An LP-based approach to the Train Unit Assignment Problem

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Input:

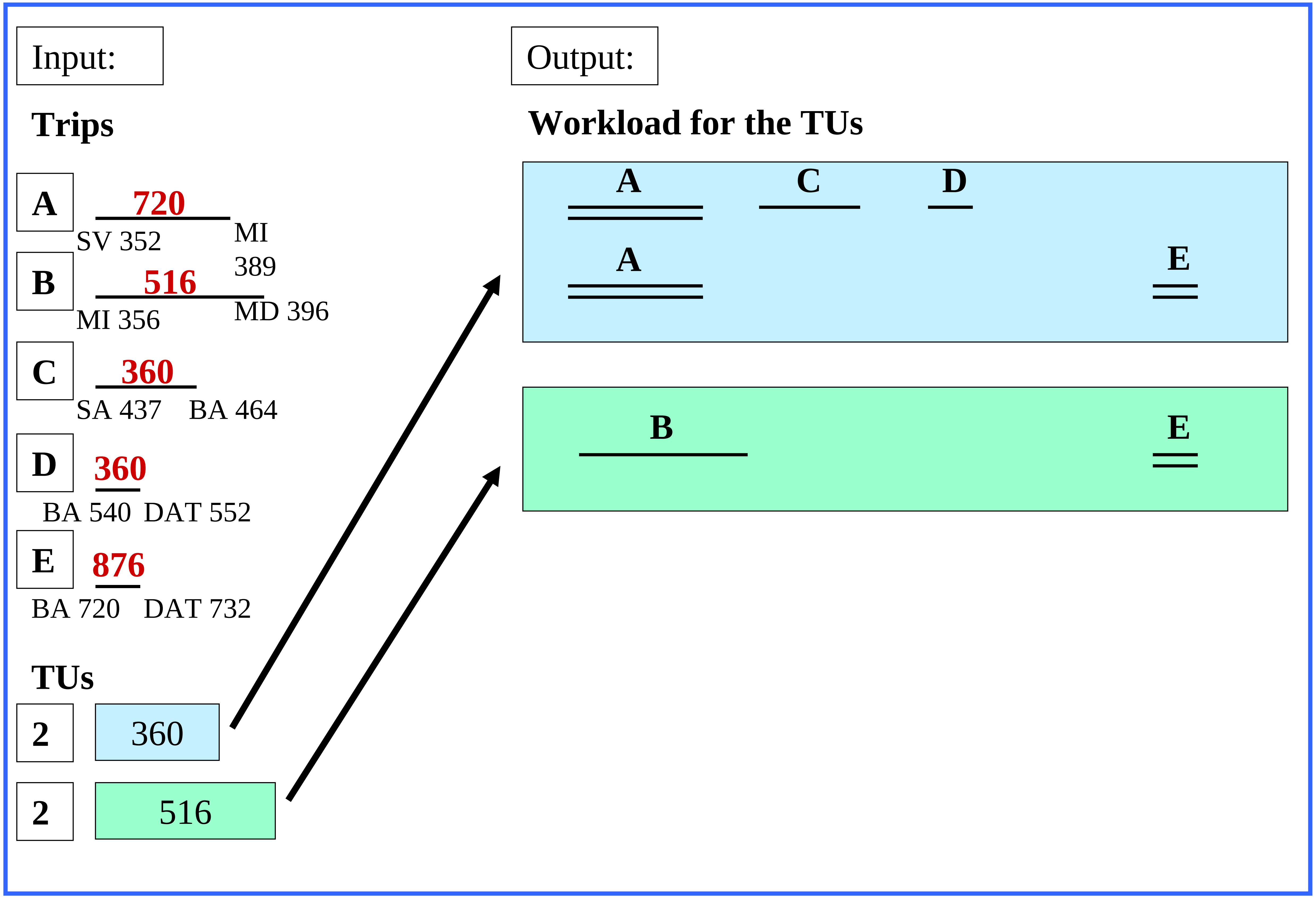
- timetabled train trips with a required number of passenger seats
- train unit types

Output:

Find a minimum cost assignment of the train units (TUs) to the trips, satisfying a set of **constraints**

Constraints:

- required number of passenger seats for each trip (possibly combining TUs)
- maximum number of TUs that can be assigned to each trip (generally 2)
- maximum number of TUs available
- sequencing constraints between trips


Input:
Trips
A
720
SV 352
MI 389
B
516
MI 356
MD 396
C
360
SA 437
BA 464
D
360
BA 540
DAT 552
E
876
BA 720
DAT 732
TUs
2
360
2
516
Output:
Workload for the TUs
A
C
D
A
E
B
E

Graph Representation

n # *of trips*

p # *of train unit types*

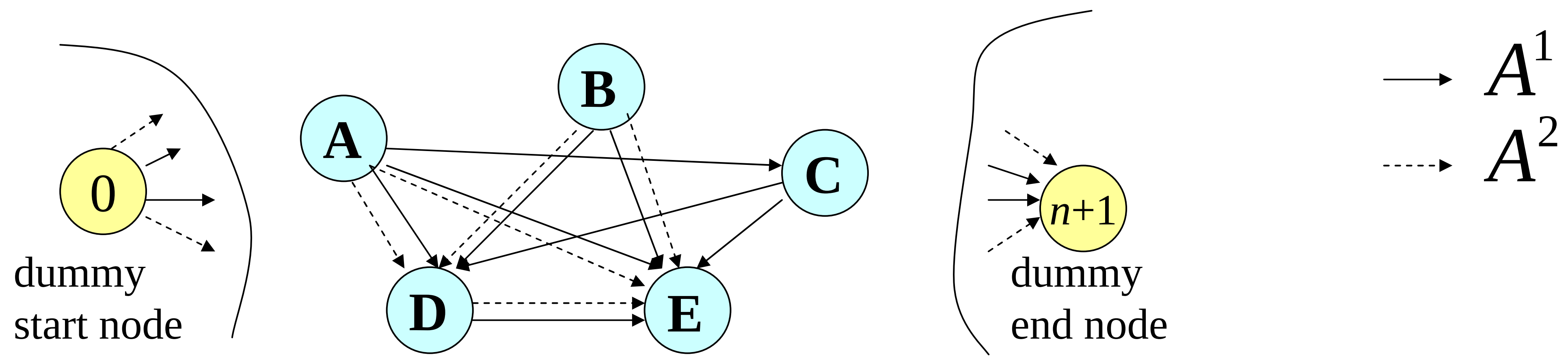
$G = (V, A)$ directed acyclic multigraph

V each ***node*** corresponds to a trip (and dummy nodes 0 and n+1)

A is partitioned into p subsets $A^1 \cup A^2 \cup \ldots \cup A^p$

A^k is associated with TUs of type k

arc $(i, j)^k \in A^k$ exists iff a TU of type k can be assigned to i and then to j within the same day



ILP Formulation

P^k **collection of paths** from 0 to n+1 in (V, A^k) $k = 1,\dots, p$

x_P integer variable that denotes the **number of times that path *P* is selected in the solution**

$P \in P^k, k = \{1,\dots, p\}$

$P_j^k \subseteq P^k$ subcollection of paths in P^k that visit trip *j*,

$k = 1,\dots, p, \quad j = 1,\dots, n$

r_j required # of passenger seats

u_j maximum # of TUs that can be assigned to the trip

d^k # of available TUs

c^k cost for each TU used

s^k capacity

$$\min \sum_{k=1}^{p} \sum_{P \in P^k} c^k x_P,$$

min of the total cost of the paths

$$\sum_{P \in P^k} x_P \le d^k, \quad k = 1,\dots, p,$$

max # of paths for TU of type *k*

$$\sum_{k=1}^{p} \sum_{P \in P_j^k} s^k x_P \ge r_j, \quad j = 1,\dots, n,$$

capacity constraints

$$\sum_{k=1}^{p} \sum_{P \in P_j^k} x_P \le u_j, \quad j = 1,\dots, n,$$

max # of TU for trip *j*

$$x_P \ge 0, \text{ integer}, \ , k = 1,\dots, p, \quad P \in P^k.$$

# of times that path *P* is selected in the solution

Column Generation

The model has an exponential number of variables and can be solved by column generation techniques.

$J_P \subseteq \{1,\dots,n\}$ set of trips visited by a path $P \in P^k$

Dual Problem

$$\max -\sum_{k=1}^{p} d^k \alpha^k + \sum_{j=1}^{n} r_j \beta_j - \sum_{j=1}^{n} u_j \gamma_j,$$

$$-\alpha^k + \sum_{j \in J_P} s^k \beta_j - \sum_{j \in J_P} \gamma_j \le c^k, \quad k = 1,\dots,p, \quad P \in P^k,$$

$$\alpha^k, \beta_j, \gamma_j \ge 0, \quad k = 1,\dots,p, \quad j = 1,\dots,n.$$

The column generation problem calls for $k \in \{1,\dots,p\}$ and $P \in P^k$ such that

$$\sum_{j \in J_P} (s^k \overline{\beta_j} - \overline{\gamma_j}) > c^k + \overline{\alpha}^k$$

It can be solved as a **maximum profit path** from 0 to n+1 in (V, A^k) with node profits $s^k \overline{\beta}_j - \overline{\gamma}_j$ for each $j \in V \setminus \{0, n+1\}$

"Strong" capacity constraints

$u_j = 2, \quad j = 1,\ldots,n$

<u>An example</u>

$r_j = 1032$

$p = 8$

$$w_j^k \equiv \sum_{P \in P_j^k} x_P$$

number of TUs of type *k* assigned to a trip *j*

$k = (1,\ldots,p), \quad j = (1,\ldots,n)$

$$1150 w_j^1 + 1044 w_j^2 + 786 w_j^3 + 702 w_j^4 + 543 w_j^5 + 517 w_j^6 + 515 w_j^7 + 360 w_j^8 \geq 1032$$

weak

$$2w_j^1 + 2w_j^2 + w_j^3 + w_j^4 + w_j^5 + w_j^6 + w_j^7 + w_j^8 \geq 2,$$

~~$2w_j^1 + 2w_j^2 + 2w_j^3 + w_j^4 + w_j^5 + w_j^6 + w_j^7 + w_j^8 \geq 2,$~~

$$2w_j^1 + 2w_j^2 + 2w_j^3 + 2w_j^4 + w_j^5 + w_j^6 + w_j^7 \geq 2,$$

~~$2w_j^1 + 2w_j^2 + 2w_j^3 + 2w_j^4 + 2w_j^5 + w_j^6 + w_j^7 \geq 2,$~~

$$2w_j^1 + 2w_j^2 + 2w_j^3 + 2w_j^4 + 2w_j^5 + 2w_j^6 \geq 2$$

strong

“Strong” capacity constraints

The described capacity constraints are the following:

$$\sum_{k=1}^{p} s^k w_j^k \ge r_j, \quad j = (1,\dots, n),$$

$$\sum_{k=1}^{p} w_j^k \le u_j, \quad j = (1,\dots, n)$$

$$u_j = 2, \quad j = 1,\dots, n$$

Assuming $s^1 \ge s^2 \ge \dots \ge s^p \ge s^{p+1} = 0$

For a trip j define:

h_j such that $s^{h_j} \ge r_j, \quad s^{h_j+1} < r_j$

t_j such that $2s^{t_j} \ge r_j, \quad 2s^{t_j+1} < r_j$

$f_j(k)$ such that $s^k + s^{f_j(k)} \ge r_j, \quad s^k + s^{f_j(k)+1} < r_j$

$$\sum_{l=1}^{k-1} 2w_j^l + \sum_{l=k}^{f_j(k)} w_j^l \ge 2,$$

$$j = 1,\dots, n \quad k = h_j + 1,\dots, t_j + 1$$

The derived strong inequalities describe the convex hull of the capacity constraints.

An LP-based Heuristic Algorithm

1. solve the current LP (CPLEX) ⇒ z^*
2. constructive heuristic (based on the optimal dual solution)
3. refine the solution found
4. if there are dual constraints violated ⇒ add some of the corresponding primal variables to the current LP

   else ⇒ fixing
5. if the current LP is infeasible ⇒ stop

   if $z^* =$ value of the incumbent solution ⇒ stop

   else goto 1.

Experimental Results on the Case Study

528 trips

8 train unit types

Methods implemented in C, computational tests on a Pentium IV, 3.2 GHz, 1 Gb Ram, Cplex 9.0

Practitioners' solution 72

model	**LB**	**time** (sec)
LP rel.	**57**	414
LP rel. (strong)	**62**	125

Heur	**time** (sec)
63	1027

Future research

- consider other real-life constraints in the problem (maintenance, cyclicity)
- find robust solutions w.r.t. uncertainties in the timetables of the trips due to possible delays
- test the algorithm on other real-life instances