# Projecting Systems of Linear Inequalities in Binary Variables

Egon Balas
Carnegie Mellon University

# Projecting out continuous variables from a system of linear inequalities

$$Q' = \{(u, x) \in \mathbb{R}^{p+q} : Au + Bx \leq d,\ x \in T\},\ A_{m\times p}, B_{m\times q}$$

- The projection of $Q'$ onto the $x$-space is defined as

$$P_x(Q') := \{x \in \mathbb{R}^q \cap T : \exists u \in \mathbb{R}^p \text{ with } (u, x) \in Q'\}$$

- Projection cone:

$$W' = \{v \in \mathbb{R}^m : vA = 0,\ v \geq 0\}$$

- $P_x(Q') = \{x \in \mathbb{R}^q \cap T : (vB)x \leq vd \text{ for all } v \in \text{extr } W'\}$

  (Special case when $A$ has a single column: Fourier elimination)

# Projecting out 0-1 variables from a system of linear inequalities

$$Q := \{(u,x) \in \mathbb{R}^{p+q} : Au + Bx \le d,\ u \in \{0,1\}^p,\ x \in T\}$$

- The projection of $Q$ onto the $x$-space is defined as

$$P_x(Q) := \{x \in \mathbb{R}^q \cap T : \exists u \in \{0,1\}^p \text{ with } (u,x) \in Q\}$$

In general, $P_x(Q)$ is nonconvex; we are also interested in

$$P_x^c(Q) := \text{ conv } P_x(Q).$$

In general,

$$P_x^c(Q) \underset{\neq}{\subset} P_x(Q')$$

Example:

$$\begin{array}{rcrcrcl} u & + & x_1 & & & = & 1 \\ & & x_1 & + & x_2 & = & 1 \\ u & & & + & x_2 & = & 1 \\ & & & & u, x_1, x_2 & \geq & 0 \end{array} \tag{1}$$

$$Q' = \{(u, x_1, x_2) \in \mathbb{R}^3 : (1) \text{ holds}\}$$

$$Q = \{(u, x_1, x_2) \in \mathbb{R}^3 : (1) \text{ holds}, u \in \{0, 1\}\}$$

$$P_x(Q') = \left\{ \left( \frac{1}{2}, \frac{1}{2} \right) \right\}; \quad P_x(Q) = \{\emptyset\}$$

# Projecting out $u \in \{0,1\}^p$

Let $a_j$ be the $j$-th column of $A$.
For any $S \subseteq \{1, \ldots, p\}$, let $a(S) := \sum(a_j : j \in S)$
Let $\mathcal{S} := \{S \subseteq \{1, \ldots, p\},\ S \neq \emptyset\}$
Theorem. If $T$ is convex, then

$$P_x^c(Q) = \{x \in \mathbb{R}^q \cap T : (vB)x \leq vd \text{ for all } v \in \text{ extr}\, W\}$$

where

$$W = \left\{ v \in \mathbb{R}^m \;\middle|\; \begin{array}{lclcll} vB & - & w^S B & = & 0 & S \in \mathcal{S} \\ vd & - & w^S(d - a(S)) & = & 0 & S \in \mathcal{S} \\ & & v, w^S \geq 0,\ S \in \mathcal{S} & & & \end{array} \right\}.$$

# Proof outline.

$(u, x) \in Q$ iff $x \in \mathbb{R}^q \cap T$ satisfies

$$\bigvee_{S \in \mathcal{S}'} (Bx \leq d - a(S)) \tag{1}$$

where $\mathcal{S}' := \mathcal{S} \cup \{\emptyset\}$, i.e.

$$P_x(Q) = \{x \in \mathbb{R}^q \cap T : Bx \leq d - a(S) \text{ for some } S \in \mathcal{S}'\}.$$

Hence

$$\begin{aligned} P_x^c(Q) = \{x \in \mathbb{R}^q \cap T : x \quad -\sum (y^S : S \in \mathcal{S}') &= 0 \\ By^S - (d - a(S))y_0^S &\leq 0 \quad S \in \mathcal{S}' \\ \sum_{S \in \mathcal{S}'} y_0^S &= 1\} \end{aligned} \tag{2}$$

Projecting out $(y^S, y_0^S)$, $S \in \mathcal{S}'$, it follows that $x \in P_x^c(Q)$ iff it satisfies

$$\alpha x \leq \beta$$

for every $(\alpha, \beta)$ such that

$$\begin{array}{rcrcll} \alpha & & - & w^S B & = & 0 \\ & -\beta & + & w^S(d - a(S)) & = & 0 \\ & & & w^S \geq 0 & & S \in \mathcal{S}'. \end{array}$$

Setting $v := w^\emptyset$ and substituting $vB$ for $\alpha$ and $vd$ for $\beta$ yields $W$.
$\square$

# Easy case

- $Q$ is down-monotone in $u$:

$$((u,x) \in Q,\ 0 \le y \le u) \Rightarrow (y,x) \in Q$$

$$(\text{e.g. } A \ge 0)$$

Proposition. If $Q$ is down-monotone in $u$,

$$P_x(Q) = \{x \in \mathbb{R}^q \cap T : Bx \le d\}$$

i.e. $P_x(Q)$ is the restriction of $Q$ to $u = 0$.

# Proof.

- If $\hat{x}$ satisfies $Bx \leq d$, then $(\hat{u}, \hat{x})$, with $\hat{u} = 0$, satisfies $Au + Bx \leq d$; hence $\hat{x} \in P_x(Q)$.
- If $B_i \hat{x} > d_i$ for some $i$, then $A_i u + B_i \hat{x} > d_i$, i.e. $(u, \hat{x}) \notin Q$ for all $u \in \{0, 1\}^p$. □

Example. Set packing:

$$Q = \{(u, x) \in \{0, 1\}^{p+q} : Au + Bx \leq 1\}, \ a_{ij}, b_{k\ell} \in \{0, 1\}$$

$$P_x(Q) = \{x \in \{0, 1\}^q : Bx \leq 1\}$$

# Easy case:

- $Q$ is up-monotone in $u$:

$$((u, x) \in Q,\ \ u \leq y \leq 1) \Rightarrow (y, x) \in Q$$

(where $Q$ is defined with $\geq$ instead of $\leq$).

Proposition. If $Q$ is up-monotone in $u$,

$$P_x(Q) = \{x \in \mathbb{R}^q \cap T : Bx \geq d - A1\},$$

i.e. $P_x(Q)$ is the restriction of $Q$ to $u = 1$.

# Proof.

If $\bar{x}$ satisfies $Bx \geq d - A1$, then $(\bar{u}, \bar{x})$, with $\bar{u} = 1$, satisfies $Au + Bx \geq d$; hence $\bar{x} \in P_x(Q)$.

Conversely, if $\hat{x} \in P_x(Q)$, there exists $\hat{u} \in \{0,1\}^p$ with $(\hat{u}, \hat{x}) \in Q$, hence $\hat{x}$ satisfies $Bx \geq d - A\hat{u} \geq d - A1$. □

Example. Set covering:

$$Q = \{(u,x) \in \{0,1\}^{p+q} : Au + Bx \geq 1\}, \; a_{ij}, b_{k\ell} \in \{0,1\}\}$$

$$P_x(Q) = \{x \in \{0,1\}^q : B_i x \geq 1, \text{ for all } i \text{ s.t. } A_i = 0\}$$

Hard case: Set partitioning

Not monotone

Example: Set packing

$$\begin{array}{llllll} x_1 & & & +x_4 & & \leq 1 \\ & x_2 & +x_3 & & & \leq 1 \\ x_1 & & +x_3 & +x_4 & & \leq 1 \\ x_1 & & & & +x_5 & \leq 1 \end{array}$$

Projecting out $x_1$ and $x_4$ yields

$$\begin{aligned} x_2 + x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_5 &\leq 1 \end{aligned}$$

Example: Set covering

$$\begin{array}{llllll} x_1 & & & +x_4 & & \geq 1 \\ & x_2 & +x_3 & & & \geq 1 \\ x_1 & & +x_3 & +x_4 & & \geq 1 \\ x_1 & & & & +x_5 & \geq 1 \end{array}$$

Projecting out $x_1$ and $x_4$ yields

$$x_2 + x_3 \geq 1$$

# Eliminating one 0-1 variable

$$Q \quad := \quad \{(x_0, x) \in \{0,1\}^{n+1} : a_{i0}x_0 + a_i x \geq b_i, \\ i \in M,\ a_i \in \mathbb{Z}^n,\ i \in M\}$$

$$Q' \quad := \quad \{Q : x_0 \in \{0,1\} \text{ is replaced by } 0 \leq x_0 \leq 1\}$$

$$M^+ \quad := \quad \{i \in M : a_{i0} > 0\}$$

$$M^- \quad := \quad \{i \in M : a_{i0} < 0\}$$

$$M^0 \quad := \quad \{i \in M : a_{i0} = 0\}$$

Fourier elimination:

$$\begin{aligned} P_x(Q') = \{x \in \{0,1\}^n : a_i x \quad &\geq \quad b_i - a_{i0}, && i \in M^+ \\ a_i x \quad &\geq \quad b_i, && i \in M^- \cup M_0 \\ \sum_{j=1}^{n} (a_{i0}a_{kj} - a_{k0}a_{ij})x_j \quad &\geq \quad a_{i0}b_k - a_{k0}b_i, && i \in M^+, k \in M^-\} \end{aligned}$$

Let $F(Q') := P_x(Q') \setminus P_x(Q)$

Theorem. $F(Q')$ is the set of those $x \in P_x(Q')$ for which

$\exists i \in M^+$, $k \in M^-$ such that $a_{i0} \geq 2$, $a_{k0} \leq -2$, and

$$\begin{array}{rcccccl} 1 & \leq & b_i & - & a_i x & \leq & a_{i0} + \lfloor a_{i0}/a_{k0} \rfloor \\ a_{k0}+1 & \leq & b_k & - & a_k x & \leq & \lfloor a_{k0}/a_{i0} \rfloor \end{array} \qquad (3)$$

Corollary. If $a_{i0} \in \{1, -1, 0\}$ for all $i \in M$, then $F(Q') = \emptyset$, i.e.

$$P_x(Q') = P_x(Q)$$

# Why Fourier elimination does not work for Set Partitioning

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | | 1 | | | = | 1 |
| | 1 | | 1 | | 1 | | = | 1 |
| | | 1 | 1 | | | 1 | = | 1 |
| 1 | | 1 | | 1 | | | $\le$ | 1 |
| | | | 1 | | 1 | | $\le$ | 1 |
| | | 1 | 1 | | | 1 | = | 1 |
| | | 1 | −1 | 1 | −1 | | = | 0 |
| 2 | | | | 1 | | | $\le$ | 1 |
| | | | 1 | | | 1 | $\le$ | 1 |
| | | | −1 | 1 | −1 | | $\le$ | 0 |
| | | | 1 | | 1 | | $\le$ | 1 |
| | | | 2 | −1 | 1 | 1 | $\le$ | 1 |
| | | | −2 | 1 | −1 | −1 | $\le$ | 1 |

# Projection in Propositional Calculus

- Satisfiability:
  Is a set of clauses $C_i$, $i \in M$, in literals $x_j, \bar{x}_j$, $j \in N$,

$$C_i \qquad x_1 \vee \bar{x}_2 \vee \cdots \vee x_n \qquad i \in M$$

  satisfiable?

- If $\exists k \in N$ (and no other $h \in N$), s.t.

$$x_k \in C_i,\ \bar{x}_k \in C_j \text{ for some } i \neq j,$$

  then

$$C := C_i \cup C_j \setminus \{x_k, \bar{x}_k\}$$

  is the resolvent of $C_i$ and $C_j$
  Resolution (which replaces $C_i, C_j$ by $C$) is a projection method.

- IP formulation:

$$\sum_{j\in N_i^+} x_j \quad + \quad \sum_{j\in N_i^-} (1-x_j) \quad \geq \quad 1 \qquad i \in M$$
$$x_j \in \{0,1\},\ j \in N$$

If $x_k \in C_i$ and $(1-x_k) \in C_h$, $h \neq i$, then Fourier elimination combines $C_i$ with $C_h$ to obtain

$$\sum(x_j : j \in N_i^+ \cup N_h^+ \setminus \{k\}) + \sum((1-x_j) : j \in N_i^- \cup N_h^- \setminus \{k\}) \geq 1$$

which is the resolvent $C$ of $C_i$ and $C_h$.

# Another representation

$Q := \{(u,x) \in \{0,1\}^{p+q} : Au + Bx = d\},$

$a(S) := \sum(a_j : j \in S), S \subseteq \{1,\dots,p\}$

$$\begin{array}{rcrcll} x & - & \sum(y^S : S \in \mathcal{S}') & = & 0 & \\ & & By^S - (d - a(S))y_0^S & = & 0 & S \in \mathcal{S}' \\ & & \sum_{S \in \mathcal{S}'} y_0^S & = & 1 & \end{array} \qquad (2)$$

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$$\begin{array}{cccccccccl} x - y^{S_1} & & -y^{S_2} & & \dots & -y^{S_q} & & = 0 & & d^S := d - a(S) \\ By^{S_1} & -d^{S_1}y_0^{S_1} & & & & & & = 0 & & \\ & & By^{S_2} & -d^{S_2}y_0^{S_2} & & & & = 0 & & \\ & & & & \ddots & & & \vdots & & \\ & & & & & By^{S_q} & -d^{S_q}y_0^{S_q} & = 0 & & \\ & y_0^{S_1} & & +y_0^{S_2} & \cdots & & +y_0^{S_q} & = 1 & & \{1,\dots,q\} = \mathcal{S}' \end{array}$$

$$
\begin{array}{lllllllll}
x - y^{S_1} & & -y^{S_2} & & \dots & -y^{S_q} & & = 0 & d^S := d - a(S) \\
By^{S_1} & -d^{S_1} y_0^{S_1} & & & & & & = 0 & \\
& & By^{S_2} & -d^{S_2} y_0^{S_2} & & & & = 0 & \\
& & & & \ddots & & & \vdots & \\
& & & & & By^{S_q} & -d^{S_q} y_0^{S_q} & = 0 & \\
& y_0^{S_1} & & +y_0^{S_2} & \dots & & +y_0^{S_q} & = 1 & \{1, \dots, q\} = \mathcal{S}'
\end{array}
$$

The system (2) implies

$$
\begin{aligned}
Bx - \sum(d^S \lambda_S : S \in \mathcal{S}') &= 0 \\
\sum(\lambda_S : S \in \mathcal{S}') &= 1 \qquad (3) \\
\lambda_S &\geq 0, \; S \in \mathcal{S}'
\end{aligned}
$$

When are (2) and (3) equivalent?

Let $B_{m \times q}, \quad m \leq q$

Theorem. (2) and (3) are equivalent if
for every $m \times m$ nonsingular submatrix $D$ of $B$
and every convex combination $d(\lambda)$ of $d^S$ with weights $\lambda_S, S \in \mathcal{S}'$,

$$D^{-1}d(\lambda) \geq 0 \text{ implies } D^{-1}d^S \geq 0 \text{ whenever } \lambda_S > 0.$$

Cases when the condition holds:

- $x$ belongs to one of several boxes (multidimensional intervals)
- $x$ belongs to one of several simplices
- $x$ belongs to one of several network (or multiple network) polyhedra
- Application: derivation of Ball, Liu and Pulleyblank's compact linear characterization of 2-terminal Steiner-tree polyhedra