Strategic Planning with Start-Time Dependent Variable Costs: A Case Study in Solving Nonlinear Integer Models

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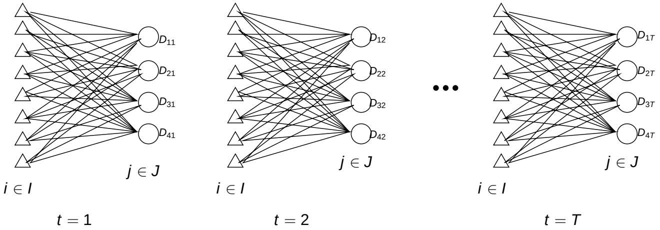
Motivating Application: Production and Distribution Planning

Minimize costs to meet demand over the planning horizon.



Motivated by problem in upstream oil and gas industry: long-term strategic development of wells and transportation network.

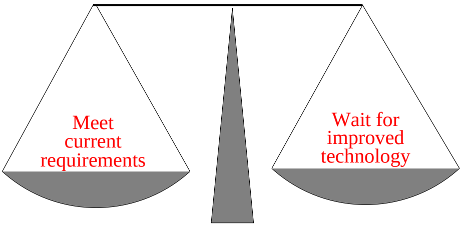
Planning horizon as long as 20 years

Key Challenge: Start-time Dependent Costs

Problem characteristics: (Example: Development of oil fields)

- Planning a set of unique activities over a multi-period horizon
- To start an activity, must install available technology, and is an all or nothing decision
- Once installed, technology cannot be changed
- Variable and fixed costs depend on installed technology, which may improve over time

Key Trade-off



Generic Multiple-Period Planning Model

- Time Horizon: T periods
- Activities to Plan: $a \in A$
- Decision Variables:

  x_{at} Level of activity a in period t

  $$y_{at} = \begin{cases} 1 & \text{if activity } a \text{ starts in period } t \\ 0 & \text{otherwise} \end{cases}$$

- Cost Data:

  f_{at} Fixed cost if activity a starts in period t

  c_{at} Variable cost over entire horizon if activity a starts in period t

$f_{at} \geq 0$ and $c_{at} \geq 0$

A Mixed Integer Bilinear Formulation: Constraints

- Activity levels must be zero until started:

$$x_{at} \leq \sum_{s=1}^{t} M_a y_{as} \quad \forall t$$

- Start each activity at most once:

$$\sum_{t} y_{at} \leq 1$$

- Meet system demands (links activities):

$$\mathbf{x} \in C$$

- Binary restrictions and bounds on activity levels:

$$y_{at} \in \{0, 1\} \qquad 0 \leq x_{at} \leq M_a$$

A Mixed Integer Bilinear Formulation: Objective

- Minimize cost over the planning horizon

$$\sum_{a \in A} \sum_{s=1}^{T} \left(f_{as} + c_{as} \sum_{t=s}^{T} x_{at} \right) y_{as}$$

- If $y_{as} = 1$ objective records

$$f_{as} + c_{as} \sum_{t=s}^{T} x_{at}$$

- Formulation is mixed-integer program with bilinear objective
- Production and distribution planning problem is NP-hard, even with $T = 2$ and no fixed costs

We study a single activity problem

Our approach: Develop strong formulations for

Single Activity Structure: MIBL

$$\min \quad \sum_{s=1}^{T} c_s y_s \sum_{t=s}^{T} x_t + \sum_{s=1}^{T} f_s y_s$$

$$\text{s.t.} \quad x_t \leq \sum_{s=1}^{t} M y_s \qquad \forall t$$

$$x \geq 0,\ y \in \{0,1\}^T,\ \sum_t y_t \leq 1$$

Structure is simple, but not trivial.

Formulation 1: A Simple Linearization of MIBL

- Introduce auxiliary variables to obtain linear objective:

$$\sum_{s=1}^{T} c_s \underbrace{y_s \sum_{t=s}^{T} x_t}_{z_s} + \sum_{s=1}^{T} f_s y_s \quad \Rightarrow \quad \sum_{s=1}^{T} c_s z_s + \sum_{s=1}^{T} f_s y_s$$

- Simple Linearization: $z_s \geq 0$ and

$$z_s \geq \sum_{t=s}^{T} x_t - (1 - y_s)(T - s + 1)M \quad \forall s$$

- Alternative: Extended linearization with $O(T^2)$ variables and constraints

Tightened Linearization Inspired by Lot Sizing (LS)

- Problem with simple linearization: when y is fractional, can have positive activity levels ($x_t > 0$) and pay no variable costs ($z_s = 0 \ \forall s$)
- Lot sizing: for each time period t, cumulative production should exceed cumulative demand
- Similarly, cumulative amount we pay for up to period t should exceed cumulative activity performed

$$\sum_{s=1}^{t} z_s \geq \sum_{s=1}^{t} x_s \qquad \forall t$$

- These constraints significantly improve relaxation

Tightened Linearization Inspired by Lot Sizing (LS)

- Difference from lot sizing: we must pay for all at rate determined by start period
$$z_s \leq (T - s + 1) M y_s \qquad \forall s$$
- Valid formulation obtained by also enforcing
$$x \in [0, M]^T \quad y \in \{0,1\}^T \quad \sum_t y_t \leq 1$$
- We have characterized an exponential class of inequalities which define the convex hull of this formulation

Formulation 2: Linearize in (x, y) space

Simple trick to move nonlinear objective into constraints:

$$\begin{aligned} \min \quad & \mu + \sum_{s=1}^{T} f_s y_s \\ \text{s.t.} \quad & \mu \geq \sum_{s=1}^{T} c_s y_s \sum_{t=s}^{T} x_t \\ & x_t \leq M \sum_{s=1}^{t} y_s \quad \forall t \\ & x \geq 0,\ y \in \{0,1\}^T,\ \sum_t y_t \leq 1 \end{aligned}$$

F : the set of (μ, x, y) feasible to the above constraints

- For fixed binary vector, constraints become linear.
- There are only finitely many feasible binary vectors

  $\Rightarrow$ Convex hull of F is a polyhedron

Explicit Characterization of the Convex Hull

Theorem

$conv(F)$ is given by the linear inequalities defining F, and

$$\mu \geq \sum_{t=1}^{T} c_{i_t} x_t - M \sum_{t=1}^{T} \sum_{s=t}^{T} (c_{i_s} - c_t)^+ y_t$$

for all $i_t \in \{1, \ldots, t\}, t = 1, \ldots, T$.

- Exponential, but separation can be done in polynomial time.
- We can add just T of the inequalities to obtain a valid formulation when we drop the nonlinear inequality

$$\mu \geq \sum_{s=1}^{T} c_s y_s \sum_{t=s}^{T} x_t$$

$\Rightarrow$ New formulation (LBL) with optional cuts

Formulation 3: Treat the start time decision *implicitly*

For $x \in [0, M]^T$ define $h(x)$ by

$$
\begin{aligned}
h(x) = \min \quad & \sum_{s=1}^{T} c_s y_s \sum_{t=s}^{T} x_t + \sum_{s=1}^{T} f_s y_s \\
\text{s.t.} \quad & \sum_{s=1}^{t} M y_s \geq x_t \quad \forall t \\
& y \in \{0,1\}^T, \ \sum_t y_t \leq 1 \\
= \min \quad & \left\{ f_s + c_s \sum_{t=s}^{T} x_t : 1 \leq s \leq \min\{k : x_k > 0\} \right\}
\end{aligned}
$$

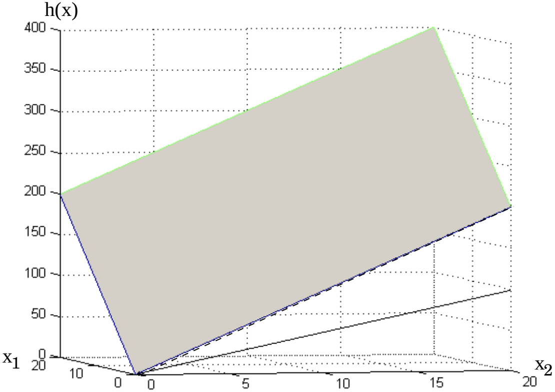
New formulation

$$\min \left\{ h(x) : x \in [0, M]^T \right\}$$

Example Cost Function $h(x)$ for $T = 2$

$c_1 = 10 \quad c_2 = 5$
$f_1 = 0 \quad f_2 = 0$
$M = 20$

$$h(x_1, x_2) = \begin{cases} 5x_2 & \text{if } x_1 = 0 \\ 10(x_1 + x_2) & \text{if } x_1 > 0 \end{cases}$$



Concave minimization formulation

Theorem

h is concave over $[0, M]^T$.

- Concave minimization is hard! $\Rightarrow$ Specialized branch-and-cut
- First step, move non-convex objective into constraints

$$\min \left\{ \mu : \mu \geq h(x), x \in [0, M]^T \right\}$$

- Use branching to enforce the nonlinear constraint
- Use cutting planes to approximate the non-convex feasible region

Concave minimization: initial relaxation and branching

- Relaxation obtained by linear lower bound on objective function:

$$\mu \geq h(x) \geq \sum_{t=1}^{T} \min \{c_s : s \leq t\} x_t$$

- Branching on start-time s:
  - Left branch: $s \leq k$. Update objective lower bound
  - Right branch: $s > k$. Update objective lower bound *and* fix $x_t = 0$, $t = 1, \ldots, k$
- Objective lower bounds can be updated so that when start-time is fixed by branching, lower bound is exact $\Rightarrow$ finite termination

Concave minimization: branching example

Data: $c_1 = 6, c_2 = 4, c_3 = 3, f_1 = f_2 = f_3 = 10, M = 10$

Initial relaxation: $\mu \geq 6x_1 + 4x_2 + 3x_3$

Current solution: $x_1 = 5, x_2 = 5, x_3 = 10, \mu = 80$

True cost: $h(x) = 10 + 120 = 130$

- First branch: start-time less than 3
  - Updated relaxation: $\mu \geq 10 + 6x_1 + 4x_2 + 4x_3$
  - Lower bound with current solution: 100
- Second branch: start-time 3 or later
  - Relaxation remains the same
  - Restrict $x_1 = x_2 = 0$
  - Possible next solution: $x_3 = 10, \mu = 30, h(x) = 40$
    (or infeasible)

Concave minimization: cutting planes

Feasible region:

$$E = \{(\mu, x) : \mu \geq h(x),\ x \in [0, M]^T\}$$

Theorem

- *Given a solution* (μ, x) *it is possible to check whether* $(\mu, x) \in conv(E)$ *by solving an explicit polynomial size linear program.*
- *If* $(\mu, x) \notin conv(E)$ *the solution to this linear program yields a valid inequality which cuts off this solution.*

- Using these cutting planes is critical

Concave minimization: cutting plane example

Data:	$c_1 = 6, c_2 = 4, c_3 = 3, f_1 = f_2 = f_3 = 10, M = 10$
Initial relaxation:	$\mu \geq 6x_1 + 4x_2 + 3x_3$
Current solution:	$x_1 = 5, x_2 = 5, x_3 = 10, \mu = 80$
True cost:	$h(x) = 10 + 120 = 130$

- Linear program yields the valid inequality

$$\mu \geq -20 + 9x_1 + 5x_2 + 6x_3$$

- For current solution yields: $\mu \geq 110$
- At the point $x = (0, 10, 10)$ yields $\mu \geq 90 = h(x)$
- Similarly tight at 3 other extreme points of $[0, 10]^3$

Summary of Formulations

	LS	LBL	CM
Variables	x, y, z	x, y	x
	$\longrightarrow$ Decreasing size $\longrightarrow$		
Implementation Requirements		Cuts*	Cuts and Branching
	$\longrightarrow$ Increasing complexity $\longrightarrow$		

*Optional, but helpful.

Results for small instances

- Production and distribution planning instances
- Randomly generated, with characteristics similar to real data
- Each entry is an average over five instances, run for at most 1 hour

Fixed Cost?	$(\|I\|, \|J\|, T)$	Simple Lin Opt Gap	LS Ave Time (s)
No	$(10, 5, 10)$	78.9%	30
	$(15, 5, 10)$	76.8%	46
	$(10, 10, 10)$	55.6%	70
Yes	$(10, 5, 10)$	46.0%	17
	$(15, 5, 10)$	49.6%	162
	$(10, 10, 10)$	38.8%	47

Tests on Larger Instances

- Two hour time limit
- Sizes range from 50 suppliers, 10 customers, 10 periods to 100 suppliers, 20 customers, and 20 periods
- Formulation sizes for largest instance:

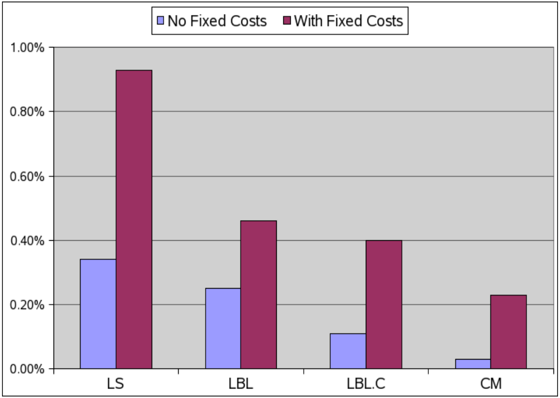
	Vars $\approx$ Rows
Lot sizing inspired linearization (LS)	126,000
Linear formulation based on bilinear (LBL)	86,000
Concave minimization (CM)	44,000

- All formulations are sparse

Average Optimality Gap

LBL.C $\Rightarrow$ LBL with new cuts

Average Gap From Best Upper Bound



Ave Root LP Solve Times	LS	LBL	CM
Without Fixed Costs	554.1	195.0	13.2
With Fixed Costs	1132.5	730.6	66.1

Special Case: Nondecreasing Activities

- Replace $x \in [0, M]^T$ with

$$x \in X = \{x : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_T \leq M\}$$

- The nondecreasing constraint was present in the motivating application.
- The extreme points of X

$$\begin{aligned} M^1 &= (M, M, \cdots, M, M) \\ M^2 &= (0, M, \cdots, M, M) \\ &\vdots \\ M^T &= (0, 0, \cdots, 0, M) \\ M^{T+1} &= (0, 0, \cdots, 0, 0) \end{aligned}$$

Special Case: Concave Minimization

Feasible region:

$$E' = \{(\mu, x) : \mu \geq h(x),\ x \in X\}$$

Theorem

conv(E') is given by $x \in X$ and

$$\mu \geq \sum_{t=1}^{T} (h(M^t) - h(M^{t+1}))x_t / M.$$

Proof: the bound is valid and tight at all extreme points and *h* is concave.

Convex hull for mixed-integer linear formulations still requires exponentially many inequalities

Computational results with non-decreasing activities

Preliminary results on instances *without* fixed costs

$(\lvert I\rvert, \lvert J\rvert, T)$	Time(s) or * Gap	
	LS	CM
$(30, 10, 10)$	* 0.34%	129.4
$(40, 10, 10)$	* 0.11%	24.8
$(50, 10, 10)$	* 0.21%	27.7
$(20, 10, 15)$	* 0.78%	699.9
$(30, 10, 15)$	* 0.37%	1335.5
$(40, 10, 15)$	* 0.26%	234.5
$(50, 10, 20)$	* 0.70%	* 0.02%
$(100, 10, 20)$	* 2.43%	* 0.08%

* Did not finish after limit of 1 hour.

These results are a single instance for each size.

Extensions and Further Work

- Allow dependence of variable costs on both start time *and* period in which activity occurs
- Consider more complicated constraints on a single activity. For example,
  - Time-dependent upper bounds
  - Production ramping constraints
- Uncertainty in technological improvements and system demands

Questions?

Related, but different

Capacity Expansion

- Plan when and how much capacity to install
- Variable costs may depend on when capacity installed
- Capacity installation is a continuous decision
- Li and Tirupati (1994), Rajagopalan (1998)

Dynamic Facility Location

- Plan when and where to install facilities
- Variable costs depend only on the period in which activity occurs
- Van Roy and Erlenkotter (1982), Shulman (1991)

Maximum Gap From Best Upper Bound