MIR, Mixing and Network Dual MIPs: a (progress?) report

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Outline

Mixing Sets

The mixing set $X^M(s, z, b)$ consists of

$$s + z_l \geq b_l \text{ for } l = 1, \ldots, n$$
$$s \in \mathbb{R}^1_+, \; z \in \mathbb{Z}^n_+.$$

Let $f_l = b_l - \lfloor b_l \rfloor$ for all l.
A tight extended formulation for $\mathrm{conv}(X^M(s, z, b))$ is:

$$s = \sum_{i=1}^n f_i \delta_i + \mu$$
$$z_t + \mu + \sum_{\{i: f_i \geq f_t\}} \delta_i \geq \lfloor b_t \rfloor + 1 \text{ for } t = 1, \ldots, n$$
$$\sum_{i=0}^n \delta_i = 1$$
$$\delta \in \mathbb{R}^{n+1}_+, \mu \in \mathbb{R}^1_+, z \in \mathbb{R}^n_+.$$

A Network Dual Extended Formulation

Suppose wlog $f_1 \geq f_2 \cdots \geq f_n$. Let $\mu_0 = \mu$ and $\mu_t = \mu + \sum_{i:f_i \geq f_t} \delta_i$ giving:

$$
\begin{gathered}
s = \sum_{i=0}^{n} (f_i - f_{i+1}) \mu_i \\
z_t + \mu_t \geq \lfloor b_t \rfloor + 1 \text{ for } t = 1, \dots, n \\
-\mu_{j-1} + \mu_j \geq 0 \text{ for } j = 1, \dots, n \\
\mu_0 - \mu_n \geq -1 \\
\mu \in \mathbb{R}^{n+1}, \mu_0 \geq 0, z \in \mathbb{R}^n_+.
\end{gathered}
$$

This is a network dual matrix, and the extreme points are obviously integer.

Mixing Inequalities

Let $T \subseteq \{1,\ldots,K\}$ with $|T| = t$, and suppose that $i_1,\ldots i_t$ is an ordering of T such that $0 = f_{i_0} \le f_{i_1} \le f_{i_2} \le \ldots \le f_{i_t} < 1$. Then the mixing inequalities

$$s \ge \sum_{\tau=1}^{t}(f_{i_\tau} - f_{i_{\tau-1}})(\lfloor b_{i_\tau} \rfloor + 1 - y_{i_\tau})$$

and

$$s \ge \sum_{\tau=1}^{t}(f_{i_\tau} - f_{i_{\tau-1}})(\lfloor b_{i_\tau} \rfloor + 1 - y_{i_\tau}) + (1 - f_{i_t})(\lfloor b_{i_1} \rfloor - y_{i_1})$$

are valid for X_K^{MIX}.

Example of Mixing Inequalities

Consider the set

$$X = \{(s,y) \in \mathbb{R}^1_+ \times \mathbb{Z}^3 : s + y_1 \geq 1.4, s + y_2 \geq 2.6, s + y_3 \geq 0.7\}.$$

With $|T| = 1$,

$$s \geq 0.4(2 - y_1), s \geq 0.6(3 - y_2), s \geq 0.7(1 - y_3)\ ,$$

then the type 1 mixing inequalities with $|T| > 1$

$$\begin{array}{lllllll} s & \geq & 0.4(2-y_1) & + & (0.6-0.4)(3-y_2) & & \\ s & \geq & 0.4(2-y_1) & & & + & (0.7-0.4)(1-y_3) \\ s & \geq & & & 0.6(3-y_2) & + & (0.7-0.6)(1-y_3) \\ s & \geq & 0.4(2-y_1) & + & (0.6-0.4)(3-y_2) & + & (0.7-0.6)(1-y_3) \end{array}$$

and the type 2 mixing inequalities with $|T| > 1$ have an additional term.

Mixing Sets and Lot-sizing

Discrete Lot-Sizing with Constant Capacity and Initial Stock Variable

$$s_0 + \sum_{u=1}^{t} Cy_u \geq \sum_{u=1}^{t} d_u \quad t = 1, \dots, n$$
$$s_0 \in \mathbb{R}^1_+, y \in \{0,1\}^n_+$$

$$s_0/C + z_t \geq d_{1t}/C \equiv \sum_{u=1}^{t} d_u/C \; t = 1, \dots, n$$
$$s_0 \in \mathbb{R}^1_+, z \in \mathbb{Z}^n$$
$$0 \leq z_t - z_{t-1} \leq 1 \; \forall \; t$$

Lot-Sizing and Wagner-Whitin Costs

$$\min \textstyle\sum_t (p'_t x_t + h'_t s_t + q_t y_t)$$
$$s_{t-1} + x_t = d_t + s_t \ \forall\ t$$
$$x_t \leq C y_t \ \forall\ t$$
$$s, x \geq 0, y \in \{0,1\}^n$$

Wagner-Whitin Costs: $h_t = p'_t + h'_t - p'_{t+1} \geq 0 \ \forall\ t$

$$\min \textstyle\sum_t (h_t s_t + q_t y_t) + constant$$
$$s_{t-1} + C \textstyle\sum_{u=t}^{l} y_u \geq d_{tl} \ \forall\ 1 \leq t \leq l \leq n$$
$$s \geq 0, y \in \{0,1\}^n$$

Lot-Sizing and Wagner-Whitin Costs: Computation

The convex hull of the Wagner-Whitin relaxation is the intersection of the convex hulls of n discrete lot-sizing (mixing) sets.

Even if the costs are not Wagner-Whitin, the Wagner-Whitin relaxation typically closes almost 100% of the gap.

Mixing Set Extensions 1

Continuous Mixing set

$$s + r_t + z_t \geq b_t \quad t = 1, \ldots, n$$
$$s \in \mathbb{R}^1_+, r \in \mathbb{R}^n_+, z \in \mathbb{Z}^n$$

Mixing Set With Flows

$$s + x_t \geq b_t \quad t = 1, \ldots, n$$
$$x_t \leq z_t \quad t = 1, \ldots, n$$
$$s \in \mathbb{R}^1_+, r \in \mathbb{R}^n_+, z \in \mathbb{Z}^n$$

Mixing Set Extensions 2

Mixing set with Stock VUB

$$s + z_t \geq b_t \quad t = 1, \ldots, n$$
$$s \leq uw$$
$$s \in \mathbb{R}^1_+, w \in \{0, 1\}, z \in \mathbb{Z}^n$$

Divisible Mixing Set

$$s + C_t z_t \geq b_t \quad t = 1, \ldots, n$$
$$s \in \mathbb{R}^1_+, z \in \mathbb{Z}^n$$

where $C_1 | \cdots | C_n$.

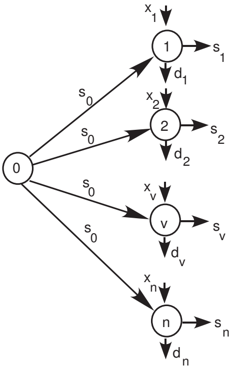
Mixing Set with Flows

$$s + x_t \geq b_t \quad t = 1, \ldots, n$$
$$x_t \leq z_t \quad t = 1, \ldots, n$$
$$s \in \mathbb{R}^1_+, x \in \mathbb{R}^n_+, z \in \mathbb{Z}^n$$

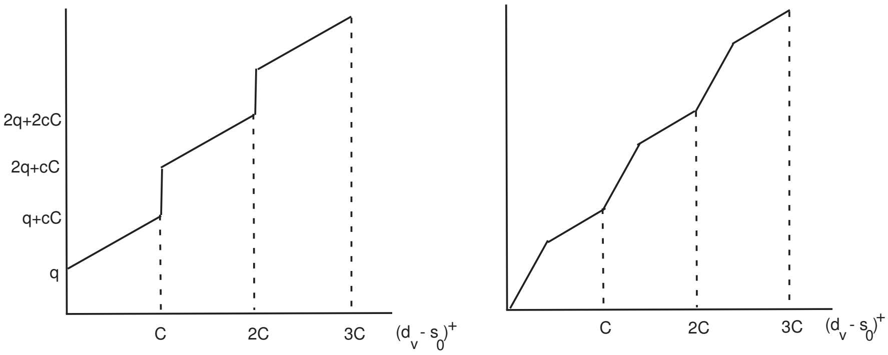
Let $\sigma_t = s + x_t - b_t$ where $b_1 \leq \cdots \leq b_n$.
Rewrite as $\sigma_t + x_k \geq b_k - b_t \quad t < k \leq n$
Convex Hull:

$$\bigcap_{t=1}^{n} \mathrm{conv}(\sigma_t + y_k \geq b_k - b_t \quad t \leq k \leq n, \sigma \in \mathbb{R}^1_+, y \in \mathbb{Z}_+)$$
$$\cap (0 \leq x \leq y)$$

Discrete News-vendor Problem



Discrete News-vendor: Recourse Cost Functions



a) Batch Costs

b) Batch Costs or Linear Costs

Mixing set with Stock VUB

$X^{MIX-SVUB}$.

$$s + z_t \geq b_t \quad t = 1, \ldots, n$$
$$s \leq uw$$
$$s \in \mathbb{R}^1_+, w \in \{0, 1\}, z \in \mathbb{Z}^n$$

Mixing Set with Stock VUB: Valid Inequalities

$$
\begin{aligned}
\mathrm{conv}(X^{MIX-SVUB}) = \{(s,y,w) \in \mathbb{R}^1_+ \times \mathbb{R}^n_+ \times [0,1] : \\
s + y_t + (1-f_t)w \geq \lceil b_t \rceil \text{ if } f_t > 0 \\
s + y_t \geq b_t \text{ if } f_t = 0 \\
s \leq uw \\
y_t + (\lceil b_t \rceil - \lceil b_t - u \rceil^+)w \geq \lceil b_t \rceil \\
s \geq \textstyle\sum_{t=1}^r (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - y_{i_t}) \\
s \geq \textstyle\sum_{t=1}^r (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - y_t) + (1 - f_{i_r})(\lfloor b_{i_1} \rfloor - y_{i_1} + 1 - w) \\
\text{if } f_{i_1} > 0 \\
s \geq \textstyle\sum_{t=1}^r (f_{i_t} - f_{i_{t-1}})(\lfloor b_{i_t} \rfloor + 1 - y_t) + (1 - f_{i_r})(\lfloor b_{i_1} \rfloor - y_{i_1}) \text{ if } f_{i_1} = 0 \\
\}
\end{aligned}
$$

Mixing Set with Stock VUB: Extended Formulation

$$s = \sum_{i=1}^{n} f_i \delta_i + \mu$$
$$\sum_{i=0}^{n} \delta_i = 1$$
$$y_t + \mu + \sum_{i: f_i \geq f_t} \delta_i \geq \lfloor b_t \rfloor + 1$$
$$s \leq uw$$
$$\delta_0 + w \geq 1$$
$$y_t + (\lceil b_t \rceil - \lceil b_t - u \rceil^+) w \geq \lceil b_t \rceil$$
$$\delta \in \mathbb{R}_+^{n+1}, \mu \in \mathbb{R}_+^1, y \in \mathbb{R}_+^n, w \in [0,1]$$

Lot-Sizing with Stock Bounds and Fixed Costs

Atamtürk, A. and S. Küçükyavuz

$$X^{WW-CC-SUB} = X^{WW-CC} \cap \{s \in \mathbb{R}^n : s \leq u\}$$

Theorem

$$\mathrm{conv}(X^{WW-CC-SUB}) = \mathrm{conv}(X^{WW-CC})$$
$$\cap\{(s,y) \in \mathbb{R}^n \times \mathbb{R}^n : Y_{kt} \geq \lceil \frac{d_{kt} - u_{k-1}}{C} \rceil, s \leq u\}.$$

Now let
$X^{WW-CC-SVUB} = X^{WW-CC} \cap \{(s,w) \in \mathbb{R}^n \times \{0,1\}^n : s \leq uw\}$.

Conjecture

$$\text{conv}(X^{WW-CC-SVUB}) = \cap_k \text{conv}(X_k^{MIX-SVUB}(s_{k-1}, y, w_{k-1})).$$

In practice this formulation is very effective, even when the costs are not Wagner-Whitin.

Wagner-Whitin Lot-Sizing with Varying Capacities

Consider the set $X^{WW-C} =$

$$\{(s,y) \in \mathbb{R}_+^{n+1} \times \{0,1\}^n : s_{t-1} + \sum_{u=t}^{l} C_u y_u \geq d_{tl} \text{ for } 1 \leq t \leq l \leq n\}.$$

$$\text{Let } \delta_{tl} = \min\{s_{t-1} + C_t \sum_{u=t}^{l} y_u : (s,y) \in X^{WW-C}\}.$$

- The inequality $s_{t-1} + C_t \sum_{u=t}^{l} y_u \geq \delta_{tl}$ is valid for X^{WW-C}.
- The Varying Capacity relaxation

$$\min \sum_t (h_t s_t + q_t y_t)$$
$$s_{t-1} + C_t \sum_{u=t}^{l} y_u \geq \delta_{tl} \ \forall\, t, l,$$
$$s \in \mathbb{R}_+^{n+1}, y \in \{0,1\}^n$$

- This relaxation is the intersection of n mixing sets. The convex hull of this set is:

$$\bigcap_{t=1}^{n} \text{conv}\left(X^{MIX}(s_{t-1}/C_t, (y_t, \ldots, y_n), (\delta_{tt}, \ldots, \delta_{tn})/C_t)\right).$$

- When the C_t are nondecreasing, i.e. $C_t \leq C_{t+1}$ for all t, then the $\{\delta_{tl}\}$ can be calculated in polynomial time.
- **Theorem** This relaxation solves $WW - C$ with non-decreasing capacities and non-increasing set-up costs.

Divisible Mixing Sets

$$s + C_t z_t \geq b_t \quad t = 1, \ldots, n$$
$$s \in \mathbb{R}^1_+, z \in \mathbb{Z}^n$$

where $C_1 | \cdots | C_n$.

- Polynomial Algorithm for Optimization (de Farias and Zhao)
- Compact Extended Formulation (Conforti and Di Summa)

The Network Dual Problem

$$MIX^{2TU} = \{x : Ax \geq b, x_i \text{ integer}, i \in I\}$$

where A is a totally unimodular (TU, for short) matrix with at most 2 nonzero entries per row with and I is a nonempty subset of the column indices of A.

Equivalent Network Dual Problem

MIX^{2TU} is equivalent to

$$x_i - x_j \geq b_{ij} \; \forall i, j,$$
$$u_j \geq x_j \geq l_j \; \forall j,$$
$$x_i \text{ integer}, i \in I$$

Hence the term Network Dual!

Main Theorem

Theorem

- *There exists an extended formulation for* $\mathrm{conv}(MIX^{2TU})$ *that consists entirely of network dual and bound constraints with integer right hand sides.*
- *Suppose that* $\mathcal{F}$ *is the list of all possible fractional values of the continuous variables* x_i $i \in N \setminus I$ *in all extreme points of* $\mathrm{conv}(MIX^{2TU})$*. The extended formulation is compact (polynomial) if the length of the list* $\mathcal{F}$ *is of polynomial size.*
- *If* $\mathcal{F}$ *is of polynomial size,* $OPT(MIX^{2TU}) \in \mathcal{P}$*.*

The Extended Formulation

$$x_i = \sum_{\ell=0}^{k} \mu_\ell^i (f_\ell - f_{\ell+1})$$

$$\mu_k^i - \mu_0^i = 1,$$

$$\mu_\ell^i - \mu_{\ell-1}^i \geq 0, 1 \leq \ell \leq k$$

$$\mu_{\ell_{l_i}}^i \geq \lfloor l_i \rfloor + 1$$

$$\mu_{\ell_{u_i}}^i \leq \lfloor u_i \rfloor$$

$$\mu_{k_t}^i - \mu_t^j \geq \lfloor l_{ij} \rfloor + 1, \; 1 \leq t \leq k_{ij}$$

$$\mu_{k_t}^i - \mu_t^j \geq \lfloor l_{ij} \rfloor, \; k_{ij} < t \leq k$$

$$y_i - y_j \geq \beta_{ij}$$

Network Dual: The continuous mixing set with flows

The continuous mixing set with flows *CFLOWMIX* is

$$s + r_j + x_j \geq b_j, j \in N$$
$$x_j \leq y_j, j \in N$$
$$s \geq 0, r_j \geq 0, x_j \geq 0, y_j \geq 0 \text{ integer }, j \in N.$$

We use $y_j = y_j, s = s, \sigma_j = s + r_j, x_j = x_j, j \in N$ to transform it into following set *FLOW*:

$$\sigma_j + x_j \geq b_j, j \in N$$
$$x_j \leq y_j, j \in N$$
$$s \geq 0, \sigma_j - s \geq 0, x_j \geq 0, y_j \geq 0 \text{ integer }, j \in N.$$

The constraint matrix is a TU matrix with at most 2 nonzeroes per row, *FLOW* is a mixed integer set of the type X^{2TU}.
The list $\mathcal{F} = \{f(b_j)\, j \in N, f(b_i - b_j), i, j \in N, 0\}$ is complete.

Network Dual: Wagner-Whitin Lot-sizing with Backlogging

The set *LOT*

$$s_k + r_t + \sum_{u=k}^{t} y_u \geq b_t - b_k, k, t \in N, t > k$$
$$s_k \geq 0, r_t \geq 0, y_t = \{0, 1\}, k, t \in N.$$

represents the dominant of the feasible solutions of a lot sizing problem with constant capacities and backlogging.

The linear transformation:
$z_t = \sum_{u=1}^{t} y_u, \sigma_k = s_k - z_k, \rho_t = r_t + z_t, k, t \in N$ maps LOT into the following mixed-integer set:

$$\sigma_k + \rho_t \geq b_t - b_k, k, t \in N, t > k$$
$$\sigma_k + z_k \geq 0, k \in N$$
$$\rho_t - z_t \geq 0, t \in N$$
$$1 \geq z_t - z_{t-1} \geq 0, t \in N$$
$$z_t \text{ integer}, t \in N$$

The above mixed-integer set is of the type MIX^{2TU}.
Van Vyve has shown that the list
$\mathcal{F} = \{f(b_j) j \in N, f(b_i - b_j), i, j \in N, 0\}$ is complete for LOT.

Network Dual: Bipartite cover inequalities.

Given a bipartite graph $G = (U, V; E)$, let (I, L) be a partition of $U \cup V$ with $I \neq \emptyset$ and let $BIP(I, L)$ be the mixed integer set:

$$x_i + x_j \geq b_{uv}, i \in U, j \in V$$
$$x_i \geq 0, i \in L, x_i \geq 0 \text{ integer}, i \in I$$

The set $BIP(I, L)$ is obviously a set of the type MIX^{2TU}.

- Miller and Wolsey show that for the set $BIP(U, V)$, the list $\{f(b_{uv}), u \in U, v \in V, 0\}$ is complete.
- Conforti, Gerards and Zambelli give a formulation in the x-space of the set $BIP(I, L)$ under the condition:

$$f(b_{uv}) = 0, \frac{1}{2}, u \in U, v \in V.$$

Some Open Questions

- Is there a polynomial algorithm for the generalization of the divisible mixing set:

$$s + \sum_{k=1}^{K} C_k z_{kt} \geq b_t \quad t = 1, \ldots, n$$
$$s \in \mathbb{R}^1_+, z \in \mathbb{Z}^{Kn}_+?$$

- Does the conjecture for $WW - CC - SVUB$ hold?
- Is there a theorem relating the value of the $WW - CC$ bound to the $LS - CC$ bound?
- The effect of bounds on s in other simple MIPs?