Local Cuts from the Bidirected Graph Relaxation

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The basic Problem

We consider the problem

$$\begin{aligned} \max c_x x + c_y y \\ s.t. Ax + By \leq b \\ x \in \{0,1\}^{n_1}, y \in \mathbb{R}_+^{n_2} \end{aligned} \quad (1)$$

where $A \in \mathbb{Q}^{m \times n_1}$, $B \in \mathbb{Q}^{m \times n_2}$ and $b \in \mathbb{Q}^m$.

- Solved with branch and cut.
- Lots of family of cuts.
- Few general purpose cuts.
- Numerical Issues.

Questions and Goal

- How far can we go defining templates?
- Do we need more family of cuts?
- Is there a way out for both issues?

Christof and Reinelt Results [CR01]:

Facet Structure for TSP_n

n	Vertices	Facets	Classes
3	1	0	0
4	3	3	1
5	12	20	2
6	60	100	4
7	360	3,437	6
8	2,520	194,187	24
9	20,160	42,104,442	192
10	181,440	$\geq 51,043,900,866$	$\geq 15,379$

Christof and Reinelt Results [CR01]:

Facet Structure for LOP_n

n	Vertices	Facets	Classes
3	6	8	2
4	24	20	2
5	120	40	2
6	720	910	5
7	5,040	87,472	27
8	40,320	$\geq 488,602,996$	$\geq 12,231$

Christof and Reinelt Results [CR01]:

Facet Structure for $CUTP_n$.

n	Vertices	Facets	Classes
3	4	4	1
4	8	16	1
5	16	56	2
6	32	368	3
7	64	111,764	11
8	128	$\geq 217,093,472$	≥ 147
9	256	$\geq 12,246,651,158,320$	$\geq 164,506$

Christof and Reinelt Results [CR01]:

0-1 polytopes with many facets

Dimension d	Vertices v	Facets f	$\sqrt[d]{f}$
6	18	121	2.22
7	30	432	2.37
8	38	1,675	2.52
9	48	6,875	2.66
10	83	41,591	2.89
11	106	250,279	3.09
12	152	$\geq 1,975,937$	≥ 3.34
13	254	$\geq 17,464,365$	≥ 3.60

Other negative results:

- Given $P \subseteq \{0,1\}^n$ define $f(P) = \#$ facets of P, $f_n = \max\{f(P) : P \subseteq \{0,1\}^n\}$, then:
  - $f_n \leq 30(n-1)!$ [FKR00].
  - $f_n \geq \left(\frac{cn}{\log n}\right)^{n/4}$ for some positive constant c [BP01].
  - $f_n \geq \left(\frac{cn}{\log^2 n}\right)^{n/2}$ for some positive constant c [GGM05].
- Let Pn, N be the convex hull of N randomly chosen points in $B(0,1) \subset \mathbb{R}^n$, then:
  - $\left(c_1 \log \frac{N}{n}\right)^{n/2} \leq \mathbb{E}[f(P_{n,N})] \leq \left(c_2 \log \frac{N}{n}\right)^{n/2}$ for some $c_i > 0$, $n \leq N \leq 2^n$ [BMT85].
- There are too many!

How far could we go if we disregard precision?

- Use as benchmark computational results by:
  - Dash and Gunluk [DG06] (GMI).
  - Fischetti and Monaci [FM05] (GMI2).
  - Fischetti and Saturni [FS05] implementation of the K-cuts of Cornuéjols et al. [CLV03] (K-cuts).
  - Fischetti and Monaci [FM05] results using their Corner relaxation cuts (Corner).

How far could we go if we disregard precision? (continued)

- In our implementation (Iterated): allow any rank and mixture of
    - Sequence independent Lifted covers of Gu et al. [GNS00].
    - GMI cuts with complemented variables from single tableau rows.
    - K-cuts with $k = 1, \ldots, 15$.
    - GMI cuts from aggregation of two tableau rows.
    - Use rational representation of cuts with up-to 256 bits representation (cuts not representable in this form are discarded).

GAP closed at root node

Name	GMI1	GMI2	K-cuts	Corner	Iterated	Avg. Frac
10teams	57.14	57.14	100.00	0	42.85	0.21
aflow30a	-	10.87	11.15	37.08	24.74	0.18
aflow40b	-	9.79	5.48	26.08	19.85	0.15
air03	100.00	100.00	100.00	100.00	100.00	0.00
bell3a	45.10	52.92	60.43	83.79	71.11	0.33
bell5	14.53	84.90	14.53	5.05	89.03	0.10
blend2	15.98	16.48	0.00	37.93	15.93	0.30
danoint	1.73	0.24	1.74	1.74	0.00	0.11
cap6000	41.65	41.65	41.65	34.51	26.43	0.13
fiber	63.09	57.68	54.28	80.22	88.05	0.18
fixnet6	12.88	11.49	10.65	80.63	67.32	0.10
gen	60.69	64.97	59.77	38.26	70.57	0.23
gesa2	28.53	30.83	30.17	97.64	48.88	0.16
gesa2_o	31.03	31.03	30.29	97.64	47.32	0.19
gesa3	47.53	53.26	47.56	70.19	50.10	0.18
gesa3_o	50.54	54.30	60.53	70.19	46.24	0.19
harp2	24.07	28.62	29.43	15.69	28.39	0.25
manna81	100.00	29.19	100.00	57.14	100.00	0.00
marketshare1	0.00	0.00	0.00	0.00	0.00	0.21
marketshare2	0.00	0.00	0.00	0.00	0.00	0.26
mas74	6.67	6.67	7.27	33.14	0.00	0.25
mas76	6.42	6.42	7.02	33.53	0.00	0.23
mkc	6.83	7.96	6.62	0.00	4.11	0.20

GAP closed at root node

Name	GMI1	GMI2	K-cuts	Corner	Iterated	Avg. Frac
mod010	100.00	100.00	21.47	100.00	100.00	0.08
mod011	31.15	-	-	-	24.02	0.13
modglob	17.28	17.28	17.28	62.88	24.74	0.07
net12	-	27.89	7.07	6.37	26.72	0.34
noswot	100.00	0.00	0.00	0.00	0.00	0.17
opt1217	-	19.61	19.74	0.00	32.39	0.28
p0033	56.82	56.82	56.85	31.86	81.41	0.11
p0201	26.71	20.27	17.96	0.00	52.39	0.19
p0282	3.70	3.70	3.70	9.28	91.12	0.12
p0548	39.20	60.59	40.04	0.02	97.49	0.27
p2756	0.54	3.20	0.61	0.00	97.77	0.16
pk1	0.00	100.00	0.00	0.00	0.00	0.21
pp08a	54.42	53.48	52.22	24.30	81.40	0.16
pp08aCUTS	33.79	32.83	31.32	40.32	63.66	0.16
qiu	2.53	2.36	0.93	0.00	93.23	0.25
qnet1	11.91	7.44	9.87	64.31	17.41	0.19
qnet1_o	42.99	37.77	42.02	64.80	47.92	0.18
rentacar	28.07	34.62	15.53	0.00	30.19	0.06
rgn	3.15	5.02	10.37	0.00	29.02	0.24
set1ch	39.16	38.91	39.16	71.60	73.05	0.16
seymour	7.69	8.33	6.75	11.24	12.21	0.26
timtab1	-	24.08	23.51	28.17	31.08	0.24
vpm1	22.91	17.09	15.86	5.45	55.94	0.14
vpm2	10.93	8.25	10.36	17.35	31.61	0.24

What can we say?

- In most cases, we can reduce the gap using existing techniques.
- Finding violated inequalities is easy.
- Cases we don’t improve is because rational representation is too large (above 512 bits).
- Procedure can generate thousands of cuts.
- Active cuts no more than 300.
- Need a dynamic cut management scheme as in Applegate et al. [ABCC03].
- Problem is reliability (correctness) of the cuts added.

What are we looking for?

- Use idea of Applegate et al. [ABCC01] of using simpler relaxations to generate cut explicitly.
    - Use equivalence of separation/optimization.
    - Using an optimization oracle, find separating facets of a point and a polyhedron.
- Need to find a combinatorial, simple, and rich relaxation for general MIPS.
    - Use bidirected graph relaxation as defined by E.L. Johnson and M.W. Padberg [JP81].
    - It has nice properties.
    - It is an exact relaxation for vertex cover and vertex partition problems.
    - Contains clique table relaxation.

The Bidirected Graph:

- Given $P := \{x, y : x \in \{0,1\}^{n_1}, y \in \mathbb{R}^{n_2}_{+}, Ax + By \leq b\}$
- Define $G(V, E)$ where $V = \{v_i^o, v_i^1\}_{i=1}^{n_1}$, and where $(v_i^k, v_j^m) \in E$ if and only $\forall (x, y) \in P$ such that $x_i = k$ we have that $x_j = m$.
    - We say that $I \subseteq V$ is independent $\Leftrightarrow \forall u, v \in I, (u, v), (v, u) \notin E$.
    - Note that $\forall (x, y) \in P$ we have that $I = \{v_i^{x_i}\}_{i=1}^{n_1}$ is an independent set of size n_1.
    - We relax P to the set of independent sets of size n_1 of G.
    - Any sub-graph of G is a valid relaxation for P.
    - Can be (partially) derived from the original formulation using simple pre-processing techniques.

Initial Numerical Experience

cuts from the bidirected graph relaxation

Instance	GMI1	GMI2	Iterated	BD_LC	Ncuts	Time
gen	60.69	64.97	70.57	45.09	7	$10^{5.05}$
gesa2	28.53	30.83	48.88	8.42	3	$10^{2.97}$
mkc	6.83	7.96	4.11	0	40	$10^{4.26}$
p0282	3.70	3.70	91.12	85.86	13	$10^{2.05}$
p0548	39.20	60.59	97.49	59.31	11	$10^{3.72}$
p2756	0.54	3.20	97.77	*2.34	10	$10^{3.43}$
roll3000	-	-	20.85	0.11	91	$10^{4.39}$
seymour	7.69	8.33	12.21	8.53	14	$10^{4.46}$

How nice where the cuts?

- Of all cuts (203), 91% where generalized clique inequalities.
- All left hand side where ± 1.
- The relaxation can be very weak.
- Current implementation is very slow.

Final Comments:

- Proposed relaxation can bridge a lot of the GAP closed with traditional techniques.
- At the same time, it can be very weak!
- Cuts found where always very nice numerically speaking.
- Used relaxation has a lot of combinatorial structure.
- We could strengthen the relaxation by adding variables to the original problem depending on the fractional values in the current LP.
- Can they work in practice?
- Can we incorporate continuous variables?

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