Polyhedral Aspects of Symmety Breaking in Integer Programming

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Joint work with
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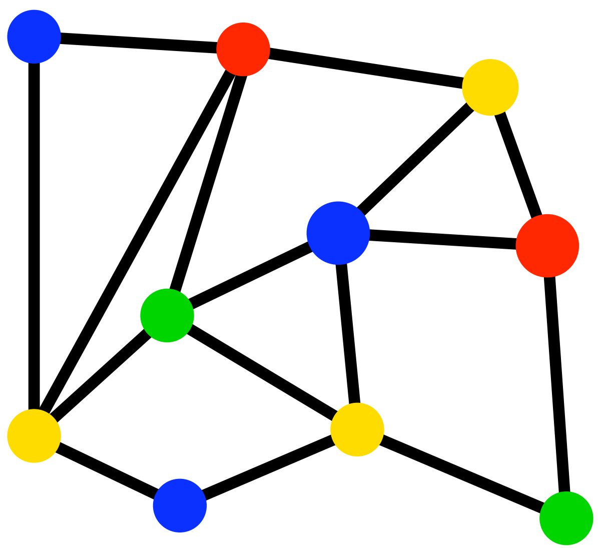
Outline

- Examples: Vertex coloring, graph partitioning
- Orbitopes
- Orbitopal fixing
- Computational results for graph partitioning
- Remarks

Example: Vertex Coloring



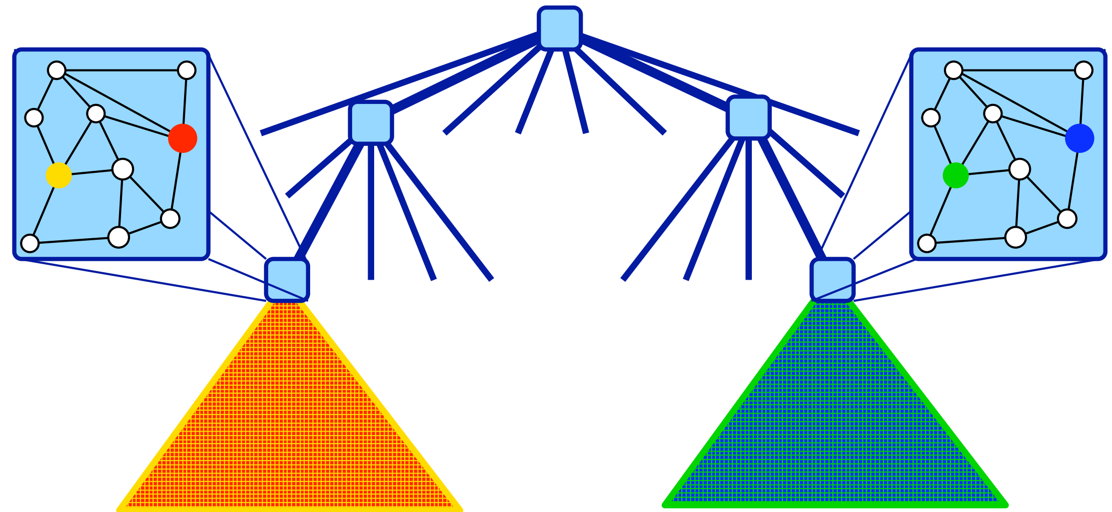
$$\min \quad \sum_{c=1}^{C_{\max}} y_c$$

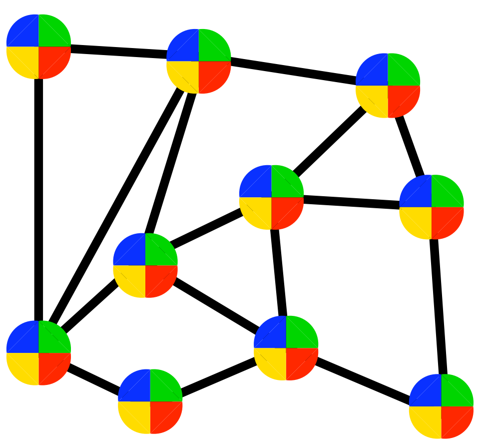
$$\begin{array}{rrcll} \text{s.t.} & \sum_c x_{v,c} & = & 1 & \forall v \\ & x_{v,c} + x_{w,c} & \leq & 1 & \forall c\ \forall \{v,w\} \in E \\ & x_{v,c} & \leq & y_c & \forall v\ \forall c \\ & x_{v,c} & \in & \{0,1\} & \forall v\ \forall c \end{array}$$

Symmetric group operates on variables
with constant objective function value on orbits.

Two problems

Unnecessarily large search space
(Branch-and-cut)

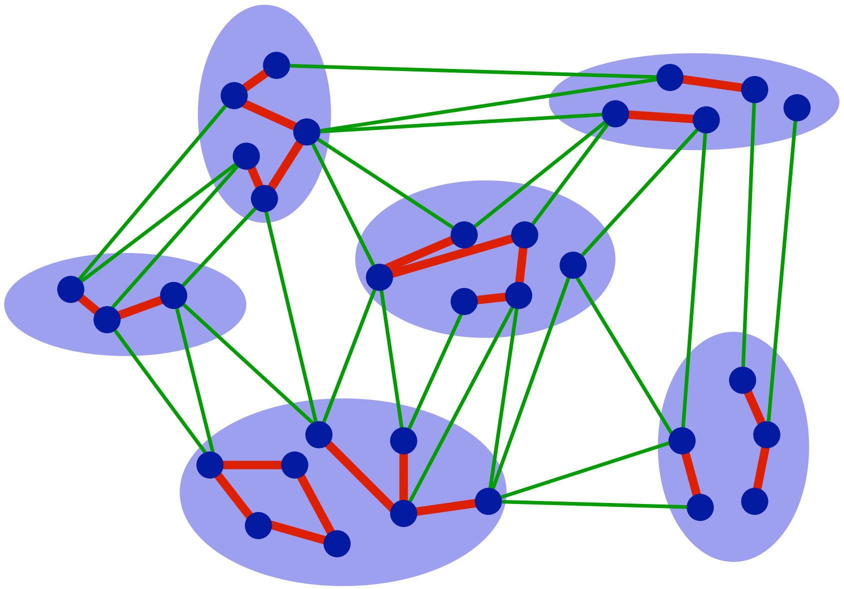




Useless LP-bounds

$$
\begin{array}{llll}
\min & \sum_{c=1}^{C_{\max}} y_c & & \\
\text{s.t.} & \sum_{c} x_{v,c} = & 1 & \forall v \\
& x_{v,c} + x_{w,c} \leq & 1 & \forall c\ \forall \{v,w\} \in E \\
& x_{v,c} \leq & y_c & \forall v\ \forall c \\
& x_{v,c} \in & [0,1] & \forall v\ \forall c
\end{array}
$$

Example: Graph Partitioning

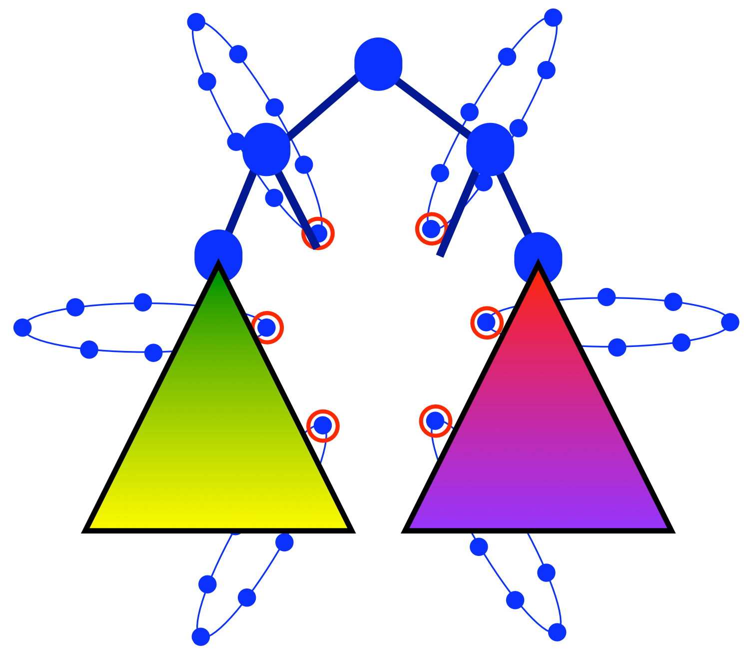


$x_{v,j} = 1$ iff v is in part j

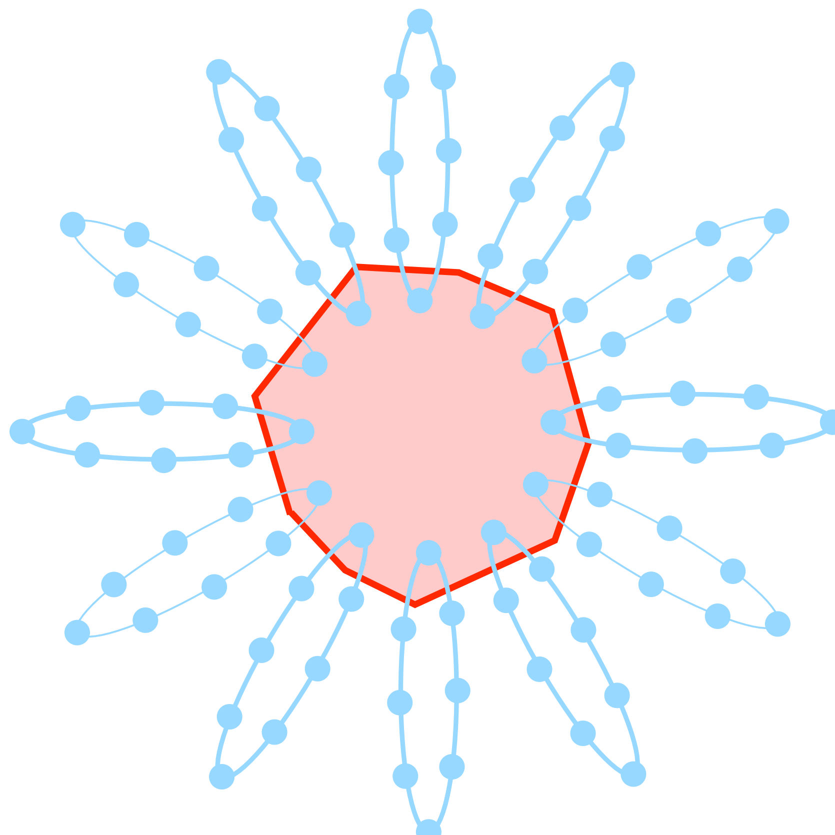
$y_{v,w} = 1$ iff v and w are in the same part

$$\sum_v x_{v,j} \leq B$$

0	0	1	0	0	0	0
0	0	0	0	0	0	1
0	0	1	0	0	0	0
0	0	0	0	0	0	1
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	0	0	0	1
0	0	0	0	0	1	0
0	0	0	0	1	0	0
0	1	0	0	0	0	0



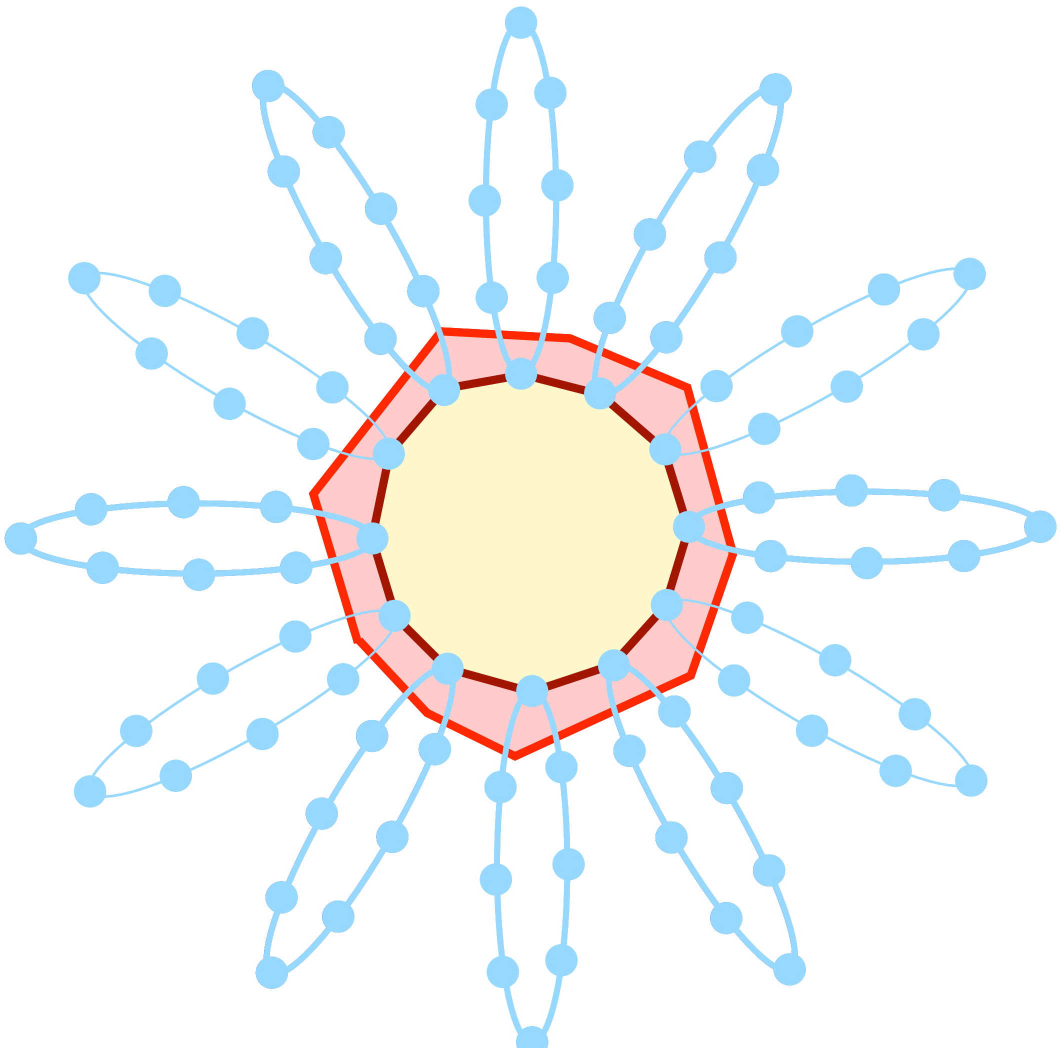
- Isomorphism pruning (Margot 2002+)
- Orbital branching (Linderoth et al. 2007)
- Additional inequalities



Outline

- ☑ Examples: Vertex coloring, graph partitioning
- ☐ Orbitopes
- ☐ Orbitopal fixing
- ☐ Computational results for graph partitioning
- ☐ Remarks

Plan



- Describe convex hull of representatives
- "Partitioning", "packing", "full"
- Symmetric / cyclic group

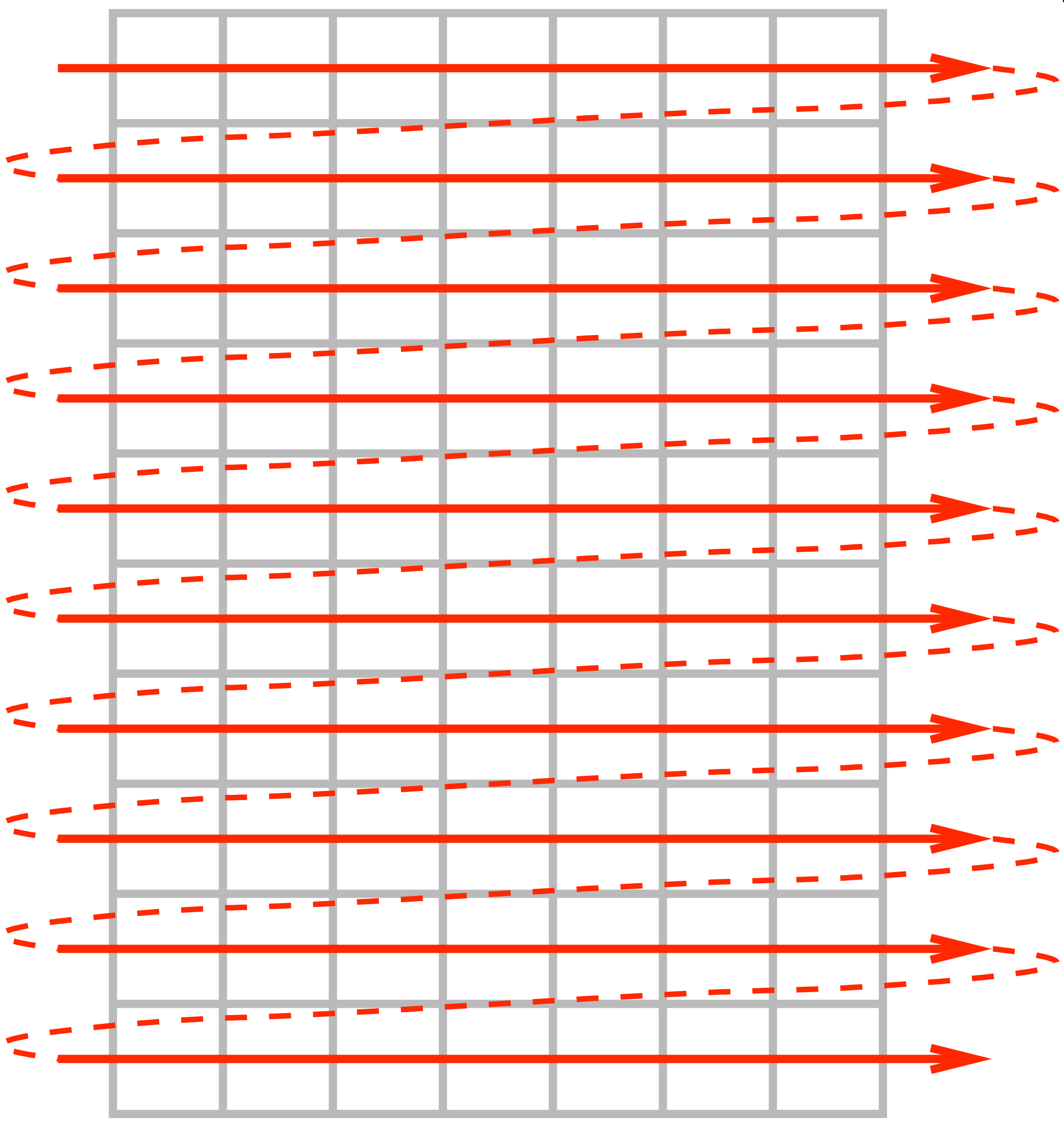
The Matrices

0	0	0	0	1	0	0
1	0	0	0	0	0	0
0	0	0	0	0	0	0
1	0	0	0	0	0	0
0	0	0	0	0	1	0
0	0	0	0	1	0	0
0	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	0	0	0	0

$$\mathcal{M}_{p,q} := \{0,1\}^{[p]\times[q]}$$

$$\mathcal{M}^{=}_{p,q} := \{x \in \mathcal{M}_{p,q} \;:\; x(\mathrm{row}_i) = 1\}$$

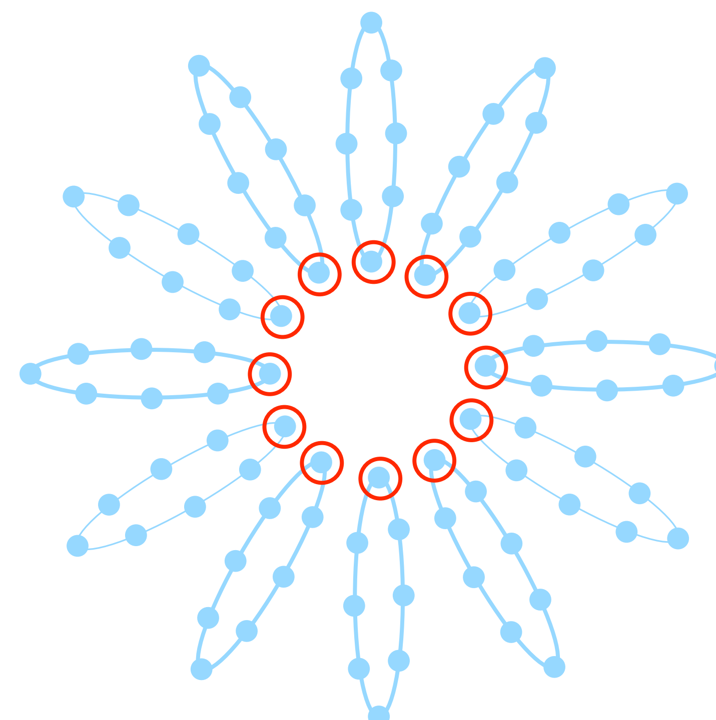
$$\mathcal{M}^{\leq}_{p,q} := \{x \in \mathcal{M}_{p,q} \;:\; x(\mathrm{row}_i) \leq 1\}$$



$\prec$: lexicographic ordering of $\mathcal{M}_{p,q}$

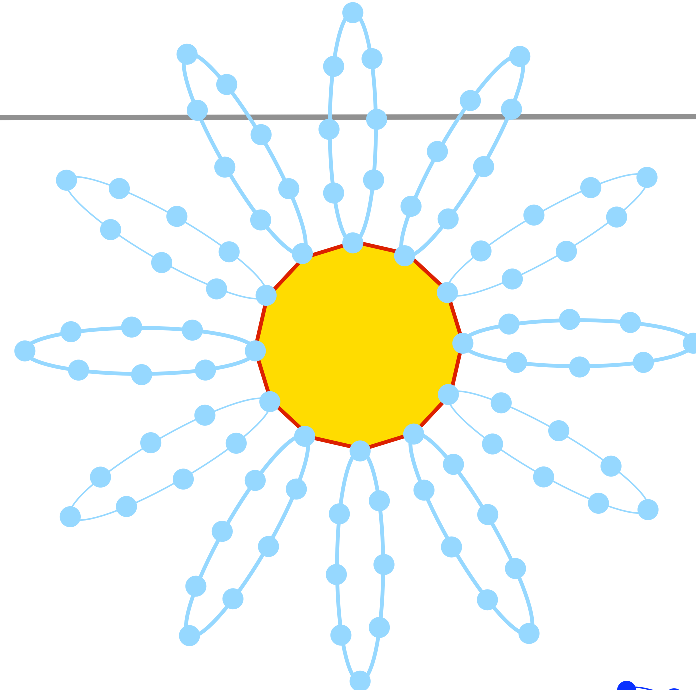
$\mathcal{M}^{\max}_{p,q}(G)$:

set of matrices that are $\prec$-maximal in their orbits

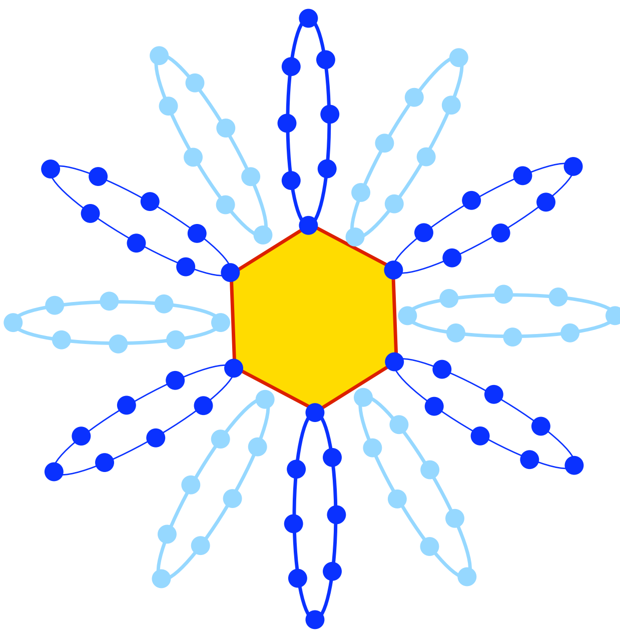


The Orbitopes

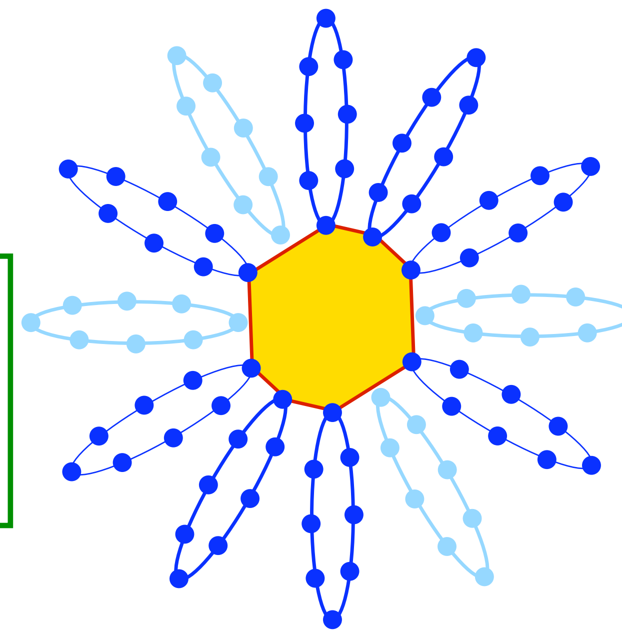
$\mathrm{O}_{p,q}(G) := \mathrm{conv}\ \mathcal{M}^{\max}_{p,q}(G)$



"full orbitope"



G :
symmetric group ($\mathfrak{S}_q$)
cyclic group ($\mathfrak{C}_q$)



$\mathrm{O}^{=}_{p,q}(G) := \mathrm{conv}(\mathcal{M}^{\max}_{p,q}(G) \cap \mathcal{M}^{=}_{p,q})$

"partitioning orbitope"

$\mathrm{O}^{\leq}_{p,q}(G) := \mathrm{conv}(\mathcal{M}^{\max}_{p,q}(G) \cap \mathcal{M}^{\leq}_{p,q})$

"packing orbitope"

	Partitioning Packing	Full	Full (q=2)
Symmetric Group	Complete Polynomial	??? Polynomial	Complete Polynomial
Cyclic Group	Complete Polynomial	??? ???	

Polytope

Optimization

Vertices of Orbitopes

$\mathfrak{S}_q$

1	0	0	0	0	0	0
0	1	0	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	1	0	0	0	0

lex sorted columns

$\mathfrak{C}_q$ (packing)

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
1	0	0	0	0	0	0
0	0	0	0	0	1	0
0	0	0	0	1	0	0
0	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	0	0	0	0

first column lex max

$\mathfrak{C}_q$ (partitioning)

1	0	0	0	0	0	0
0	0	0	0	1	0	0
1	0	0	0	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
1	0	0	0	0	0	0
0	0	0	0	1	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	0	0	1	0

Cyclic Group: Partitioning

THEOREM [K & Pfetsch 2006]

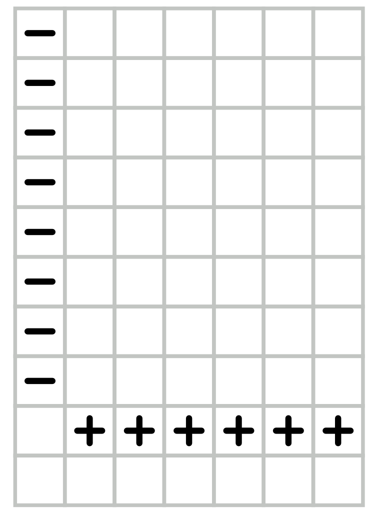
The orbitope $\mathrm{O}^{=}_{p,q}(\mathfrak{C}_q)$ is described by the following constraints:

- nonnegativity constraints
- row-sum equations
- $x_{11} = 1$ and $x_{1j} = 0$ for all $j > 1$

THEOREM [K & Pfetsch 2006]

The orbitope $O^{\leq}_{p,q}(\mathfrak{C}_q)$ is described by the following constraints:

- nonnegativity constraints
- row-sum inequalities
- $x_{1j} = 0$ for all $j > 1$
- these inequalities →

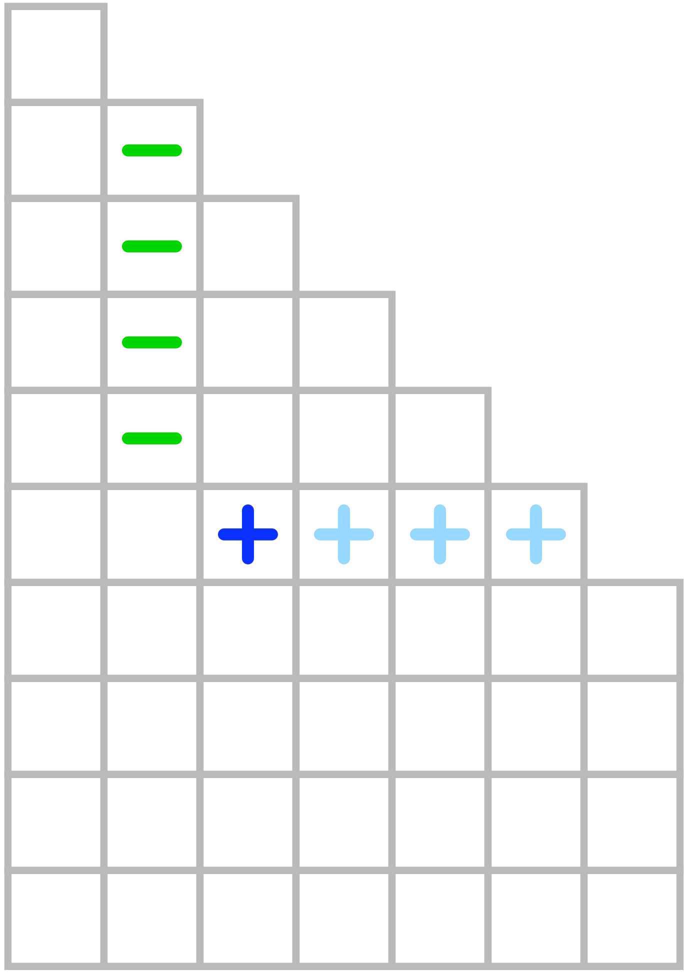


≤ 0

$$\begin{pmatrix}
 & + & & & & + & + & + & + & & & & & & & & & & & & \\
 & & + & & & & & & & + & + & + & + & & & & & & & & \\
 & & & + & & & & & & & & & & + & + & + & + & & & & \\
 & & & & + & & & & & & & & & & & & & + & + & + & + \\
- & & & & & + & + & + & + & & & & & & & & & & & & \\
- & - & & & & & & & & + & + & + & + & & & & & & & & \\
- & - & - & & & & & & & & & & & + & + & + & + & & & & \\
- & - & - & - & & & & & & & & & & & & & & + & + & + & +
\end{pmatrix}$$

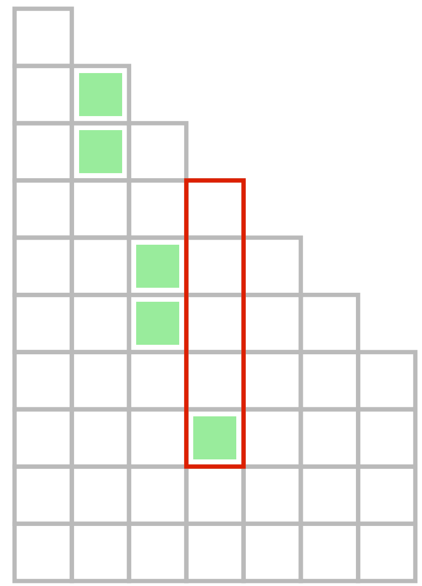
The Results so far

	Partitioning Packing	Full	Full (q=2)
Symmetric Group	Polynomial		
Cyclic Group	Complete Polynomial		

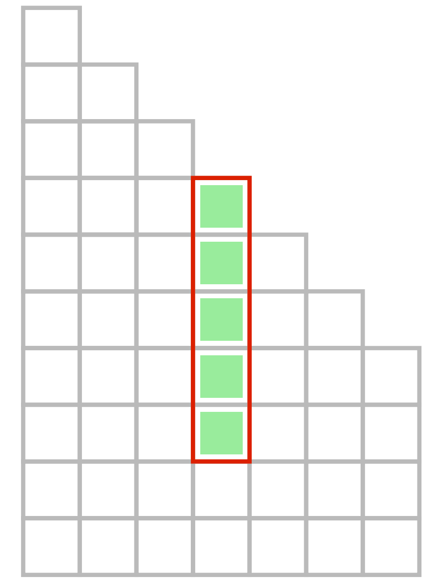


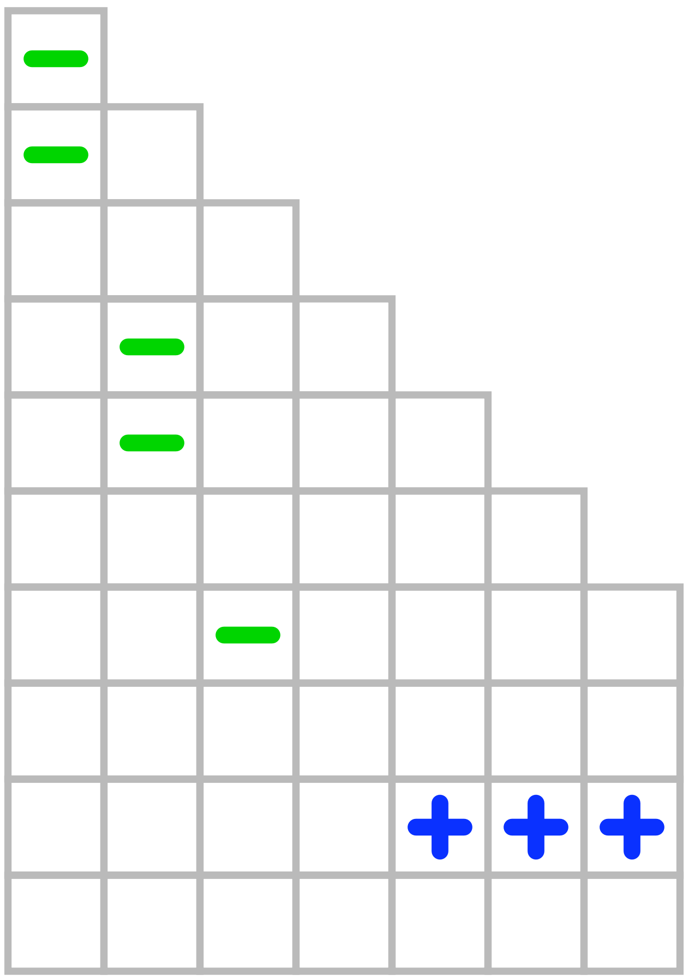
≤ 0

Shifted-Column Inequalities



shifting of

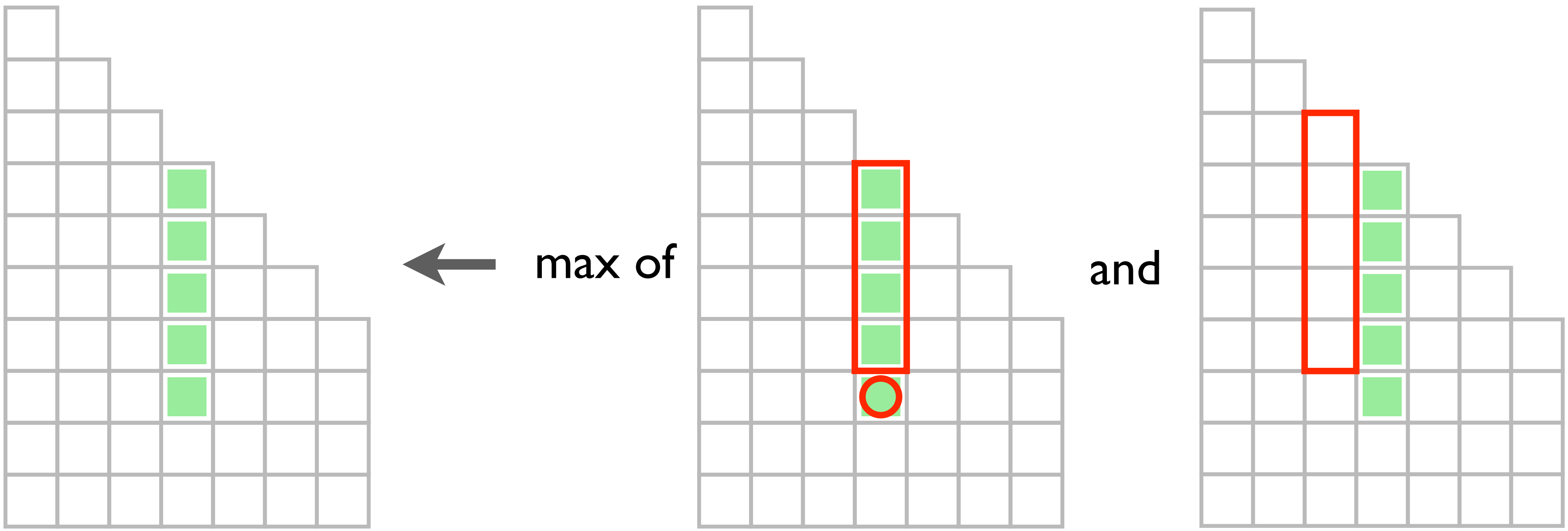




(exponentially many)
shifted columns inequalities (SCI)

THEOREM [K & Pfetsch 2006]

The separation problem for SCIs can be solved in linear time.

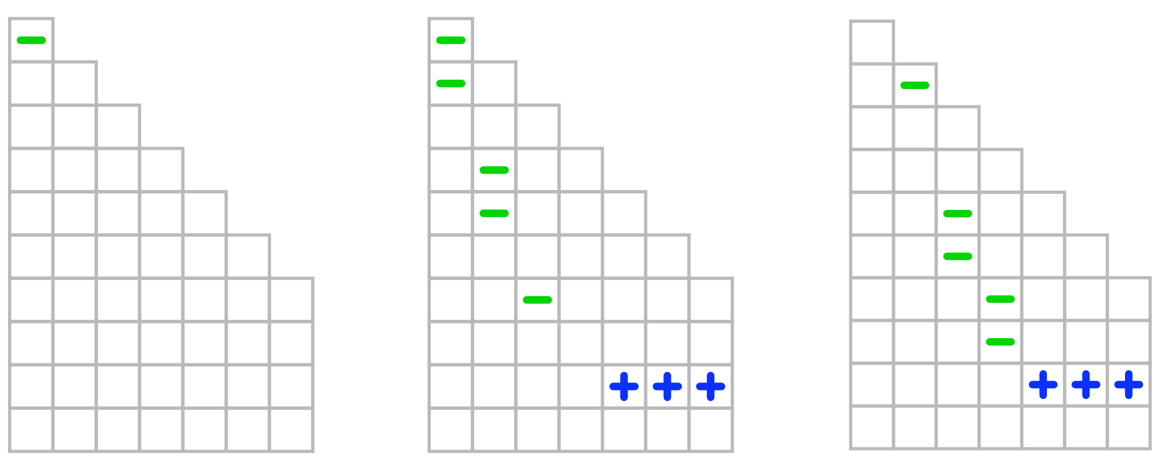


THEOREM [K & Pfetsch 2006]

The orbitope $\mathrm{O}^{=}_{p,q}(\mathfrak{S}_q)$ equals the polytope $Q(p,q)$ that is described by the following constraints:

- nonnegativity constraints
- row-sum equations
- shifted column inequalities

The description is irredundant up to:

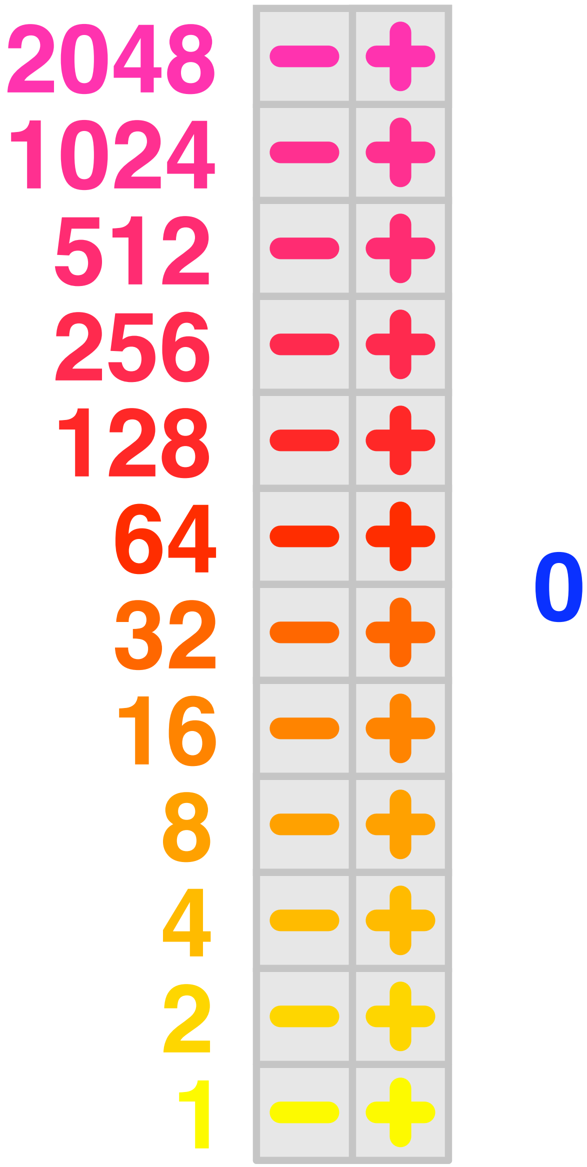


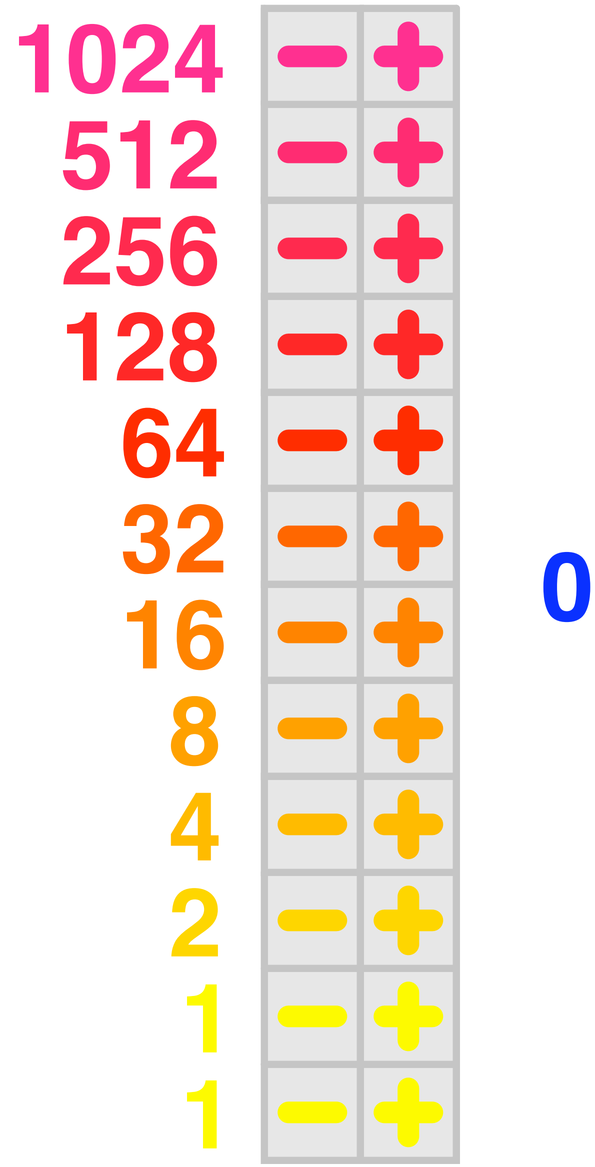
The Results so far

	Partitioning Packing	Full	Full (q=2)
Symmetric Group	Complete Polynomial		
Cyclic Group	Complete Polynomial		

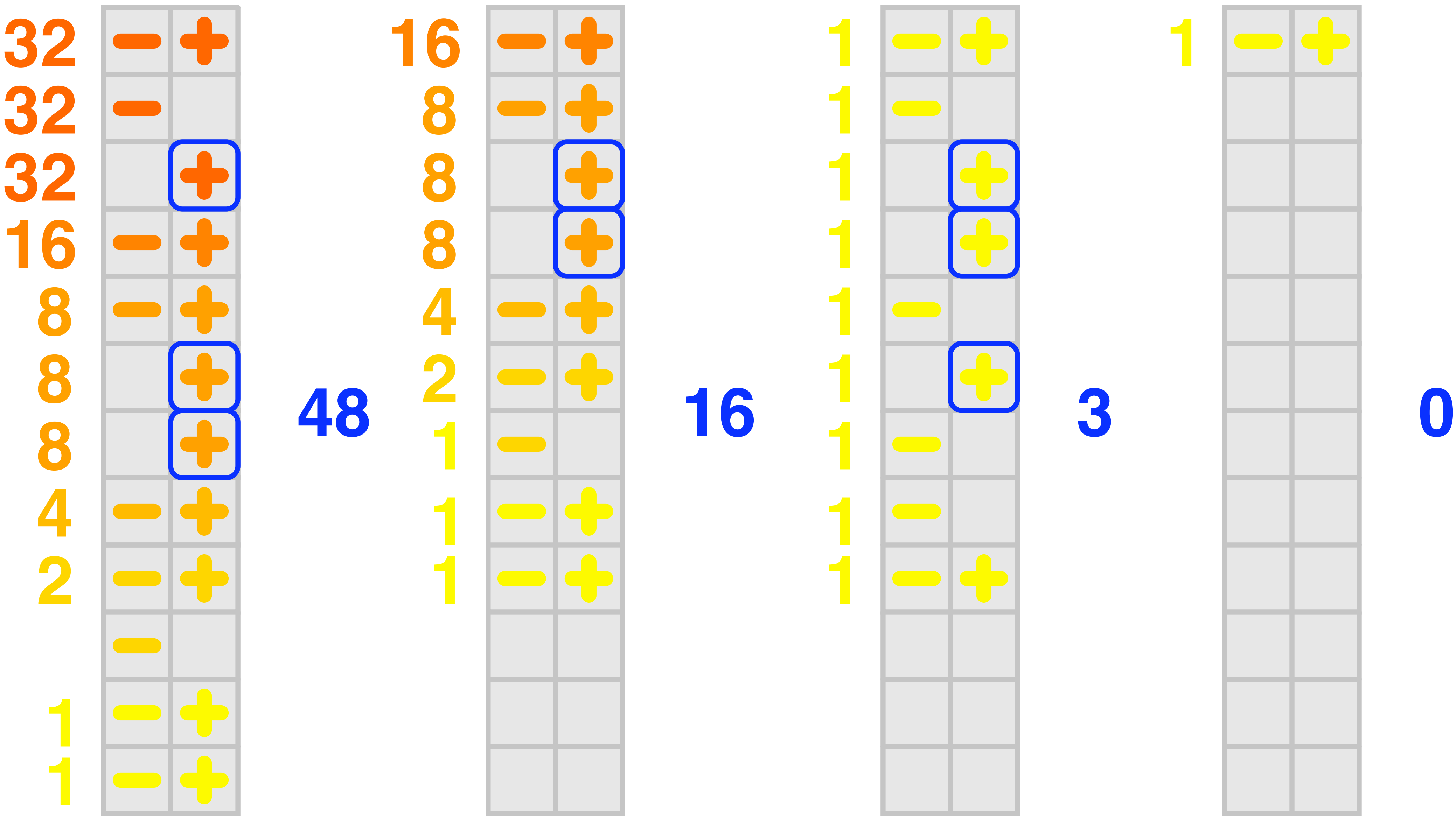
Vertices of $\mathrm{O}_{p,2}(\mathfrak{S}_2) = \mathrm{O}_{p,2}(\mathfrak{C}_2)$:

0	0
1	1
0	0
1	1
1	1
1	0
0	1
1	0
1	0
0	0
1	1
0	1


2048
1024
512
256
128
64
32
16
8
4
2
1
0


1024
512
256
128
64
32
16
8
4
2
1
1
0

Block Inequalities



Description of the Orbisack

THEOREM [K & Loos 2006]

The orbisack $\mathrm{O}_{p,2}(\mathfrak{S}_2) = \mathrm{O}_{p,2}(\mathfrak{C}_2)$ is completely described by the following constraints:

- 0/1-bounds on the variables
- block inequalities

The description is essentially irredundant.

Knowledge on Orbitopes

	Partitioning Packing	Full	Full (q=2)
Symmetric Group	Complete Polynomial	??? Polynomial	Complete Polynomial
Cyclic Group	Complete Polynomial	??? ???	

Outline

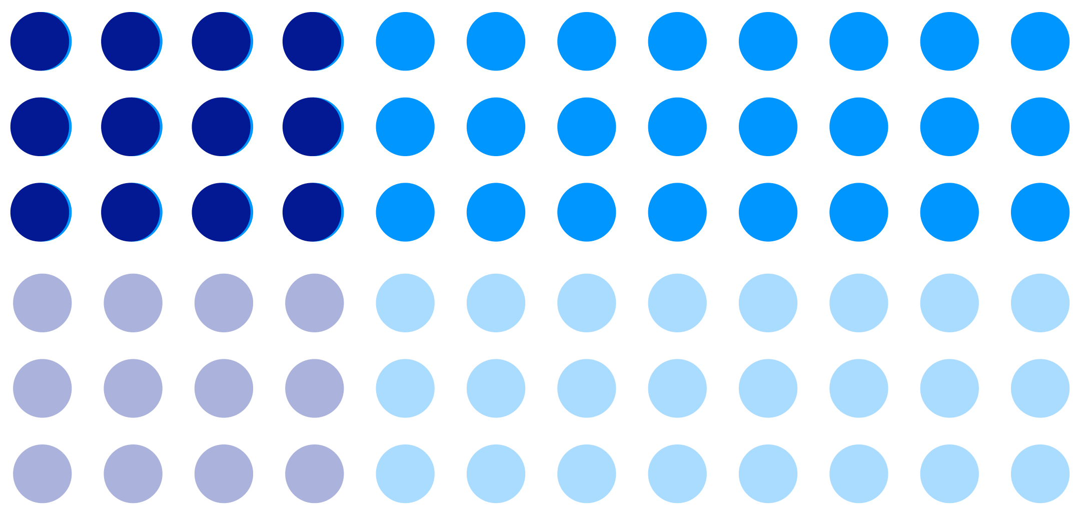
- [x] Examples: Vertex coloring, graph partitioning
- [x] Orbitopes
- [ ] Orbitopal fixing
- [ ] Computational results for graph partitioning
- [ ] Remarks

$P \subseteq \mathrm{C}_d := [0,1]^d$ polytope

0	0			1			0	1		

cube face F

$P \cap F \cap \{0,1\}^d$



$P \cap \{0,1\}^d$

$\mathrm{Fix}_F(P)$: smallest cube face containing $P \cap F \cap \{0,1\}^d$

(In general: NP-hard to compute)

Relaxation: $P \subseteq \tilde{P} \implies \mathrm{Fix}_F(P) \subseteq \mathrm{Fix}_F(\tilde{P})$

$\tilde{P}$ Knapsack: $\mathrm{Fix}_F(\tilde{P})$ easy to compute

For $P = \{x \in [0,1]^d \,:\, Ax \le b\}$:

While there is some inequality $ax \le \beta$ in $Ax \le b$ with

$$F \;\neq\; \tilde{F} \;:=\; \mathrm{Fix}_F(\{x \in [0,1]^d \,:\, ax \le \beta\})$$

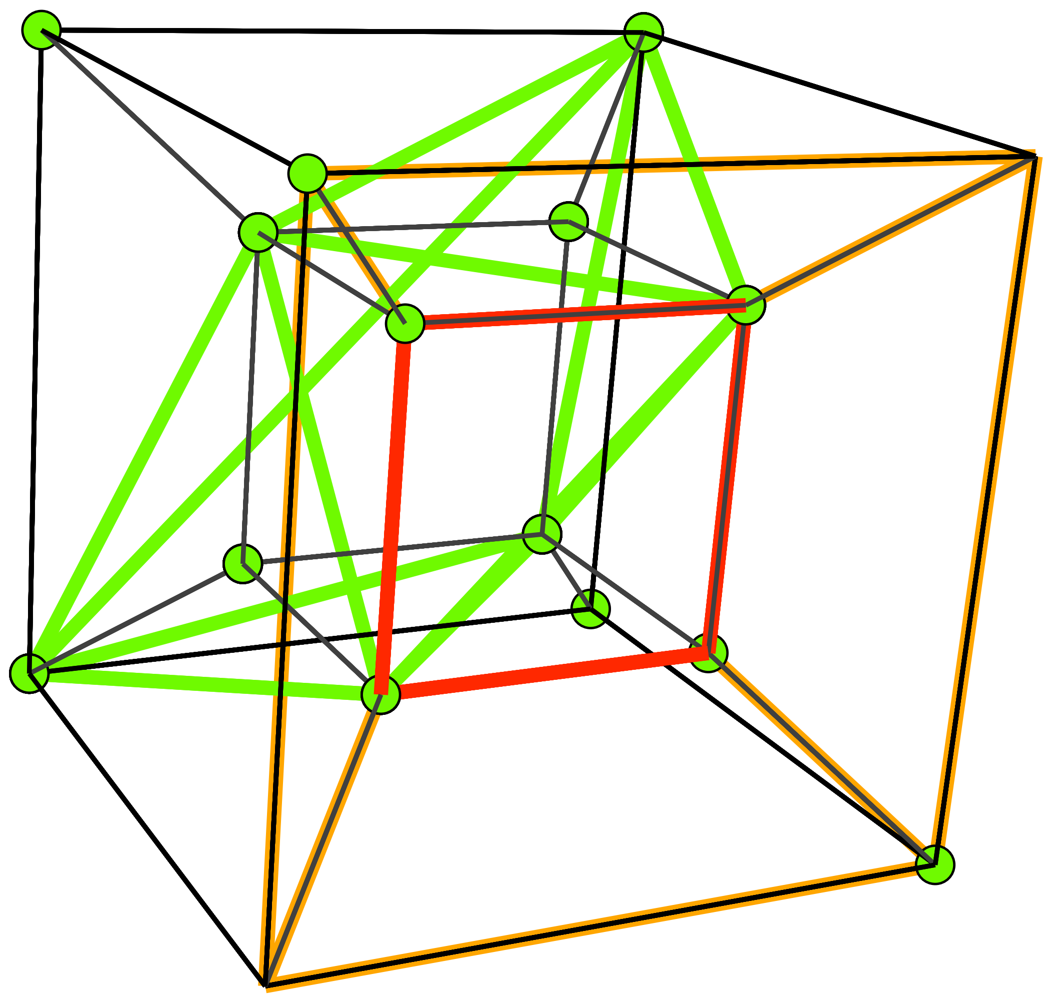
replace F by $\tilde{F}$.

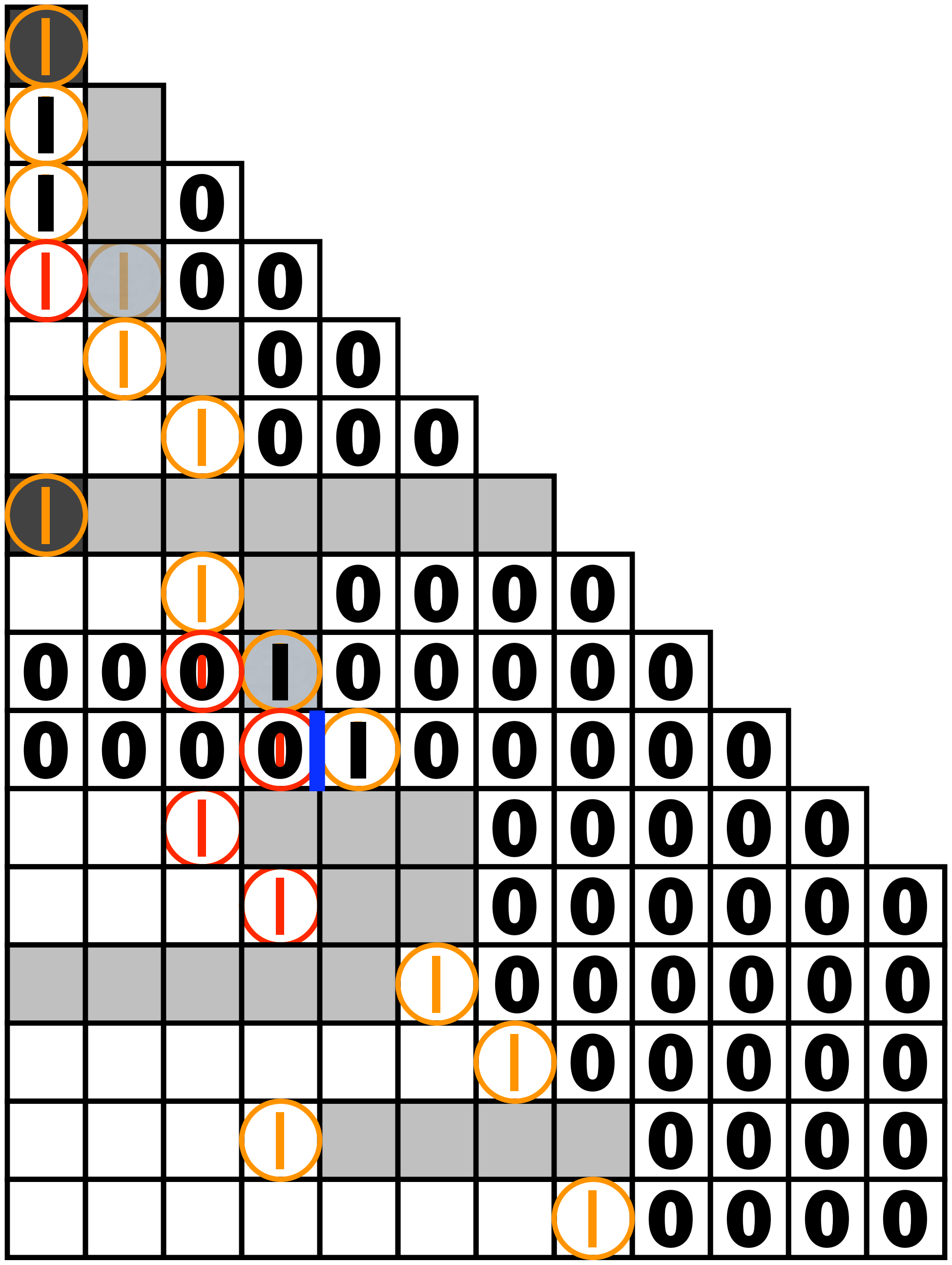
$\longrightarrow$ $\mathrm{Fix}_F(Ax \le b)$ sequential fixing

- Independent of choices

THEOREM [K, Peinhardt, Pfetsch 2007]

Sequential fixing (even with ideal descriptions) is weaker than simultaneous fixing, in general.





Orbitopal fixing of F : $\mathrm{Fix}_F(\mathrm{O}^{=}_{p,q})$

(removes all orbitopal symmetry)

THEOREM [K, Peinhardt, Pfetsch 2007]

There is a linear time algorithm for orbitopal fixing.

THEOREM [K, Peinhardt, Pfetsch 2007]

Sequential fixing (wrt affine hull) is as strong as orbitopal fixing.

Outline

- ☑ Examples: Vertex coloring, graph partitioning
- ☑ Orbitopes
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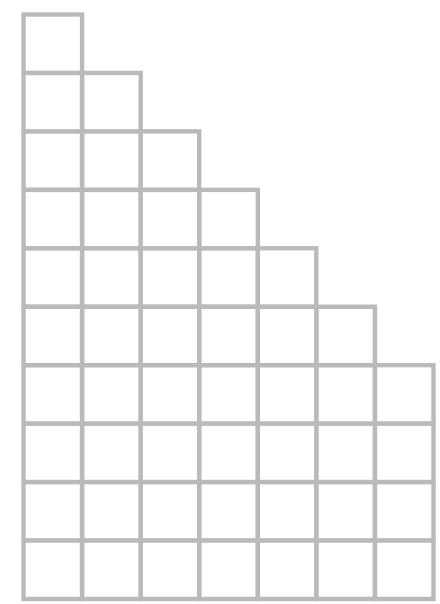
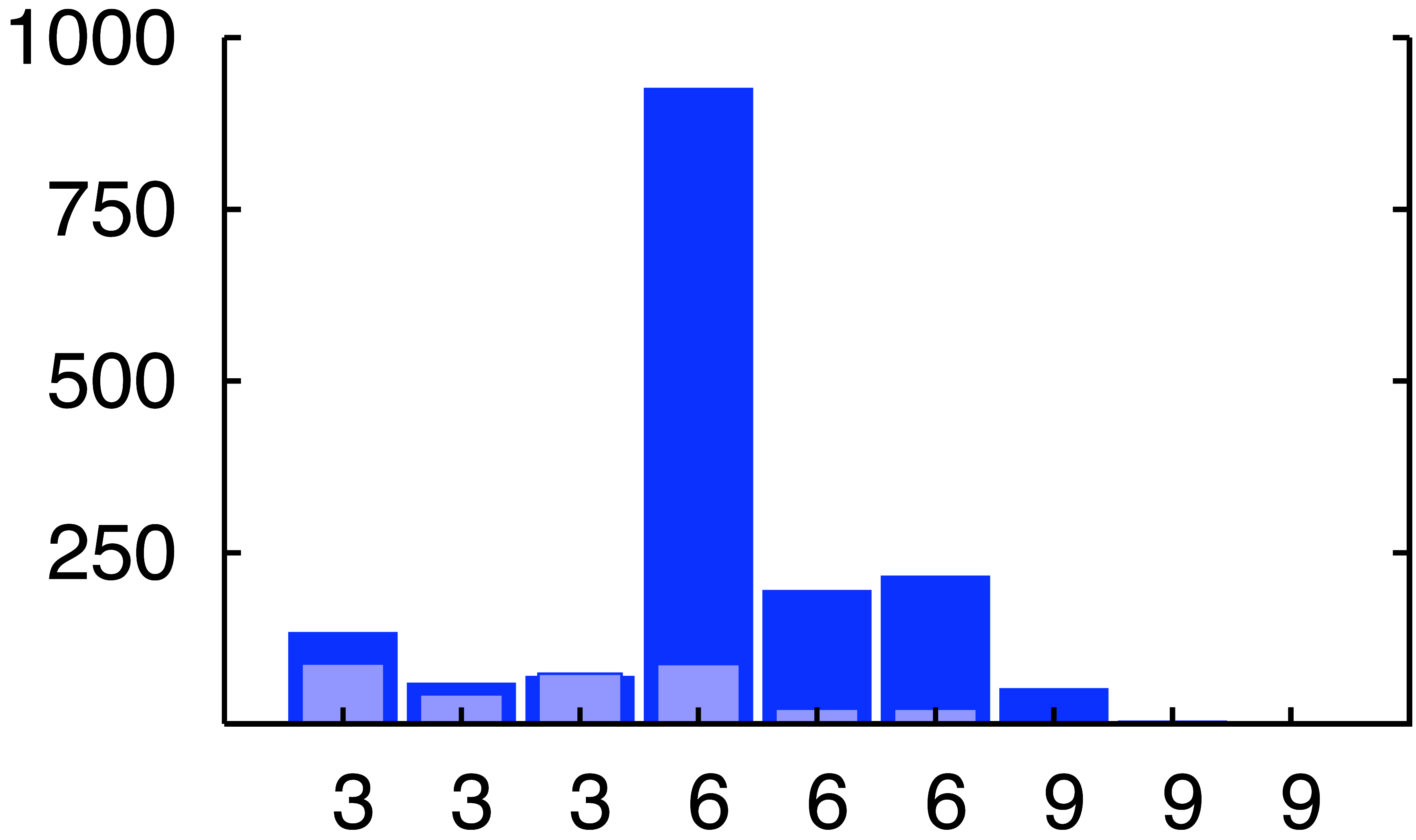
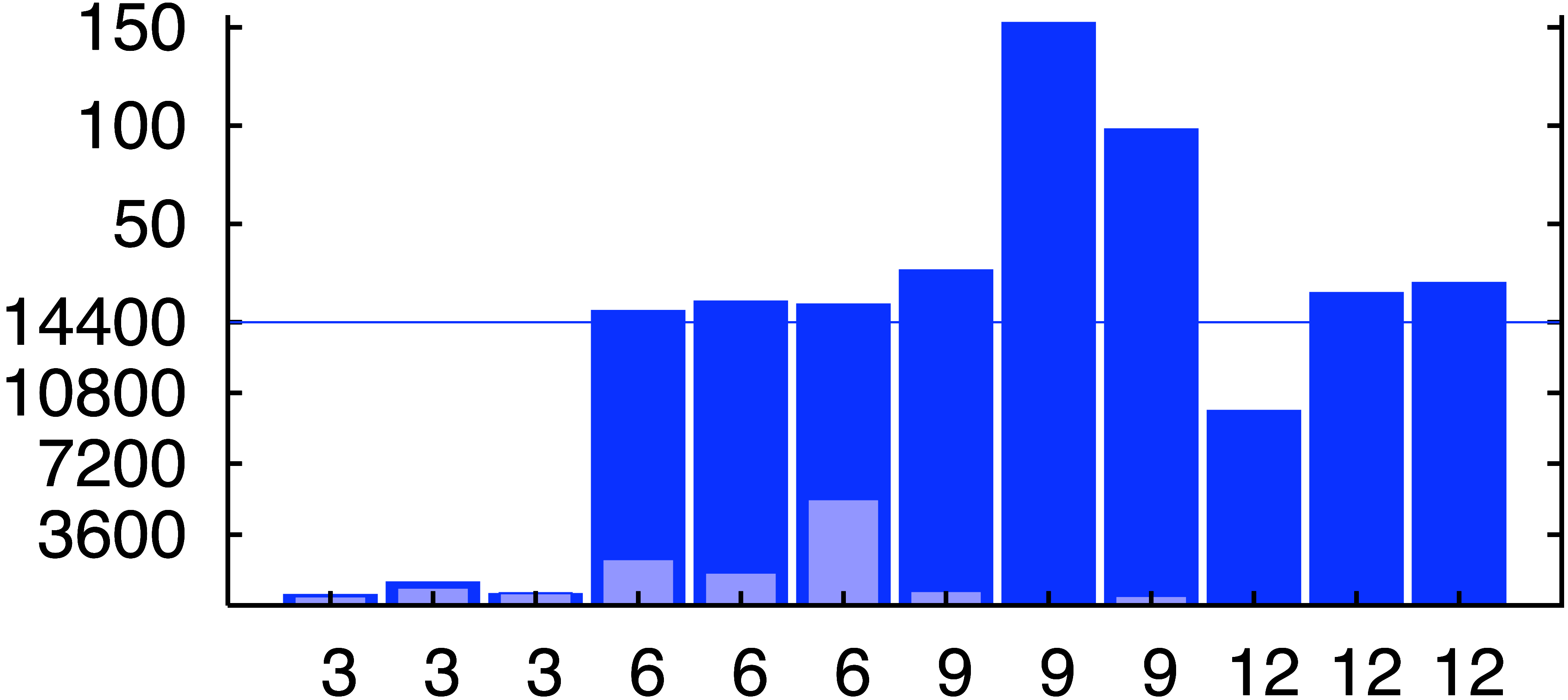
- Graph partitioning problem
- Clique inequalities (heuristic separation)
- First index plus reliability branching [Achterberg et al.]
- SCIP [Achterberg], CPLEX 10.01
- Initialization with optimal objective function value
- Triangle fixed to zero



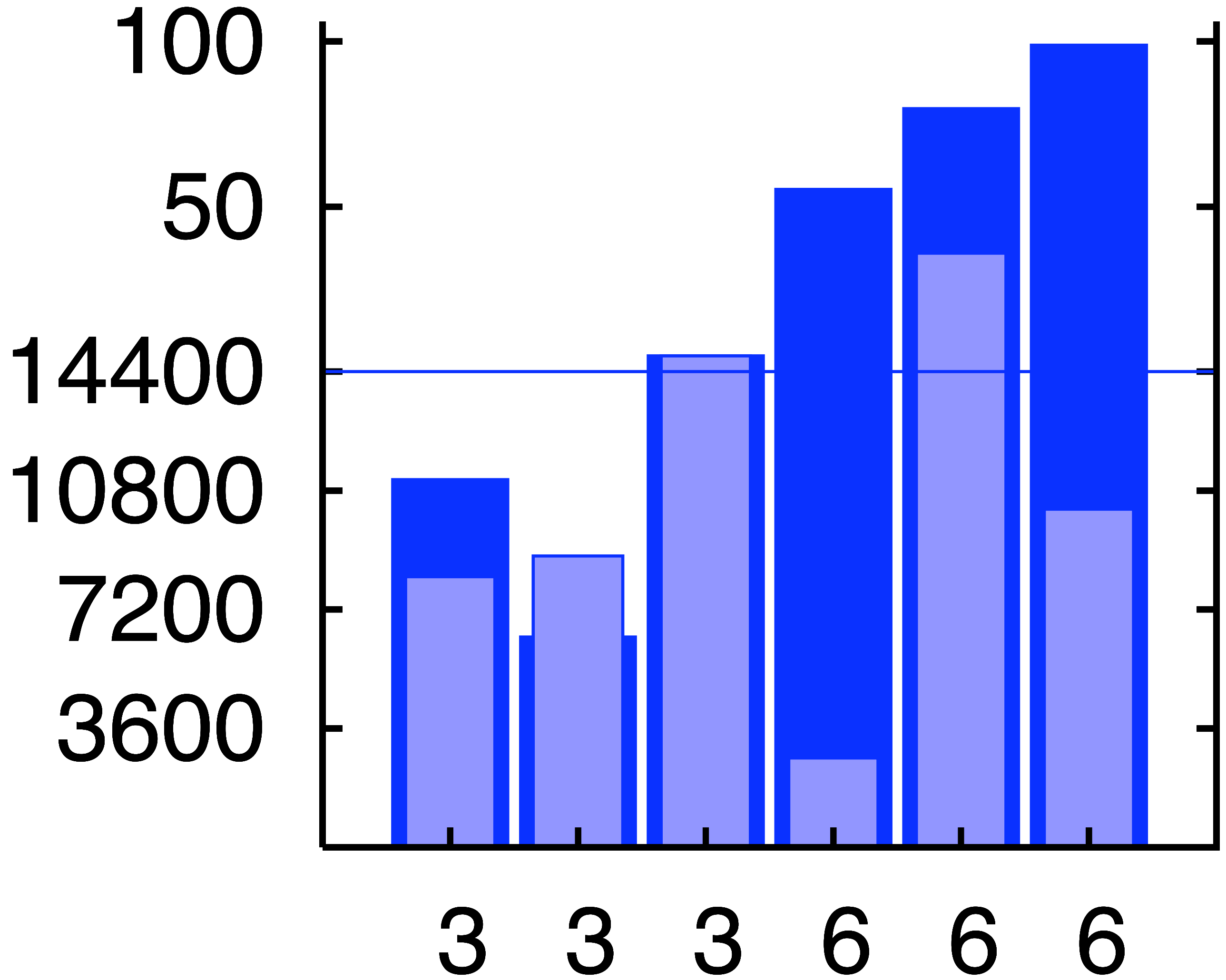
Random Instances with n=30, m=300



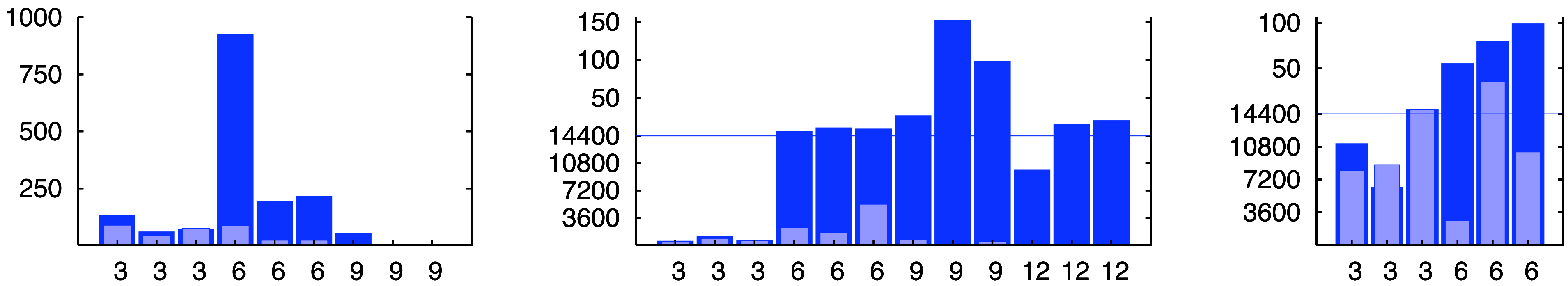
Random Instances with n=30, m=400



Random Instances with n=50, m=560



Summary on Random Instances



- Significant improvements by orbitopal symmetry breaking
- SCI separation slightly better than adding all CIs
- Orbitopal fixing slightly better than SCI separation
- Pure CPLEX: Fixed triangle similar to built-in symmetry breaking

- ☑ Examples: Vertex coloring, graph partitioning
- ☑ Orbitopes
- ☑ Orbitopal fixing
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- ☐ Remarks

Remarks

- Orbitopal Fixing: Plug-and-play tool (SCIP)
- Polyhedral structure of orbitopes:
  - Full/Covering for symmetric groups
  - Products of groups
  - Permuting rows and columns?
- Interplay with specific polyhedra
- Other systems of representatives
- Applications (e.g. metastable states)

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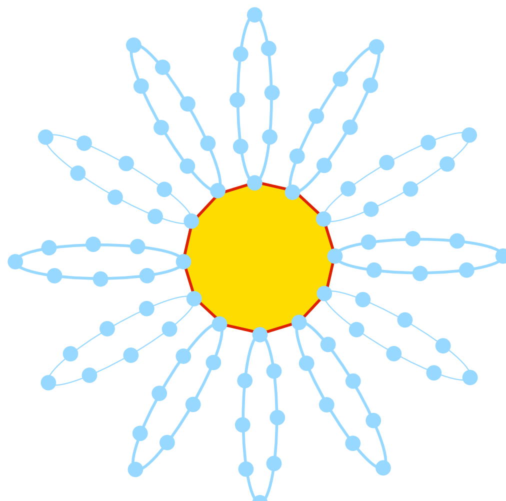

Thank you for your attention.

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