

MIR, Mixing and Network Dual MIPs: a (progress?) report

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Outline

- 1 Mixing Sets
- 2 Lot-Sizing
- 3 Extensions of Mixing Sets
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Mixing Sets

The mixing set $X^M(s, z, b)$ consists of

$$s + z_l \geq b_l \text{ for } l = 1, \dots, n$$

$$s \in \mathbb{R}_+^1, z \in \mathbb{Z}_+^n.$$

Let $f_l = b_l - \lfloor b_l \rfloor$ for all l .

A tight extended formulation for $\text{conv}(X^M(s, z, b))$ is:

$$s = \sum_{i=1}^n f_i \delta_i + \mu$$

$$z_t + \mu + \sum_{\{i: f_i \geq f_t\}} \delta_i \geq \lfloor b_t \rfloor + 1 \text{ for } t = 1, \dots, n$$

$$\sum_{i=0}^n \delta_i = 1$$

$$\delta \in \mathbb{R}_+^{n+1}, \mu \in \mathbb{R}_+^1, z \in \mathbb{R}_+^n.$$

A Network Dual Extended Formulation

Suppose wlog $f_1 \geq f_2 \geq \dots \geq f_n$. Let $\mu_0 = \mu$ and $\mu_t = \mu + \sum_{i:f_i \geq f_t} \delta_i$ giving:

$$\begin{aligned} \mathbf{s} &= \sum_{i=0}^n (f_i - f_{i+1}) \mu_i \\ \mathbf{z}_t + \mu_t &\geq \lfloor \mathbf{b}_t \rfloor + \mathbf{1} \text{ for } t = 1, \dots, n \\ -\mu_{j-1} + \mu_j &\geq \mathbf{0} \text{ for } j = 1, \dots, n \\ \mu_0 - \mu_n &\geq -\mathbf{1} \\ \mu &\in \mathbb{R}^{n+1}, \mu_0 \geq \mathbf{0}, \mathbf{z} \in \mathbb{R}_+^n. \end{aligned}$$

This is a network dual matrix, and the extreme points are obviously integer.

Mixing Inequalities

Let $T \subseteq \{1, \dots, K\}$ with $|T| = t$, and suppose that i_1, \dots, i_t is an ordering of T such that $0 = f_{i_0} \leq f_{i_1} \leq f_{i_2} \leq \dots \leq f_{i_t} < 1$. Then the mixing inequalities

$$s \geq \sum_{\tau=1}^t (f_{i_\tau} - f_{i_{\tau-1}})(\lfloor b_{i_\tau} \rfloor + 1 - y_{i_\tau})$$

and

$$s \geq \sum_{\tau=1}^t (f_{i_\tau} - f_{i_{\tau-1}})(\lfloor b_{i_\tau} \rfloor + 1 - y_{i_\tau}) + (1 - f_{i_t})(\lfloor b_{i_1} \rfloor - y_{i_1})$$

are valid for X_K^{MIX} .

Example of Mixing Inequalities

Consider the set

$$X = \{(s, y) \in \mathbb{R}_+^1 \times \mathbb{Z}^3 : s + y_1 \geq 1.4, s + y_2 \geq 2.6, s + y_3 \geq 0.7\}.$$

With $|T| = 1$,

$$s \geq 0.4(2 - y_1), s \geq 0.6(3 - y_2), s \geq 0.7(1 - y_3),$$

then the type 1 mixing inequalities with $|T| > 1$

$$s \geq 0.4(2 - y_1) + (0.6 - 0.4)(3 - y_2)$$

$$s \geq 0.4(2 - y_1) + (0.7 - 0.4)(1 - y_3)$$

$$s \geq 0.6(3 - y_2) + (0.7 - 0.6)(1 - y_3)$$

$$s \geq 0.4(2 - y_1) + (0.6 - 0.4)(3 - y_2) + (0.7 - 0.6)(1 - y_3)$$

and the type 2 mixing inequalities with $|T| > 1$ have an additional term.

Mixing Sets and Lot-sizing

Discrete Lot-Sizing with Constant Capacity and Initial Stock Variable

$$s_0 + \sum_{u=1}^t Cy_u \geq \sum_{u=1}^t d_u \quad t = 1, \dots, n$$
$$s_0 \in \mathbb{R}_+^1, y \in \{0, 1\}_+^n$$

$$s_0/C + z_t \geq d_{1t}/C \equiv \sum_{u=1}^t d_u/C \quad t = 1, \dots, n$$
$$s_0 \in \mathbb{R}_+^1, z \in \mathbb{Z}^n$$
$$0 \leq z_t - z_{t-1} \leq 1 \quad \forall t$$

Lot-Sizing and Wagner-Whitin Costs

$$\begin{aligned}
 \min \quad & \sum_t (p'_t x_t + h'_t s_t + q_t y_t) \\
 & s_{t-1} + x_t = d_t + s_t \quad \forall t \\
 & x_t \leq C y_t \quad \forall t \\
 & s, x \geq 0, y \in \{0, 1\}^n
 \end{aligned}$$

Wagner-Whitin Costs: $h_t = p'_t + h'_t - p'_{t+1} \geq 0 \quad \forall t$

$$\begin{aligned}
 \min \quad & \sum_t (h_t s_t + q_t y_t) + \text{constant} \\
 & s_{t-1} + C \sum_{u=t}^l y_u \geq d_{tl} \quad \forall 1 \leq t \leq l \leq n \\
 & s \geq 0, y \in \{0, 1\}^n
 \end{aligned}$$

Lot-Sizing and Wagner-Whitin Costs: Computation

The convex hull of the Wagner-Whitin relaxation is the intersection of the convex hulls of n discrete lot-sizing (mixing) sets.

Even if the costs are not Wagner-Whitin, the Wagner-Whitin relaxation typically closes almost 100% of the gap.

Mixing Set Extensions 1

Continuous Mixing set

$$\begin{aligned} s + r_t + z_t &\geq b_t \quad t = 1, \dots, n \\ s \in \mathbb{R}_+^1, r &\in \mathbb{R}_+^n, z \in \mathbb{Z}^n \end{aligned}$$

Mixing Set With Flows

$$\begin{aligned} s + x_t &\geq b_t \quad t = 1, \dots, n \\ x_t &\leq z_t \quad t = 1, \dots, n \\ s \in \mathbb{R}_+^1, r &\in \mathbb{R}_+^n, z \in \mathbb{Z}^n \end{aligned}$$

Mixing Set Extensions 2

Mixing set with Stock VUB

$$\begin{aligned}s + z_t &\geq b_t \quad t = 1, \dots, n \\s &\leq uw \\s \in \mathbb{R}_+^1, w &\in \{0, 1\}, z \in \mathbb{Z}^n\end{aligned}$$

Divisible Mixing Set

$$\begin{aligned}s + C_t z_t &\geq b_t \quad t = 1, \dots, n \\s \in \mathbb{R}_+^1, z &\in \mathbb{Z}^n\end{aligned}$$

where $C_1 | \dots | C_n$.

Mixing Set with Flows

$$s + x_t \geq b_t \quad t = 1, \dots, n$$

$$x_t \leq z_t \quad t = 1, \dots, n$$

$$s \in \mathbb{R}_+^1, \mathbf{x} \in \mathbb{R}_+^n, \mathbf{z} \in \mathbb{Z}^n$$

Let $\sigma_t = s + x_t - b_t$ where $b_1 \leq \dots \leq b_n$.

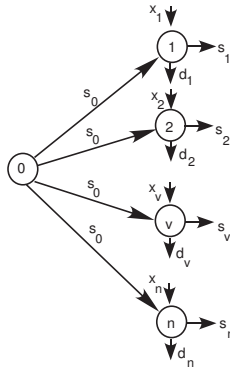
Rewrite as $\sigma_t + x_k \geq b_k - b_t \quad t < k \leq n$

Convex Hull:

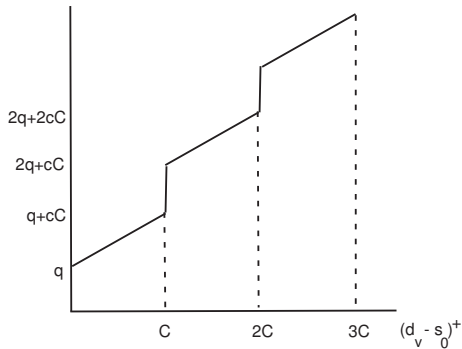
$$\bigcap_{t=1}^n \text{conv}(\sigma_t + y_k \geq b_k - b_t \quad t \leq k \leq n, \sigma \in \mathbb{R}_+^1, \mathbf{y} \in \mathbb{Z}_+)$$

$$\cap(0 \leq \mathbf{x} \leq \mathbf{y})$$

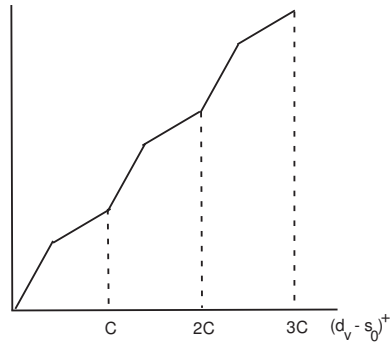
Discrete News-vendor Problem



Discrete News-vendor: Recourse Cost Functions



a) Batch Costs



b) Batch Costs or Linear Costs

Mixing set with Stock VUB

$X^{MIX-SVUB}$.

$$s + z_t \geq b_t \quad t = 1, \dots, n$$

$$s \leq uw$$

$$s \in \mathbb{R}_+^1, w \in \{0, 1\}, z \in \mathbb{Z}^n$$

Mixing Set with Stock VUB: Valid Inequalities

$$\text{conv}(X^{\text{MIX-SVUB}}) = \{(\mathbf{s}, \mathbf{y}, \mathbf{w}) \in \mathbb{R}_+^1 \times \mathbb{R}_+^n \times [0, 1] :$$

$$\mathbf{s} + \mathbf{y}_t + (1 - f_t)\mathbf{w} \geq \lceil \mathbf{b}_t \rceil \text{ if } f_t > 0$$

$$\mathbf{s} + \mathbf{y}_t \geq \mathbf{b}_t \text{ if } f_t = 0$$

$$\mathbf{s} \leq \mathbf{u}\mathbf{w}$$

$$\mathbf{y}_t + (\lceil \mathbf{b}_t \rceil - \lceil \mathbf{b}_t - \mathbf{u} \rceil^+)\mathbf{w} \geq \lceil \mathbf{b}_t \rceil$$

$$\mathbf{s} \geq \sum_{t=1}^r (f_{i_t} - f_{i_{t-1}})(\lceil \mathbf{b}_{i_t} \rceil + 1 - \mathbf{y}_{i_t})$$

$$\mathbf{s} \geq \sum_{t=1}^r (f_{i_t} - f_{i_{t-1}})(\lceil \mathbf{b}_{i_t} \rceil + 1 - \mathbf{y}_t) + (1 - f_{i_r})(\lceil \mathbf{b}_{i_1} \rceil - \mathbf{y}_{i_1} + 1 - \mathbf{w})$$

if $f_{i_1} > 0$

$$\mathbf{s} \geq \sum_{t=1}^r (f_{i_t} - f_{i_{t-1}})(\lceil \mathbf{b}_{i_t} \rceil + 1 - \mathbf{y}_t) + (1 - f_{i_r})(\lceil \mathbf{b}_{i_1} \rceil - \mathbf{y}_{i_1}) \text{ if } f_{i_1} = 0$$

Mixing Set with Stock VUB: Extended Formulation

$$\begin{aligned} \mathbf{s} &= \sum_{i=1}^n f_i \delta_i + \mu \\ \sum_{i=0}^n \delta_i &= 1 \\ \mathbf{y}_t + \mu + \sum_{i: f_i \geq f_t} \delta_i &\geq \lfloor \mathbf{b}_t \rfloor + \mathbf{1} \\ \mathbf{s} &\leq \mathbf{u} \mathbf{w} \\ \delta_0 + \mathbf{w} &\geq \mathbf{1} \\ \mathbf{y}_t + (\lfloor \mathbf{b}_t \rfloor - \lfloor \mathbf{b}_t - \mathbf{u} \rfloor^+) \mathbf{w} &\geq \lfloor \mathbf{b}_t \rfloor \\ \delta \in \mathbb{R}_+^{n+1}, \mu \in \mathbb{R}_+^1, \mathbf{y} \in \mathbb{R}_+^n, \mathbf{w} \in [0, 1] \end{aligned}$$

Lot-Sizing with Stock Bounds and Fixed Costs

Atamtürk, A. and S. Küçükyavuz

$$X^{WW-CC-SUB} = X^{WW-CC} \cap \{s \in \mathbb{R}^n : s \leq u\}$$

Theorem

$$\text{conv}(X^{WW-CC-SUB}) = \text{conv}(X^{WW-CC}) \\ \cap \{(s, y) \in \mathbb{R}^n \times \mathbb{R}^n : Y_{kt} \geq \lceil \frac{d_{kt} - u_{k-1}}{C} \rceil, s \leq u\}.$$

Now let

$$X^{WW-CC-SVUB} = X^{WW-CC} \cap \{(s, w) \in \mathbb{R}^n \times \{0, 1\}^n : s \leq uw\}.$$

Conjecture

$$\text{conv}(X^{WW-CC-SVUB}) = \bigcap_k \text{conv}(X_k^{\text{MIX-SVUB}}(s_{k-1}, y, w_{k-1})).$$

In practice this formulation is very effective, even when the costs are not Wagner-Whitin.

Wagner-Whitin Lot-Sizing with Varying Capacities

Consider the set $X^{WW-C} =$

$$\{(s, y) \in \mathbb{R}_+^{n+1} \times \{0, 1\}^n : s_{t-1} + \sum_{u=t}^l C_u y_u \geq d_{tl} \text{ for } 1 \leq t \leq l \leq n\}.$$

$$\text{Let } \delta_{tl} = \min \left\{ s_{t-1} + C_t \sum_{u=t}^l y_u : (s, y) \in X^{WW-C} \right\}.$$

- The inequality $s_{t-1} + C_t \sum_{u=t}^l y_u \geq \delta_{tl}$ is valid for X^{WW-C} .
- The Varying Capacity relaxation

$$\begin{aligned} & \min \sum_t (h_t s_t + q_t y_t) \\ & s_{t-1} + C_t \sum_{u=t}^l y_u \geq \delta_{tl} \quad \forall t, l, \\ & s \in \mathbb{R}_+^{n+1}, y \in \{0, 1\}^n \end{aligned}$$

- This relaxation is the intersection of n mixing sets. The convex hull of this set is:

$$\bigcap_{t=1}^n \text{conv} \left(X^{MIX}(s_{t-1}/C_t, (y_t, \dots, y_n), (\delta_{tt}, \dots, \delta_{tn})/C_t) \right).$$

- When the C_t are nondecreasing, i.e. $C_t \leq C_{t+1}$ for all t , then the $\{\delta_{tt}\}$ can be calculated in polynomial time.
- **Theorem** This relaxation solves $WW - C$ with non-decreasing capacities and non-increasing set-up costs.

Divisible Mixing Sets

$$s + C_t z_t \geq b_t \quad t = 1, \dots, n$$
$$s \in \mathbb{R}_+^1, z \in \mathbb{Z}^n$$

where $C_1 | \dots | C_n$.

- Polynomial Algorithm for Optimization (de Farias and Zhao)
- Compact Extended Formulation (Conforti and Di Summa)

The Network Dual Problem

$$MIX^{2TU} = \{x : Ax \geq b, x_i \text{ integer}, i \in I\}$$

where A is a totally unimodular (TU, for short) matrix with at most 2 nonzero entries per row with and I is a nonempty subset of the column indices of A .

Equivalent Network Dual Problem

MIX^{2TU} is equivalent to

$$\begin{aligned}x_i - x_j &\geq b_{ij} \quad \forall i, j, \\u_j &\geq x_j \geq l_j \quad \forall j, \\x_i &\text{ integer, } i \in I\end{aligned}$$

Hence the term Network Dual!

Main Theorem

Theorem

- *There exists an extended formulation for $\text{conv}(MIX^{2TU})$ that consists entirely of network dual and bound constraints with integer right hand sides.*
- *Suppose that \mathcal{F} is the list of all possible fractional values of the continuous variables x_i , $i \in N \setminus I$ in all extreme points of $\text{conv}(MIX^{2TU})$. The extended formulation is compact (polynomial) if the length of the list \mathcal{F} is of polynomial size.*
- *If \mathcal{F} is of polynomial size, $OPT(MIX^{2TU}) \in \mathcal{P}$.*

The Extended Formulation

$$\begin{aligned}
 x_i &= \sum_{\ell=0}^k \mu_{\ell}^i (f_{\ell} - f_{\ell+1}) \\
 \mu_k^i - \mu_0^i &= 1, \\
 \mu_{\ell}^i - \mu_{\ell-1}^i &\geq 0, \quad 1 \leq \ell \leq k \\
 \mu_{\ell_{l_i}}^i &\geq \lfloor l_i \rfloor + 1 \\
 \mu_{\ell_{u_i}}^i &\leq \lfloor u_i \rfloor \\
 \mu_{k_t}^i - \mu_t^j &\geq \lfloor l_{ij} \rfloor + 1, \quad 1 \leq t \leq k_{ij} \\
 \mu_{k_t}^i - \mu_t^j &\geq \lfloor l_{ij} \rfloor, \quad k_{ij} < t \leq k \\
 y_i - y_j &\geq \beta_{ij}
 \end{aligned}$$

Network Dual: The continuous mixing set with flows

The continuous mixing set with flows $CFLOWMIX$ is

$$s + r_j + x_j \geq b_j, j \in N$$

$$x_j \leq y_j, j \in N$$

$$s \geq 0, r_j \geq 0, x_j \geq 0, y_j \geq 0 \text{ integer}, j \in N.$$

We use $y_j = y_j, s = s, \sigma_j = s + r_j, x_j = x_j, j \in N$ to transform it into following set $FLOW$:

$$\sigma_j + x_j \geq b_j, j \in N$$

$$x_j \leq y_j, j \in N$$

$$s \geq 0, \sigma_j - s \geq 0, x_j \geq 0, y_j \geq 0 \text{ integer}, j \in N.$$

The constraint matrix is a TU matrix with at most 2 nonzeros per row, $FLOW$ is a mixed integer set of the type X^{2TU} .

The list $\mathcal{F} = \{f(b_j) \mid j \in N, f(b_j - b_i), i, j \in N, 0\}$ is complete.

Network Dual: Wagner-Whitin Lot-sizing with Backlogging

The set LOT

$$s_k + r_t + \sum_{u=k}^t y_u \geq b_t - b_k, k, t \in N, t > k$$
$$s_k \geq 0, r_t \geq 0, y_t = \{0, 1\}, k, t \in N.$$

represents the dominant of the feasible solutions of a lot sizing problem with constant capacities and backlogging.

The linear transformation:

$z_t = \sum_{u=1}^t y_u$, $\sigma_k = s_k - z_k$, $\rho_t = r_t + z_t$, $k, t \in N$ maps *LOT* into the following mixed-integer set:

$$\sigma_k + \rho_t \geq b_t - b_k, k, t \in N, t > k$$

$$\sigma_k + z_k \geq 0, k \in N$$

$$\rho_t - z_t \geq 0, t \in N$$

$$1 \geq z_t - z_{t-1} \geq 0, t \in N$$

$$z_t \text{ integer, } t \in N$$

The above mixed-integer set is of the type MIX^{2TU} .

Van Vyve has shown that the list

$\mathcal{F} = \{f(b_j)j \in N, f(b_i - b_j), i, j \in N, 0\}$ is complete for *LOT*.

Network Dual: Bipartite cover inequalities.

Given a bipartite graph $G = (U, V; E)$, let (I, L) be a partition of $U \cup V$ with $I \neq \emptyset$ and let $BIP(I, L)$ be the mixed integer set:

$$\begin{aligned} x_i + x_j &\geq b_{uv}, i \in U, j \in V \\ x_i &\geq 0, i \in L, x_i \geq 0 \text{ integer}, i \in I \end{aligned}$$

The set $BIP(I, L)$ is obviously a set of the type MIX^{2TU} .

- Miller and Wolsey show that for the set $BIP(U, V)$, the list $\{f(b_{uv}), u \in U, v \in V, 0\}$ is complete.
- Conforti, Gerards and Zambelli give a formulation in the x -space of the set $BIP(I, L)$ under the condition:

$$f(b_{uv}) = 0, \frac{1}{2}, u \in U, v \in V.$$

Some Open Questions

- Is there a polynomial algorithm for the generalization of the divisible mixing set:

$$\begin{aligned} \mathbf{s} + \sum_{k=1}^K \mathbf{C}_k \mathbf{z}_{kt} &\geq \mathbf{b}_t \quad t = 1, \dots, n \\ \mathbf{s} &\in \mathbb{R}_+^1, \mathbf{z} \in \mathbb{Z}_+^{Kn?} \end{aligned}$$

- Does the conjecture for $WW - CC - SVUB$ hold?
- Is there a theorem relating the value of the $WW - CC$ bound to the $LS - CC$ bound?
- The effect of bounds on \mathbf{s} in other simple MIPs?