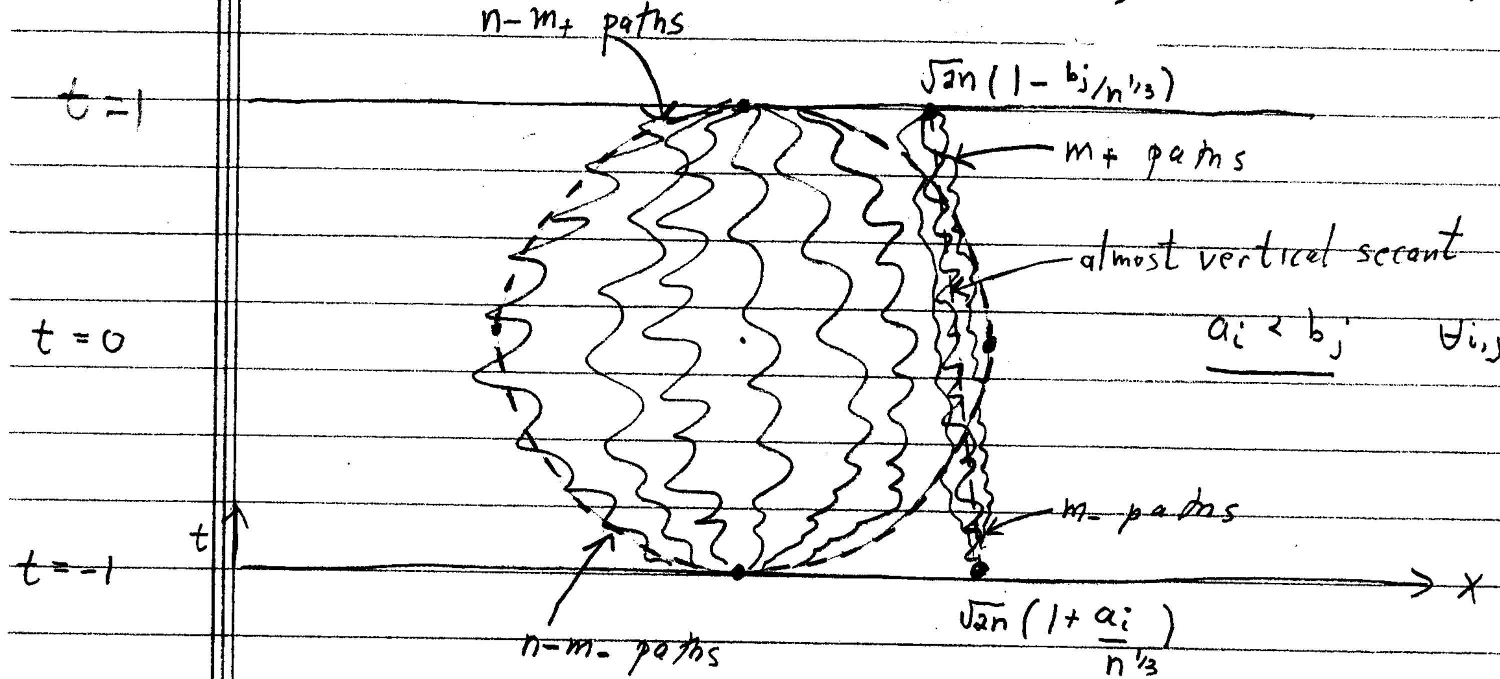
then the corresponding extended kernel is :

$$K^{(m)}(\xi_1,\tau_1;\xi_2,\tau_2) = \frac{1}{(2\pi i)^2}\int dw \int_{b-\tau} dw' \, \frac{e^{-w^3/3 + \xi_1 w + w'^3/3 - \xi_2 w'}}{(w+\tau_2)-(w'+\tau_1)} \prod_1^m \left(\frac{w - b_i + \tau_2}{w' - b_i + \tau_1}\right)$$

$$-\frac{H(\tau_2-\tau_1)}{\sqrt{4\pi(\tau_2-\tau_1)}} \exp\left(-\frac{(\xi_2-\xi_1)^2}{4(\tau_2-\tau_1)} + \frac{(\tau_2+\tau_1)(\xi_2-\xi_1)}{2} - \frac{(\tau_2-\tau_1)(\tau_1+\tau_2)^2}{4}\right)$$

Airy Process with Past and Future Outliers

(Ferrari, van Moerbeke + A, '08)



$\underline{a_i < b_j} \quad \forall i,j$

$$\lim_{n\to\infty} \mathbb{P}^{(0,b)}_{(0,a)}\left(\text{all } x_i\left(\frac{\tau}{n^{1/3}}\right) \in \sqrt{2n} + \frac{E^c - \tau^2}{\sqrt{2}\, n^{1/6}} \;\middle|\; \begin{array}{l} n-m_- \text{ paths start at } 0 \\ n-m_+ \text{ paths end at } 0 \\ m_- \text{ paths start at } \sqrt{2n}\left(1+\frac{a_i}{n^{1/3}}\right) \\ \qquad 1 \le i \le m_- \\ m_+ \text{ paths end at } \sqrt{2n}\left(1-\frac{b_j}{n^{1/3}}\right) \\ \qquad 1 \le j \le m_+ \\ \text{nonintersecting paths} \end{array} \right)$$

$$= \det\left(I - K_\tau^{(m)}(\xi,\eta)\right)_{L^2(E)} \qquad m = (m_-, m_+)$$

with

$$K_\tau^{(m)}(\xi,\eta) = \frac{1}{(2\pi i)^2} \int_{\Gamma_{a-\tau}} dw \int_{\Gamma_{b-\tau}} dw' \, \frac{e^{-\frac{w^3}{3} + \xi w - \frac{w'^3}{3} + \eta w'}}{w - w'} \prod_{i=1}^{m_-}\left(\frac{w' - a_i + \tau}{w - a_i + \tau}\right) \prod_{j=1}^{m_+}\left(\frac{w - b_j + \tau}{w' - b_j + \tau}\right)$$

with extended kernel:

$$K^{(m)}(\xi_1,\tau_1;\xi_2,\tau_2)=\frac{1}{(2\pi i)^2}\int_{a-\tau} d\omega \int_{b-\tau} d\omega' \, \frac{e^{-\omega^3/3+\xi_1\omega+\omega'^3/3-\xi_2\omega'}}{(\omega+\tau_2)-(\omega+\tau_1)} \prod_{i=1}^{m_-}\left(\frac{\omega'-a_i+\tau_1}{\omega-a_i+\tau_2}\right)\prod_{j=1}^{m_+}\left(\frac{\omega-b_j+\tau_2}{\omega'-b_j+\tau_1}\right)$$

$$-\frac{H(\tau_2-\tau_1)}{\sqrt{4\pi(\tau_2-\tau_1)}}\exp\left(-\frac{(\xi_2-\xi_1)^2}{4(\tau_2-\tau_1)}+\frac{(\tau_2+\tau_1)}{2}(\xi_2-\xi_1)-\frac{(\tau_2-\tau_1)}{4}(\tau_1+\tau_2)^2\right)$$

Remark 1: Borodin + Pêche ('08) derived this kernel as a scaling limit of correlation kernels related to the directed percolation model in a quadrant.

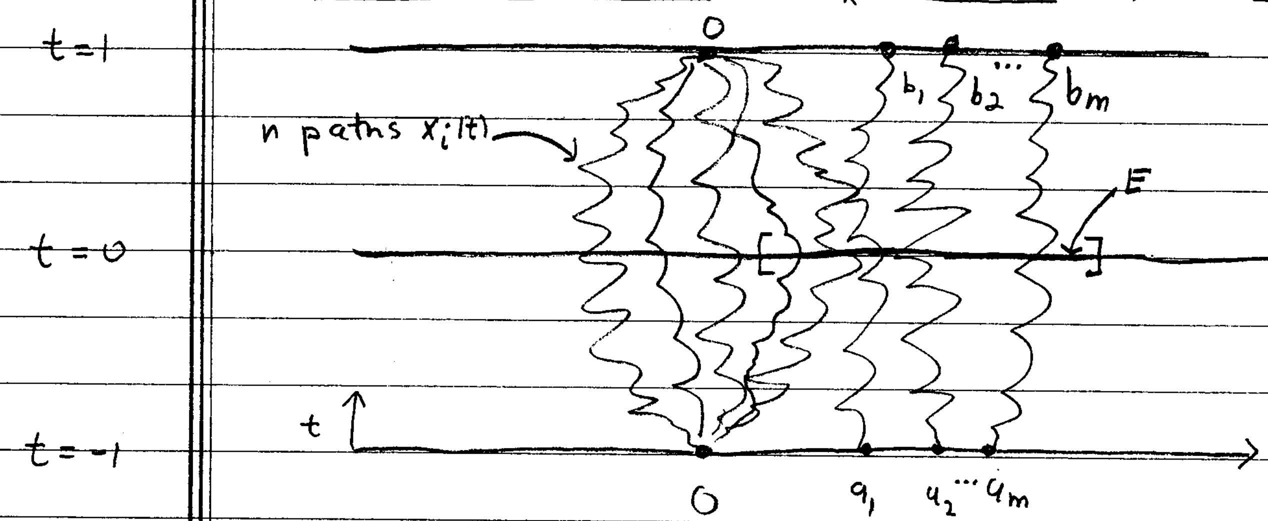
Remark 2: Noteworthy case: $m_-=m_+=m$, $a_i\equiv a$, $b_i\equiv b$.

The we have $\underline{n-m}$ Brownian motions going from 0 to 0, $\underline{m}$ going from $\sqrt{2n}(1+a/n^{1/3})$ to $\sqrt{2n}\,(1-b/n^{1/3})$

$$\sqrt{2n}(1+a/n^{1/3}) = \sqrt{2n}+\sqrt{2}\,a\,n^{1/6} \qquad \sqrt{2n}(1-b/n^{1/3}) = \sqrt{2n}-\sqrt{2}\,b\,n^{1/6}$$

Remark 3: Why is the deviation of the outliers from $\sqrt{2n}$, $O(n^{1/6})$ rather than just $O(1)$? In the $O(1)$ case, the Brownian outliers would behave as nonintersecting Brownian bridges conditioned to stay above the deterministic shape and we would just see the Airy process, while in the $O(n^{1/6})$ case (if $a<b$), the Brownian bridges are forced for a sufficiently large time to interact with the "bulk" of the $n-m$ bridges from 0 to 0 and so we see a new process.

II Sketch of Proof: a_k - all distinct, b_ℓ - all distinct



Simplicity:
$m_- = m_+ = m$
$t = 0$

$n+m$ - noninter. Br. motion

$$\mathbb{P}\left(\text{all } x_i(0)\in E \;\middle|\; \begin{array}{l} n \text{ paths go from } 0 \text{ to } 0 \\ m \text{ paths go from } a_k \text{ to } b_\ell \quad 1\le k,\ell\le m \\ \text{noninter. Br. motion}\end{array}\right)$$

$$= \lim_{\substack{a_{m+1},\dots,a_{m+n}\to 0\\ b_{m+1},\dots,b_{m+n}\to 0}} \frac{1}{Z_{n+m}} \int_{E^{n+m}} \det\big(p(a_i,x_j,1)\big)_{1\le i,j\le n+m} \det\big(p(x_j,b_i,1)\big)_{1\le i,j\le m+n} \prod dx_j$$

$$= \frac{1}{Z'_{n,m}} \int_{E^{n+m}} \det\begin{pmatrix} \left(e^{a_i x_j} e^{-x_j^2/2}\right)_{\substack{1\le i\le m\\ 1\le j\le m+n}} \\ \left(x_j^{i-1} e^{-x_j^2/2}\right)_{\substack{1\le i\le n\\ 1\le j\le m+n}} \end{pmatrix} \cdot \det(a\leftrightarrow b) \prod_j dx_j$$

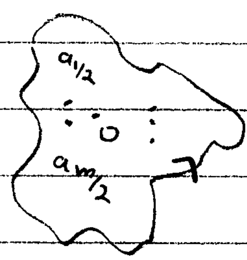
(L'Hôpital)

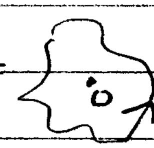
$$= \frac{1}{Z'_{n,m}} \int_{E^{n+m}} \det\left(\varphi_i^{(n)}(x_j,a)\right)_{1\le i,j\le m+n} \cdot \det(a\leftrightarrow b) \prod_j dx_j$$

(row operations)

$$\varphi_k^{(n)}(x,a) = \begin{cases} \varphi^{(n)}(x,a_k) = \dfrac{e^{-x^2/2}}{2\pi i}\displaystyle\oint_{\Gamma_{0,\frac{a}{2}}} dz\, \frac{e^{-z^2+2xz}}{z^n(z-a_k/2)} & 1 \le k \le m \\[2ex] \dfrac{H_{k-m-1}(x)\, e^{-x^2/2}}{c_{k-m-1}} = \dfrac{(k-m-1)!}{c_{k-m-1}} \dfrac{e^{-x^2/2}}{2\pi i} \displaystyle\oint_{\Gamma_0} dz\, \frac{e^{-z^2+2xz}}{z^{k-m}} & m+1 \le k \le m+n \end{cases}$$

$$e^{-z^2+2xz} = \sum_0^\infty \frac{z^j}{j!} H_j(x), \qquad \int_{\mathbb{R}} e^{-x^2} dx\, \frac{H_k(x)}{c_k} \frac{H_\ell(x)}{c_\ell} = \delta_{k\ell}$$

$\Gamma_{0,\frac{a}{2}} =$ 

$\Gamma_0 =$ 

$\Longrightarrow$

1) $$\mu := \left(\mu_{k\ell} = \int_{\mathbb{R}} \varphi_k^{(n)}(x,a)\, \varphi_\ell^{(n)}(x,b)\, dx \right)_{1 \le k,\ell \le m} = \begin{pmatrix} \bar{\mu} & 0 \\ 0 & I_n \end{pmatrix}$$

with

$$\bar{\mu}_{k\ell} = \frac{2^n \sqrt{\pi}}{2\pi i} \oint_{\Gamma_{0,\frac{a_k b_\ell}{2}}} \frac{dz\, e^z}{z^n(z - a_k b_\ell/2)}$$

2) $$P(\text{all } x_i(0) \in E) = \det(I - K_{n,m})_{L^2(E^c)}$$

with

$$K_{n,m}(x,y) = K_n^{\text{Hermite}}(x,y) + \sum_{i,j=1}^m \varphi^{(n)}(y,b_i)(\bar{\mu}^{-1})_{ij}\, \varphi^{(n)}(x,a_j)$$

Main Ingredients:

$$M_{k+1,\ell+1} = \frac{2^{n+k+\ell}}{k!\,\ell!} \left(\frac{\partial}{\partial a}\right)^k \left(\frac{\partial}{\partial b}\right)^\ell \frac{\sqrt{\pi}}{2\pi i} \oint_{\Gamma_{0,\frac{ab}{2}}} \frac{dz\, e^z}{z^n (z - ab/2)},$$

and saddlepoint analysis.

Identity:

$$-\det \begin{pmatrix} 0, & 1, & 1+x, & (1+x)^2, & \ldots, & (1+x)^{m-1} \\ 1 & & & & & \\ 1+y & & \binom{k+\ell}{k}_{0 \le k,\ell \le m-1} & & & \\ (1+y)^2 & & & & & \\ \vdots & & & & & \\ (1+y)^{m-1} & & & & & \end{pmatrix} = \frac{(xy)^m - 1}{(xy) - 1}$$

altogether yield kernel formula.

Remark: Limiting bi-orthogonal basis is:

$$\sqrt{\tilde{b}-\tilde{a}} \oint_{\tilde{a}} \frac{dw}{2\pi i}\, e^{-w^3/3 + \xi w}\, \frac{(w-\tilde{b})^{j-1}}{(w-\tilde{a})^j}$$

$$1 \le j \le m$$

$$\sqrt{\tilde{b}-\tilde{a}} \oint_{\tilde{b}} \frac{dw'}{2\pi i}\, e^{w'^3/3 - \eta w'}\, \frac{(w'-\tilde{a})^{j-1}}{(w'-\tilde{b})^j}$$

Asymptotics : (Saddlepoint Analysis)

Consider the scaling for large n:

$$a_i = \sqrt{2n}\left(1 + \frac{\tilde{a}_i}{n^{1/3}}\right), \quad b_i = \sqrt{2n}\left(1 - \frac{\tilde{b}_i}{n^{1/3}}\right)$$

all $\tilde{a}_i < \tilde{b}_j$, $1 \le i, j \le m$

$$X = \sqrt{2n} + \xi / \sqrt{2}\, n^{1/6}$$

$\Rightarrow$

(1) $$\bar{\mu}^{-1} \simeq -\sqrt{2}\, n^{1/6} \left(\frac{n}{2e}\right)^n A^{-1}$$

$$A = \left(\frac{1}{\tilde{a}_k - \tilde{b}_\ell}\right)_{1 \le k, \ell \le m}$$

(2) $$\varphi^{(n)}(x, a_k) \simeq \left(\frac{2e}{n}\right)^{n/2} \frac{1}{2\pi i} \oint_{\tilde{a}} dz\, \frac{e^{-z^3/3 + \xi z}}{z - \tilde{a}_k}$$

$$\left(\frac{2e}{n}\right)^{n/2} \frac{1}{2\pi i} \oint_{\tilde{b}} dz\, \frac{e^{z^3/3 - \xi z}}{z - \tilde{b}_k}$$

and

Identity:

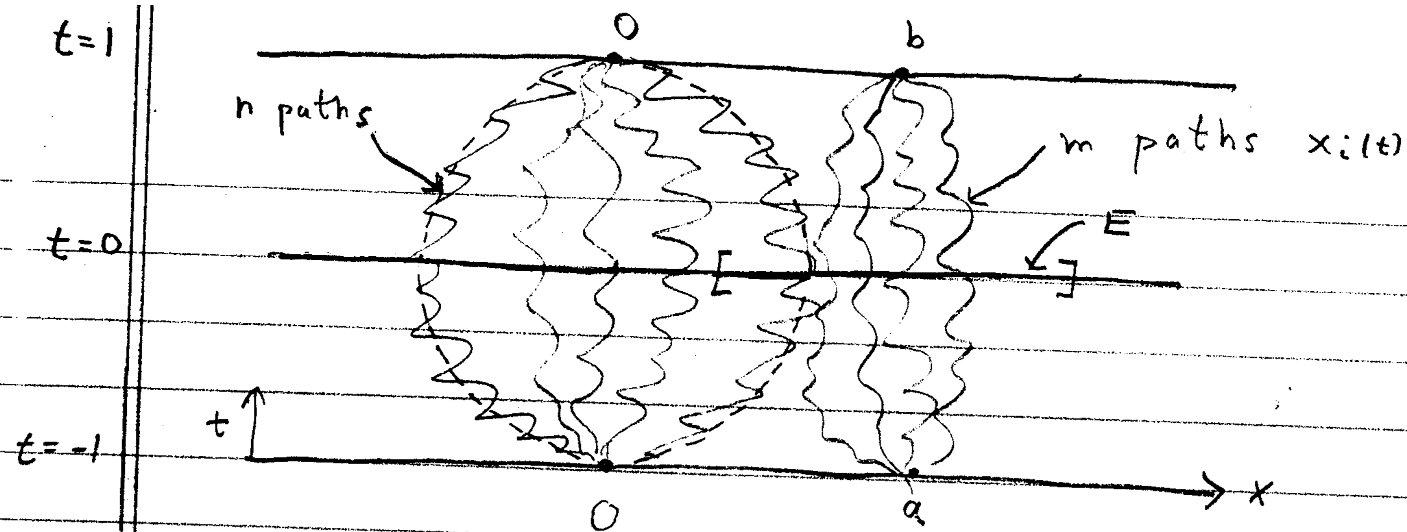
$$\sum_{1 \le i, j \le m} \frac{(A^{-1})_{ji}}{(z - \tilde{a}_i)(w - \tilde{b}_j)} = \frac{1}{w - z}\left(\prod_1^m \left(\frac{w - \tilde{a}_k}{z - \tilde{a}_k}\right)\left(\frac{z - \tilde{b}_k}{w - \tilde{b}_k}\right) - 1\right)$$

yields result for $m = m_- = m_+$ case, $t = 0$.
For $m_- \neq m_+$, let some $\tilde{a}_i \to -\infty$, $\tilde{b}_i \to +\infty$.

III. Sketch of Proof when all $a_i = a$, all $b_i = b$

Although the Airy Process with Past and Future Outliers has Kernel which makes sense when all $a_i = a$, all $b_i = b$, the proof makes no sense and while it is intuitively obvious that it must represent n Brownian motions going from 0 to 0, m going from a to b ($m_- = m_+ = m$) with $n \to \infty$ under the previous scaling, we shall give a separate an honest proof that this is indeed the case; moreover the proof in this case is quite illuminating and of independent interest. Remember, in this case, the kernel is:

$$K_\tau^{(m)}(\xi,\eta) = \frac{1}{(2\pi i)^2} \int_{a-\tau} dw \int_{b-\tau} dw' \frac{e^{-w^3_{/3} + \xi w + w'^3_{/3} + \eta w'}}{w - w'} \left(\frac{w' - a + \tau}{w - a + \tau}\right)^m \left(\frac{w - b + \tau}{w' - b + \tau}\right)^m .$$



$$\mathbb{P}\left(\text{all } x_i(0) \in E \;\middle|\; \begin{array}{l} n \text{ paths go from } 0 \text{ to } 0 \\ m \text{ go from } a \text{ to } b \\ \text{nonintersecting Brownian motion} \end{array} \right)$$

$$= \frac{1}{Z_{n,m}} \int_{E^{n+m}} \det\left(\varphi_i^{(n)}(x_j, a) \right)_{1 \le i,j \le n+m} \cdot (a \leftrightarrow b) \prod dx_j$$

$$\varphi_k^{(n)}(x,a) = \begin{cases} \dfrac{e^{-x^2/2}}{2\pi i} \displaystyle\oint_{\Gamma_{0,a/2}} dz \, \frac{e^{-z^2+2xz}}{z^n (z - a/2)^k}, & 1 \le k \le m \\[2ex] \dfrac{(k-m-1)!}{c_{k-m-1}} \dfrac{e^{-x^2/2}}{2\pi i} \displaystyle\oint_{\Gamma_0} dz \, \frac{e^{-z^2+2xz}}{z^{k-m}}, & m+1 \le k \le m+n \end{cases}$$

Compute bi-orthogonal basis:

$$\text{span}\left\{ \varphi_k^{(n)}(x,a) \right\} \longmapsto \text{span}\left\{ \tilde{\varphi}_k^{(n)}(x,a) \right\}$$

$$\text{span}\left\{ \varphi_k^{(n)}(x,b) \right\} \longmapsto \text{span}\left\{ \tilde{\varphi}_k^{(n)}(x,b) \right\}$$

Bi-orth

$\Rightarrow$

$$\tilde{\mathcal{M}} := \left(\tilde{\mu}_{ij} = \int_{\mathbb{R}} \tilde{\varphi}_i^{(n)}(x,a)\, \tilde{\varphi}_j^{(n)}(x,b)\, dx \right) = I_{n+m}$$

Since as before $\mathcal{M} = \begin{pmatrix} \mu_m & 0 \\ 0 & I_n \end{pmatrix}$,

we are reduced to solving a "classical bi-orthogonal polynomial" problem. Compute monic polynomials :

$$\langle P_i^{(1)}, P_j^{(2)} \rangle_n = h_i \delta_{ij} \qquad , \quad 0 \le i, j \le m-1$$

where

$$\langle f, g \rangle_n = \sqrt{\pi} \oint_{\Gamma_{0,a/2}} \frac{dz}{2\pi i\, z^n (z - a/2)} f\left(\frac{1}{z - a/2}\right) \oint_{\Gamma_{0,b/2}} \frac{dw\, e^{zw}}{2\pi i\, w^n (w - b/2)} g\left(\frac{1}{w - b/2}\right)$$

Then

$$\tilde{\varphi}_k^{(n)}(x,a) = \oint_{\Gamma_{0,a/2}} \frac{dz\, e^{-z^2 + 2xz}}{2\pi i\, z^n (z - a/2)} P_{k-1}^{(1)}\left(\frac{1}{z - a/2}\right)$$

$1 \le k \le m$

$$\tilde{\varphi}_k^{(n)}(x,b) = \oint_{\Gamma_{0,b/2}} \frac{dz\, e^{-z^2 + 2xz}}{2\pi i\, z^n (z - b/2)} P_{k-1}^{(2)}\left(\frac{1}{z - b/2}\right)$$

and

$$\mathbb{P}(\text{all } x_i(0) \in E) = \det(I - K_{n,m})_{L^2(E^c)}$$

with

$$K_{n,m}(x,y) = K_n^{\text{Hermite}}(x,y) + K_m(x,y)$$

By classical kernel formula:

$$K_m(x,y) = -\frac{1}{\Delta_m} \det \begin{pmatrix} 0 & \varphi_1^{(n)}(y,b), & \varphi_2^{(n)}(y,b), & \dots & , \varphi_m^{(n)}(y,b) \\ \varphi_1^{(n)}(x,a) & \mu_{11}, & \mu_{12}, & \dots & , \mu_{1m} \\ \varphi_2^{(n)}(x,a) & \mu_{21}, & \mu_{22}, & \dots & , \mu_{2m} \\ \vdots & \vdots & & & \vdots \\ \varphi_m^{(n)}(x,a) & \mu_{m1}, & \mu_{m2} & \dots & \mu_{mm} \end{pmatrix}$$

with

$$\varphi_k^{(n)}(x,a) = e^{-x^2/2} \oint_{\Gamma_{0,a/2}} \frac{dz\, e^{-z^2+2xz}}{z^n (z-a/2)^k}, \qquad \varphi_k^{(n)}(x,b) = \dots$$

and

$$\mu_{ij} = \langle u^{i-1}, v^{j-1} \rangle_n, \qquad \Delta_m = \det(\mu_{ij})_{1 \le i,j \le m}$$

Asymptotics:

Scaling: $a = \sqrt{2n}\left(1 + \frac{\tilde{a}}{n^{1/3}}\right)$, $b = \sqrt{2n}\left(1 - \frac{\tilde{b}}{n^{1/3}}\right)$, $\tilde{a} < \tilde{b}$

$x = \sqrt{2n} + \xi/\sqrt{2}\, n^{1/6}$, $n \to \infty$

$$\left\{ \begin{aligned} \mu_{k\ell} &\simeq \frac{1}{2}\left(\frac{2e}{n}\right)^n \left(\frac{\sqrt{2}}{(\tilde{b}-\tilde{a})\, n^{1/6}}\right)^{\ell+k-1} \binom{\ell+k-2}{k-1} \\ \Delta_m &\simeq \left(\frac{1}{2}\left(\frac{2e}{n}\right)^n\right)^m \left(\frac{\sqrt{2}}{(\tilde{b}-\tilde{a})\, n^{1/6}}\right)^{m^2} \end{aligned} \right.$$

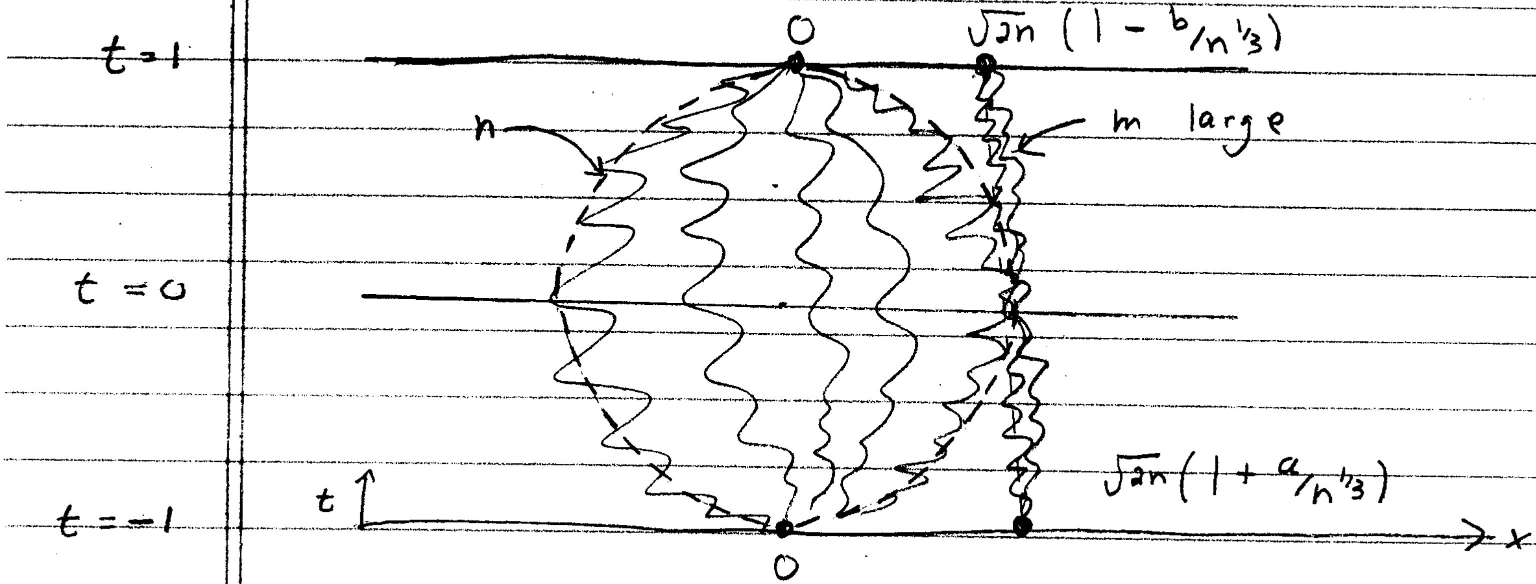
$$\left\{ \begin{aligned} \varphi_k^{(n)}(x,a) &\simeq \left(\frac{2e}{n}\right)^{n/2} \left(\frac{\sqrt{2}}{n^{1/6}}\right)^{k-1} \frac{1}{2\pi i} \oint_{\tilde{a}} \frac{dz}{(z-\tilde{a})^k}\, e^{-z^3/3 + \xi z} \\ \varphi_k^{(n)}(x,b) &\simeq \left(\frac{2e}{n}\right)^{n/2} \left(-\frac{\sqrt{2}}{n^{1/6}}\right)^{k-1} \frac{1}{2\pi i} \oint_{\tilde{b}} \frac{dz}{(z-\tilde{b})^k}\, e^{z^3/3 - \xi z} \end{aligned} \right.$$

IV Limit to the Pearcey Kernel

Consider case:

$$K_\tau(x,y) = \frac{1}{(2\pi i)^2} \int_{\tilde a} dU \int_{\tilde b} dV \, \frac{e^{-\frac{U^3}{3} + xU + \frac{V^3}{3} - yV}}{U-V} \left(\frac{U-\tilde b}{U-\tilde a}\right)^m \left(\frac{V-\tilde a}{V-\tilde b}\right)^m$$

$(a-\tau = \tilde a,\ b-\tau = \tilde b)$



as $m \to \infty$ you might expect to see Pearcey distri. as m-outliers exert more pressure on bulk??

Rescale in $K_\tau(x,y)$:

$$\left\{\begin{aligned} \tilde a &= A m^{1/3} + \tau C m^{-1/6} \\ \tilde b &= B m^{1/3} + \tau C m^{-1/6} \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= X_1 m^{2/3} + \tau X_2 m^{-1/6} + \xi X_3 m^{-\frac{1}{12}} \\ y &= X_1 m^{2/3} + \tau X_2 m^{-1/6} + \eta X_3 m^{-\frac{1}{12}} \end{aligned}\right\}$$

Parameters $A, B, \ldots, X_3$ parametrize alg. curve: $\boxed{\beta^6 = \frac{2\alpha - 3}{4\alpha^4}}$

Parameters

$$\left\{\begin{array}{ll} A = \dfrac{2\beta(1+\alpha\beta)(1-\alpha\beta+\alpha^2\beta^2)}{(\alpha-2)}, & X_1 = \beta^2(1-2\alpha) \\[2ex] B = \dfrac{2\beta(1-\alpha\beta)(1+\alpha\beta+\alpha^2\beta^2)}{(\alpha-2)}, & X_2 = \dfrac{1}{\alpha}\left(\dfrac{2(\alpha-1)}{\beta}\right)^{1/2} \\[2ex] C = -\dfrac{1}{2\alpha}\sqrt{\dfrac{2(\alpha-1)}{\beta}}, & X_3 = -\left(\dfrac{2(\alpha-1)}{\beta\alpha^2}\right)^{1/4} \end{array}\right\}$$

with

$$\beta > 0, \qquad 2 > \alpha > 3/2$$

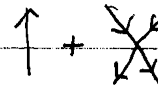
$\Rightarrow \tilde{a} < \tilde{b}$, para. real

$$\lim_{m\to\infty} K_\tau(x,y)\,dy \Big|_{\substack{x=\ldots,\ y=\ldots \\ a=\ldots,\ b=\ldots}} = K_\tau^{\text{Pearcey}}(\xi,\eta)$$

(saddle-point)

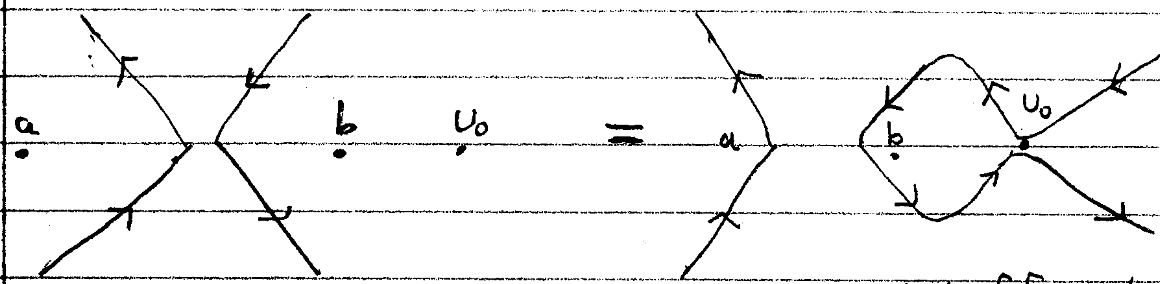
$$K_\tau^{\text{Pearcey}}(\xi,\eta) = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} dw \int dw' \, \frac{e^{-\frac{w^4}{4}+\frac{\tau}{2}w^2-\xi w+\frac{w'^4}{4}-\frac{\tau}{2}w'^2+\eta w'}}{w-w'}$$

(etc. for extended kernel)

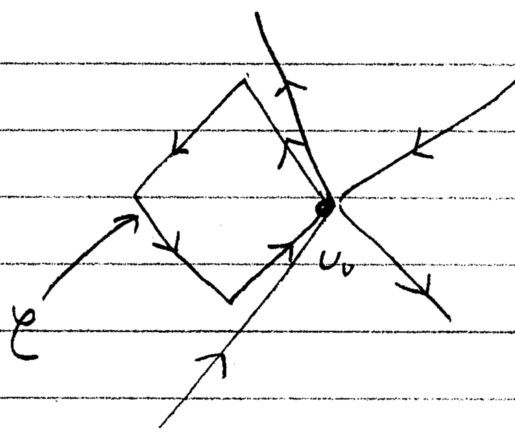
Remark: It is curious how initial contours  get deformed to Pearcey contours .

Due to saddle-point $v_0 \ni$ $\underline{v_0 > \tilde{b} > \tilde{a}}$ <u>instead of being</u> <u>**in** between $\tilde{a}, \tilde{b}$</u>, requiring deform.:

<u>Contour Deform</u>:



$+\left(\begin{array}{l}\text{stuff at } \infty \\ \text{not contributing}\end{array}\right)$

$=$ 

+ closed loop $\mathcal{C}'$ about $\mathcal{C}$ not contributing as residue contribution from $\mathcal{C}'$ is analytic function which dies when integrated about $\mathcal{C}$

+ (stuff at ∞ not contri)

<u>Contour near v_0 yields Pearcey</u>