Probabilistically Constrained Linear Programming

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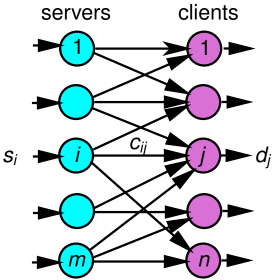
Outline

- An example
- Probabilistically constrained LPs
- A sampling based approximation scheme
- MIP for solving sampled approximations

Example

- m servers, n clients
- Ship from servers to clients to satisfy demand
- Minimize transportation cost



$$
\begin{array}{lll}
\min & \sum_{ij} c_{ij} x_{ij} & \\
\text{s.t.} & \sum_j x_{ij} \leq s_i & \forall\ i \\
& \sum_i x_{ij} \geq d_j & \forall\ j \\
& x_{ij} \geq 0 & \forall\ i, j.
\end{array}
$$

Dealing with Uncertainty

- Client demands $\tilde{d}_j$ are uncertain with a known distribution.
- Decide shipments before demand is realized.
- Modelling options:
  - Replace $\tilde{d}_j$ by a deterministic approximation, e.g., mean or quantile.
  - Satisfy demand with probability one $\rightarrow$ very conservative.
  - Allow shortages but with a quality of service or reliability guarantee.

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& \Pr\{\sum_i x_{ij} \geq \tilde{d}_j & \forall\, j\} \geq 1 - \epsilon \\
& x_{ij} \geq 0 & \forall\, i, j.
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$$

Solutions to the Example

Example with $m = 3$ and $n = 3$. Demand $\tilde{d}_j$ is i.i.d. Uniform.

Deterministic solution:

Demand	$1 - \epsilon$	Cost
mean	13.0 %	636.691
95% quantile	92.0%	874.668
98% quantile	92.5%	879.698

Stochastic solution:

$1 - \epsilon$	Cost
100 %	895.545
95 %	876.985
90 %	862.482

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Probabilistically Constrained Optimization

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & x \in X \end{aligned}$$

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- Many applications: finance, manufacturing, service, security.
- Joint probabilistic constraints.
- Special case: individual probabilistic constraints (somewhat easy).
- Assumption: Only rhs uncertainty, i.e., $\tilde{T}$ is deterministic:

$$\Pr\{Tx \geq \tilde{h}\}.$$

Challenges

1. How to check if a given solution $\hat{x}$ is feasible?
   - Requires calculating $\Pr\{\tilde{h} \leq T\hat{x}\}$.
   - Involves multidimensional integration.
   - Impossible to do exactly unless there are very few possible realizations.
2. How to optimize over the set of feasible solutions?
   - The set of feasible solutions is non-convex.

Some Existing Approaches

Tractable approximations that yield feasible solutions:

- Scenario approximation (Campi & Calafiore, Nemirovski & Shapiro)
- Bernstein approximation (Nemirovski & Shapiro)
- Robust optimization (Ben-Tal & Nemirovski, etc.)
- Conditional Value at Risk (Rockafellar & Uryasev)

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Our context:

- Very high reliability is not absolutely critical
- Decision maker seeks to explore trade-off between reliability and cost

Our Approach

1. Use sampling to simplify probabilistic structure
   - Solve problem(s) based on sample of the random data
   - Should get (near)-feasible solutions
   - Should be able to estimate quality (gaps)

2. Use mixed-integer programming to solve sample approximation
   - Sample approximation problem is *NP*-hard
   - Use and extend results for mixed-integer set known as the mixing set

Sampled Approximation

- Generate an i.i.d sample of of $\tilde{h}$: $h^1, \ldots, h^N$.

$$
\begin{aligned}
(PCP): z^*_\epsilon = \quad & \min \quad cx \\
& \text{s.t.} \quad x \in X \\
& \qquad \Pr\{Tx \geq \tilde{h}\} \geq 1 - \epsilon
\end{aligned}
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\begin{aligned}
(SA): z^N_\alpha = \quad & \min \quad cx \\
& \text{s.t.} \quad x \in X \\
& \qquad \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}\left(Tx \geq h^i\right) \geq 1 - \alpha
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Sampled Approximation

- Generate an i.i.d sample of of $\tilde{h}$: $h^1, \ldots, h^N$.

$$(PCP) : z^*_\epsilon = \min \; cx \quad \text{s.t.} \; x \in X, \; \Pr\{Tx \geq \tilde{h}\} \geq 1 - \epsilon$$

$$(SA) : z^N_\alpha = \min \; cx \quad \text{s.t.} \; x \in X, \; \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}\left(Tx \geq h^i\right) \geq 1 - \alpha$$

Remarks

- α not necessarily equal to ϵ
- $\alpha < \epsilon \Rightarrow$ Feasible solution
- $\alpha = 0 \Rightarrow$ Scenario approximation (Campi & Calafiore, Nemirovski & Shapiro)
- $\alpha > \epsilon \Rightarrow$ Lower bound

Feasibility: Exponential Convergence

X_ϵ: Set of feasible solutions to PCP
X_α^N: Set of feasible solutions to SA

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Theorem

Suppose $\alpha < \epsilon$. If X is finite and/or if h has a finite support Ξ, then

$$Pr\{X_\alpha^N \subseteq X_\epsilon\} \geq 1 - \max\{|X|, |\Xi|\} \exp(-2N(\epsilon - \alpha)^2)$$

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Suppose $\alpha < \epsilon$*. If X is finite and/or if h has a finite support* Ξ*, then*

$$Pr\{X_\alpha^N \subseteq X_\epsilon\} \geq 1 - \max\{|X|, |\Xi|\} \exp(-2N(\epsilon - \alpha)^2)$$

If $|X| \leq U^n$ and/or if $|\Xi| \leq V^m$ we obtain confidence $1 - \delta$ if

$$N \geq \frac{1}{(\epsilon - \alpha)^2}\left[\log\left(\frac{1}{\delta}\right) + \max\{n \log U, m \log V\}\right]$$

Remarks

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- Proof idea (for finite X):
  - Hoeffding's inequality: If $Y_1, \ldots, Y_N$ is an i.i.d sample of $\tilde{Y} \in [0, 1]$ and $t > 0$ then $\Pr\{\bar{Y}_N - \bar{Y} \geq t\} \leq \exp\{-2Nt^2\}$
  - For each $x \in X$ let $\tilde{Y} = \mathbb{I}\left(Tx \geq \tilde{h}\right) \ldots$ Apply Hoeffding's inequality point-wise

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- For general X and Ξ similar sample size estimates from different analysis

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  - For each $x \in X$ let $\tilde{Y} = \mathbb{I}\left(Tx \geq \tilde{h}\right) \dots$ Apply Hoeffding's inequality point-wise
- For general X and Ξ similar sample size estimates from different analysis
- If $\alpha = 0$ then we get the Campi and Calafiore result

Lower Bounding

Lemma

If $\alpha = \epsilon$ and N "large enough" then

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- Solve M SA problems with $\alpha = \epsilon$
- Let $z_{\epsilon,j}^N$ be optimal value of problem j
- Take $LB = \min\{\hat{z}_{\epsilon,j}^N : j = 1, \ldots, M\}$

Lower Bounding

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- Take $LB = \min\{\hat{z}_{\epsilon,j}^N : j = 1, \ldots, M\}$

Confidence that *LB* is a lower bound:

$$\Pr\{LB \leq z_\epsilon^*\} \geq 1 - (1/2)^M$$

e.g.: $M = 10 \Rightarrow$ Obtain lower bound with probability at least 0.999

The Method

- Solve *M* SA problems with *N* as large as possible and $\alpha \approx \epsilon$
- Conduct single *a posteriori* feasibility test on each resulting solution
- Choose best of feasible solutions found
- If none found, decrease α and repeat

The Method

- Solve *M* SA problems with *N* as large as possible and $\alpha \approx \epsilon$
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How to solve the sample approximation?

Finite Distribution Problem

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & x \in X \\ & \sum_{i=1}^{N} \pi_i \mathbb{I}\left(Tx \geq h^i\right) \geq 1 - \alpha \end{aligned}$$

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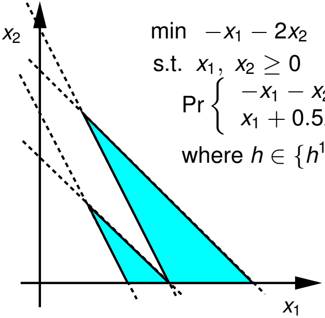
- π_i is the probability of realization h^i
- For SA: $\pi_i = 1/N$ for all i
- Assume $\alpha < 1$ and (w.l.o.g) $h^i \geq 0$ for all i
  $\Rightarrow Tx \geq 0$ for any feasible x

An example:



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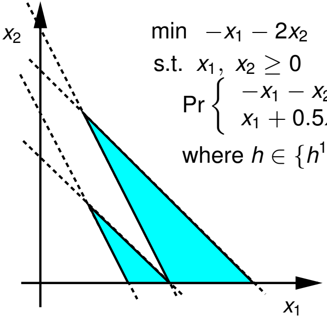
\begin{aligned}
\min \quad & -x_1 - 2x_2 \\
\text{s.t.} \quad & x_1,\ x_2 \geq 0 \\
& \Pr\left\{ \begin{array}{l} -x_1 - x_2 \geq h_1 \\ x_1 + 0.5x_2 \geq h_2 \end{array} \right\} \geq 0.5 \\
& \text{where } h \in \{h^1, \ldots, h^{10}\}
\end{aligned}
$$

Complexity

An example:



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$$\text{where } h \in \{h^1, \ldots, h^{10}\}$$

Theorem

For finite distribution, PCP (and SA) is NP-hard.

MIP Formulation

MIP Formulation

- Introduce binary variables: $z_i = 0$ enforces $Tx \geq h^i$.

$$
\begin{aligned}
\min_{x,v,z} \quad & cx \\
\text{s.t.} \quad & x \in X, \quad Tx = v \\
& v + h^i z_i \geq h^i \quad i = 1, \ldots, N \\
& \sum_{i=1}^{N} \pi_i z_i \leq \alpha \\
& z_i \in \{0, 1\} \quad i = 1, \ldots, N
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- Approach: study one row of the probabilistic constraint

$$
G_j = \Big\{ (v_j, z) : v_j + h^i_j z_i \geq h^i_j \quad i = 1, \ldots, N \\
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One row of the probabilistic constraint

$$G_j = \left\{ (v_j, z) : \quad \begin{array}{ll} v_j + h^i_j z_i \geq h^i_j & i = 1, \dots, N \\ \sum_{i=1}^{N} \pi_i z_i \leq \alpha & \end{array} \right\}$$

- Let $y = v_j$ and $h_i = h^i_j$ for $i = 1, \dots, N$
- Assume $h_1 \geq h_2 \geq \cdots \geq h_N$

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With new notation

$$G = \left\{(y, z) \in \mathbb{R} \times \mathbb{B}^N : \quad \begin{array}{ll} y + h_i z_i \geq h_i & i = 1, \ldots, N \\ \sum_{i=1}^{N} \pi_i z_i \leq \alpha & \end{array} \right\}$$

We will use results for the *mixing set*

Mixing set

$$P = \{(y, z) \in \mathbb{R} \times \mathbb{B}^N : \quad y + h_i z_i \geq h_i \quad i = 1, \dots, N\}$$

- Valid inequalities (Atamtürk, et. al (2000), Günlük and Pochet (2001))
- Extended formulation (Miller and Wolsey, 2003)

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Single row of the probabilistic constraint: mixing set + knapsack

- Be careful applying mixing set results
- New valid inequalities using knapsack

Valid inequalities

Theorem (Atamtürk, et. al (2000), Günlük and Pochet (2001))

The convex hull of the mixing set is characterized by the mixing or star inequalities:

$$y + \sum_{j=1}^{l} (h_{t_j} - h_{t_{j+1}}) z_{t_j} \geq h_{t_1} \quad \forall T = \{t_1, \ldots, t_l\} \subseteq \{1, \ldots, N\}$$

where $h_{t_{l+1}} := 0$.

Valid for G, but do not make use of the knapsack inequality.

Using the Knapsack Inequality

Single row of the probabilistic constraint

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- $p := \max\{k : \sum_{i=1}^{k} \pi_i \leq \alpha\}$ (e.g. $\pi_i = 1/N \;\Rightarrow\; p = \lfloor N\alpha \rfloor$)
- Cannot have $z_1 = z_2 = \cdots = z_{p+1} = 1$

Using the Knapsack Inequality

Single row of the probabilistic constraint

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- Cannot have $z_1 = z_2 = \cdots = z_{p+1} = 1 \Rightarrow y \geq h_{p+1}$
- Can replace mixed-integer inequalities with

$$y + (h_i - h_{p+1}) z_i \geq h_i \quad i = 1,\ldots,p$$

Using the Knapsack Inequality

New formulation for single row of the probabilistic constraint

$$G = \Big\{(y, z) : \quad y + (h_i - h_{p+1})z_i \geq h_i \quad i = 1, \ldots, p$$
$$\sum_{i=1}^{N} \pi_i z_i \leq \alpha \Big\}$$

- Stronger and more compact
- Apply star inequalities to the stronger mixing set
  $\Rightarrow$ facet-defining for conv(G)

Extended formulation for mixing set

Strengthened mixing set

$$P = \{(y, z) : \quad y + (h_i - h_{p+1})z_i \geq h_i \quad i = 1, \ldots, p\}$$

Extended Formulation: new binary variables w_i, $i = 1, \ldots, p$

$$E = \{(y, w, z) : \quad w_i \leq z_i \qquad i = 1, \ldots, p$$
$$w_1 \geq w_2 \geq \cdots \geq w_p$$
$$y + \sum_{i=1}^{p} (h_i - h_{i+1})w_i \geq h_1 \quad \}$$

Validity: $P = \text{Proj}_{(y,z)}(E)$

Strength of the Extended Formulation

Theorem (Miller and Wolsey, 2003)

The linear programming relaxation of the extended formulation is equivalent to the original MIP formulation relaxation with all star inequalities.

Comparison of formulation sizes (assuming $\pi_i = 1/N$):

	Rows	Binary Vars
Original	mp	N
Extended	$2mp$	$mp + N$

where $p = \lfloor N\alpha \rfloor$

Strengthening the Extended Formulation

- Assume $\pi_i = 1/N$ for all i: knapsack becomes $\sum_{i=1}^{N} z_i \leq p$ (where $p = \lfloor N\alpha \rfloor$)

Theorem

The convex hull of the extended formulation is given by the inequalities defining the formulation and

$$\sum_{i \in S} z_i + \sum_{i \in \{k,\ldots,p\} \setminus S} w_i \leq p - k + 1 \quad \forall S \in \mathcal{S}_k, \; k = 1, \ldots, p$$

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- Example: $k = p$, $S = \{N\} \Rightarrow z_N + w_p \leq 1$
- Separation: $O(N \log N)$

Strengthening the Extended Formulation

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- Example: $k = p$, $S = \{N\} \Rightarrow z_N + w_p \leq 1$
- Separation: $O(N \log N)$
- Can be extended to case of general π_i

Computational Tests: Sample Approximation

Probabilistic Set Covering Problem (PSC)

$$\min\left\{cx : \Pr\{Ax \geq \xi\} \geq 1 - \epsilon, x \in \{0,1\}^n\right\}$$

- Base instance: scp41 from ORlib, 200 rows, 1000 columns
- Solve sample approximations with CPLEX
- Distribution function (or cdf) of ξ can be calculated exactly
- Finite feasible region and finite distribution

Sample Approximation: Results for PSC with $\epsilon = 0.1$

Based on solving 10 instances for each combination of N and α

α	N	Ave Risk	# Feas	Min Cost	LB	Gap
0.0	60	0.122	3	407		
	70	0.089	7	407		
	80	0.068	8	406		
	90	0.067	9	400		
	100	0.072	6	408		
0.1	1000	0.107	1	387	381	1.55%
	3000	0.098	6	387	382	1.29%
	5000	0.100	4	387	382	1.29%
	7500	0.099	5	387	385	0.52%
	10000	0.097	7	387	386	0.26%

LB is valid with confidence 0.999

Computational Tests: Sample Approximation

Probabilistic Transportation Problem (PTP)

$$\min\left\{cx : x \in X, P\{\sum_{i \in I} x_{ij} \geq \tilde{d}_j, j \in D\} \geq 1 - \epsilon\right\}$$

- Base instance: randomly generated, $|I| = 40$, $|D| = 50$
- Solve sample approximations with strong MIP formulation
- Demand $\tilde{d}$ given by joint normal distribution
- A posteriori feasibility estimated with huge sample size, $N' = 250,000$
- Continuous feasible region and distribution
- Tested high and low variance cases, report just high variance case

Sample Approximation: Results for PTP with $\epsilon = 0.05$

Based on solving 10 instances for each combination of N and α

					Based on $\alpha = 0.05$	
α	N	Ave Risk	# Feas	Min Cost	LB	Gap
0.0	900	0.050	4	3.4672		
	950	0.050	6	3.4403		
	1000	0.045	9	3.4569		
0.034	5000	0.054	1	3.4064	3.3060	2.95%
	7500	0.050	5	3.3817	3.3083	2.17%
	10000	0.048	10	3.3840	3.3200	1.89%

LB is valid with confidence 0.999

Best solution with $\alpha = 0$ is 1.7% more costly than best with $\alpha > 0$

Computational Tests: MIP Techniques

- Test problem: Probabilistic transportation problem with random demands (equal probabilities)
- Instances: Randomly generated
- MIP solver: CPLEX 9.0
- Time limit: one hour

Comparison of formulations

ϵ	m	N	MIP Gap	MIP+Star Time(s)	Extended Time(s)
0.05	100	1000	0.18%	7.7	
	100	2000	1.31%	31.8	
	200	2000	1.02%	61.4	
	200	3000	2.56%	108.6	
0.10	100	1000	2.24%	34.6	
	100	2000	5.12%	211.3	
	200	2000	4.69%	268.5	
	200	3000	6.20%	812.7	

- Results are averages over 5 instances
- Lower bound using star inequalities or extended formulation is almost always exact

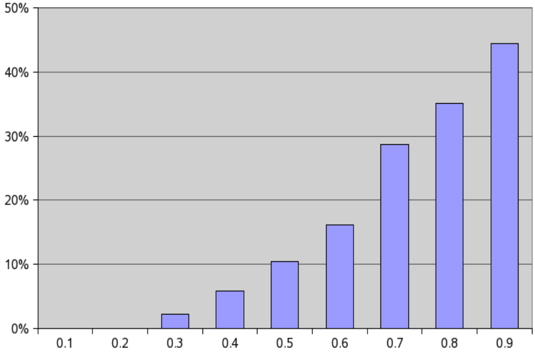
Comparison of formulations

ϵ	m	N	MIP Gap	MIP+Star Time(s)	Extended Time(s)
0.05	100	1000		7.7	1.1
	100	2000		31.8	4.6
	200	2000		61.4	12.1
	200	3000		108.6	12.4
0.10	100	1000		34.6	12.7
	100	2000		211.3	41.1
	200	2000		268.5	662.2
	200	3000		812.7	490.4

- Results are averages over 5 instances
- Lower bound using star inequalities or extended formulation is almost always exact

Effect of increasing ϵ

- Single instance, $m = 50$ and $N = 1000$, using star inequalities
- Optimality gap after one hour of computation time



ϵ

Instances with larger ϵ

New inequalities help, but instances are still hard

			Root Gap		Time(s) or Gap	
m	ϵ	N	Ext	+C	Ext	+C
25	0.30	250	1.20%	0.68%	121.2	93.9
	0.30	500	1.53%	0.58%	750.6	641.3
	0.35	250	2.24%	1.53%	563.2	408.4
	0.35	500	2.61%	1.63%	0.22%	0.06%
50	0.30	500	2.38%	2.04%	1.39%	1.43%
	0.30	1000	2.38%	1.78%	2.02%	1.69%
	0.35	500	4.28%	3.43%	3.13%	2.73%
	0.35	1000	4.18%	3.34%	3.71%	3.27%

Ext Extended formulation
+C Extended formulation with new inequalities

Concluding Remarks

- Sampled approximation + MIP quite effective scheme for PCPs
- Sampling results extend to general probabilistic constraints

$$\Pr\{g(x,\xi) \leq 0\} \geq 1-\epsilon$$

- Current/future work: MIP methods for random matrices

$$\Pr\{\tilde{T}x \geq \tilde{h}\} \geq 1-\alpha$$

- Comparison with other methods in applications