The polyhedral description of a mixed integer set

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Model:

$$X = \{(s,x,y) \in \mathbb{R} \times \mathbb{Z}^n \times \mathbb{Z}^m : s + a_1 x_i \geq b_i,\ \forall i \in N_1,\ s + a_2 y_j \geq d_j,\ \forall j \in N_2\}$$

where $N_1 = \{1,\ldots,n\}$, $N_2 = \{1,\ldots,m\}$ and $a_1, a_2 \in \mathbb{Z}_+$.

Notation: P denotes the convex hull of X

$g = gcd(a_1, a_2)$ (assume $g > 1$)

$g_1 = a_1/g$

$g_2 = a_2/g$

$\overrightarrow{r}_{ij} = g - (d_j - b_i) mod(g)$

$\overrightarrow{\rho}_{ij} = \left\lceil \frac{d_j - b_i}{g} \right\rceil$

$\overleftarrow{r}_{ji} = g - (b_i - d_j) mod(g)$

$\overleftarrow{\rho}_{ji} = \left\lceil \frac{b_i - d_j}{g} \right\rceil$

For $i \in N_1, j \in N_2$ let

$$\overrightarrow{P}^{ij} = \{ (s,x,y) : s = b_i - a_1 x_i,\ -g_1 x_i + g_2 y_j \geq \overrightarrow{\rho}_{ij},\ x_t = x_i, t \in N_1, t < i\ x_t = x_i + 1, t \in N_1, t > i,\ y_t = y_j, t \in N_1, t < j\ y_t = y_j + 1, t \in N_1, t > j\}$$

$$\overleftarrow{P}^{ji} = \{ (s,x,y) : s = d_j - a_2 y_j,\ g_1 x_i - g_2 y_j \geq \overleftarrow{\rho}_{ji},\ x_t = x_i, t \in N_1, t < i\ x_t = x_i + 1, t \in N_1, t > i,\ y_t = y_j, t \in N_1, t < j\ y_t = y_j + 1, t \in N_1, t > j\}$$

and let

$$\overline{P} = \left(\bigcup_{i \in N_1, j \in N_2} \overrightarrow{P}^{ij} \right) \bigcup \left(\bigcup_{i \in N_1, j \in N_2} \overleftarrow{P}^{ji} \right)$$

$$\overline{X} = \overline{P} \bigcap \mathbb{R} \times \mathbb{Z}^{n+m}$$

Then

$$X = \overline{X} + \mathbb{R}_+ \times \mathbb{Z}_+^{n+m} \text{ and } P = \overline{P} + \mathbb{R}_+^{n+m+1}$$

Using this relation between P and $\overline{P}$ and a result of Balas (1998) on the projection of union of polyhedra it follows that:

Proposition: *The inequality*

$$s + \sum_{j \in N_1} v_j x_j + \sum_{j \in N_2} w_j y_j \geq \alpha$$

is valid for X *if and only if there exists* $v_t \geq 0, t \in N_1, w_t \geq 0, t \in N_2, u \geq 0$ *such that*

$\sum_{t \in N_1} v_t = a_1 - g_1 u$ *(1)*

$\sum_{t \in N_2} w_t = a_2 - g_2(g - u)$ *(2)*

$\alpha \leq b_i + \overrightarrow{\rho}_{ij} u + \sum_{t \in N_1, t > i} v_t + \sum_{t \in N_2, t > j} w_t, \quad i \in N_1, j \in N_2$ *(3)*

$\alpha \leq d_j + \overleftarrow{\rho}_{ji}(g - u) + \sum_{t \in N_1, t > i} v_t + \sum_{t \in N_2, t > j} w_t, \quad i \in N_1, j \in N_2$ *(4)*

$u \leq g$ *(5)*

Let $Q = \{(\alpha, v, w, u) \in \mathbb{R}_+^{1+n+m+1} : (1), (2), (3), (4)\ (5)\}$.

Separation: Given a point $p^* = (s^*, x^*, y^*)$ then p^* is in P if and only if

$$\min\{s^* + \sum_{j \in N_1} v_j x_j^* + \sum_{j \in N_2} w_j y_j^* - \alpha : (\alpha, v, w, u) \in Q\}$$

has nonnegative optimal value

Define the bipartite graph $G = (N_1, N_2, A)$ where $N_1 = \{1,\ldots,n\}$, $N_2 = \{1,\ldots,m\}$ and A is the set of all arcs $A = \overrightarrow{A} \cup \overleftarrow{A}$ where $\overrightarrow{A} = \{(i,j) : i \in N_1, j \in N_2\}$ and $\overleftarrow{A} = \{(j,i) : i \in N_1, j \in N_2\}$. Associate with each arc $(i,j) \in \overrightarrow{A}$ the weight $\overrightarrow{r}_{ij}$, and with $(j,i) \in \overleftarrow{A}$ the weight $\overleftarrow{r}_{ji}$ (see Figure 1).

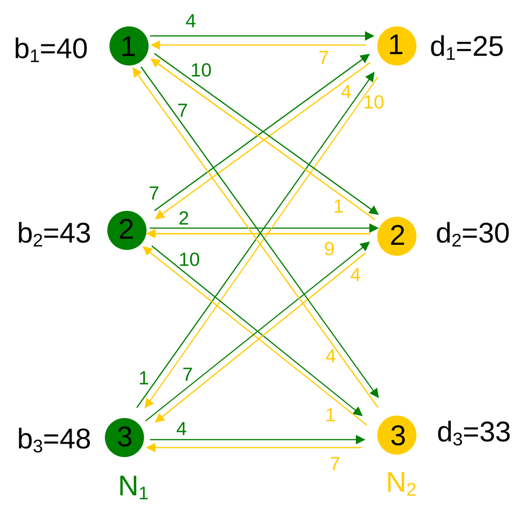

Figure 1. Bipartite graph $G = (N_1, N_2, A)$ for Example 1 (with $g = 11$).

Theorem: *Consider the bipartite graph* $G = (N_1, N_2, A)$ *and a circuit* C *in* G, *where* $C = \overrightarrow{C} \cup \overleftarrow{C}$, $\overrightarrow{C} \subset \overrightarrow{A}$, $\overleftarrow{C} \subset \overleftarrow{A}$, *with length* $\sum_{(i,j) \in \overrightarrow{C}} \overrightarrow{r}_{ij} + \sum_{(j,i) \in \overleftarrow{C}} \overleftarrow{r}_{ji} = g$. *Then the circuit-inequality*

$$s + \sum_{j \in S_1} v_j x_j + \sum_{j \in S_2} w_j y_j \geq \alpha$$

where $(\alpha, v, w, u) \in Q$ *satisfies*

$$\alpha = b_i + \overrightarrow{\rho}_{ij} u + \sum_{t \in N_1, t > i} v_t + \sum_{t \in N_2, t > j} w_t, \forall (i,j) \in \overrightarrow{C}$$

$$\alpha = d_j + \overleftarrow{\rho}_{ji}(g - u) + \sum_{t \in N_1, t > i} v_t + \sum_{t \in N_2, t > j} w_t, \forall (j,i) \in \overleftarrow{C}$$

and $v_t = 0, t \notin S_1$, $w_t = 0, t \notin S_2$ *where* S_1 *and* S_2 *are, respectively, the subsets of* N_1 *and* N_2 *in* C, *is valid for* P.

Example 1. Consider

$$X = \{(s,x,y) : s + 33x_1 \geq 40, s + 33x_2 \geq 43, s + 33x_3 \geq 48, s + 55y_1 \geq 25, s + 55y_2 \geq 30, s + 55y_3 \geq 33\}$$

The cycles with length $g = 11$ and the corresponding inequalities are:

inequality	circuit
$s + 12x_1 + 35y_1 \geq 33$	$\{(1,1),(1,1)\}$
$s + 30x_1 + 5y_2 \geq 40$	$\{(1,2),(2,1)\}$
$s + 21x_1 + 20y_3 \geq 40$	$\{(1,3),(3,1)\}$
$s + 21x_2 + 20y_1 \geq 39$	$\{(2,1),(1,2)\}$
$s + 6x_2 + 45y_2 \geq 34$	$\{(2,2),(2,2)\}$
$s + 30x_2 + 5y_3 \geq 43$	$\{(2,3),(3,2)\}$
$s + 3x_3 + 50y_1 \geq 28$	$\{(3,1),(1,3)\}$
$s + 21x_3 + 20y_2 \geq 44$	$\{(3,2),(2,3)\}$
$s + 12x_3 + 35y_3 \geq 41$	$\{(3,3),(3,3)\}$
$s + 16x_1 + 2x_2 + 24y_1 + y_2 \geq 38$	$\{(1,1),(1,2),(2,2),(2,1)\}$
$s + 25x_1 + 2x_2 + 9y_2 + y_3 \geq 42$	$\{(1,3),(3,2),(2,2),(2,1)\}$
$s + 14x_2 + 4x_3 + 24y_2 + y_3 \geq 43$	$\{(2,2),(2,3),(3,3),(3,2)\}$

Considering the circuit $\{(1,1),(1,2),(2,2),(2,1)\}$ then

$v_3 = w_3 = 0$

$v_1 + v_2 = 33 - 3u$

$w_1 + w_2 = 55 - 5(11 - u)$

$\alpha = 40 - u + v_2 + w_2$ arc $(1,1) \in \overrightarrow{A}$

$\alpha = 25 + 2(11 - u) + w_2$ arc $(1,2) \in \overleftarrow{A}$

$\alpha = 43 - u$ arc $(2,2) \in \overrightarrow{A}$

$\alpha = 30 + (11 - u) + v_2$ arc $(2,1) \in \overleftarrow{A}$

which implies $u = 5, v_1 = 16, v_2 = 2, w_1 = 24, w_2 = 1$. Hence, the circuit-inequality is

$$s + 16x_1 + 2x_2 + 24y_1 + y_2 \geq 38$$

Theorem: *Consider the circuit-inequality associated with the circuit* $C = \{(i_1 = 1, j_1 = 1), \ldots, (i_k, j_k), (j_k, i_1)\}$ *with length* g *and* $\overrightarrow{r}_{ij} > 0$ *for all* $(i,j) \in \overrightarrow{C}$ *and* $\overleftarrow{r}_{ji} > 0$ *for all* $(j,i) \in \overleftarrow{C}$:

$$s + \sum_{i \in V_1} v_i x_i + \sum_{j \in V_2} w_j y_j \geq \alpha$$

where (V_1, V_2) *denote the set of vertices in* C. *For each subset* (S_1, S_2) *of* (N_1, N_2) *such that* $V_1 \subseteq S_1, V_2 \subseteq S_2$ *the following extended-circuit inequalitiy is valid for* P

$$s + \sum_{i \in S_1} v_i' x_i + \sum_{j \in S_2} w_j' y_j \geq \alpha + \sum_{i \in S_1 \setminus V_1, i > i_k} \alpha(i) + \sum_{j \in S_2 \setminus V_2, j > j_k} \beta(j)$$

where

$$v_i' = \begin{cases} \alpha(i) - \sum_{t \in S_1, p(i) < t < i} \alpha(t) & \text{if } i \in S_1 \setminus V_1 \\ v_1 - \sum_{t \in S_1, t > i_k} v_t' & \text{if } i \in V_1, i = 1 \\ v_i - \sum_{t \in S_1, p(i) < t < i} v_t' & \text{if } i \in V_1, 1 < i < i_k \\ v_i - \sum_{t \in S_1, p(i) < t} v_t' & \text{if } i \in V_1, 1 < i = i_k \end{cases}$$

$$w_j' = \begin{cases} \beta(j) - \sum_{t \in S_2, p(j) < t < j} \beta(t) & \text{if } j \in S_2 \setminus V_2 \\ w_1 - \sum_{t \in S_2, t > j_k} w_t' & \text{if } j \in V_2, j = 1 \\ w_j - \sum_{t \in S_2, p(j) < t < j} w_t' & \text{if } j \in V_2, 1 < j < j_k \\ w_j - \sum_{t \in S_2, p(j) < t} w_t' & \text{if } j \in V_2, 1 < j = j_k \end{cases}$$

and

$$p(i) = \min\{t \in V_1 : t < i\}, i \in N_1 \setminus \{1\}$$

$$p(j) = \min\{t \in V_2 : t < j\}, j \in N_2 \setminus \{1\}$$

$$\alpha(i) = \min_{j \in V_2} \left\{ \min\left\{ b_i + \overrightarrow{\rho}_{ij} u, d_j + \overleftarrow{\rho}_{ji}(g - u) \right\} + \sum_{t \in V_1, t > i} v_t + \sum_{t \in V_2, t > j} w_t - \alpha \right\}, \forall i \in S_1 \setminus V_1$$

$$\beta(j) = \min_{i \in V_1} \left\{ \left\{ d_j + \overleftarrow{\rho}_{ji}(g - u), b_i + \overrightarrow{\rho}_{ij} u \right\} + \sum_{t \in V_1, t > i} v_t + \sum_{t \in V_2, t > j} w_t - \alpha \right\}, \forall j \in S_2 \setminus V_2.$$

Example 1 (cont.) Consider the circuit-inequality

$$s + 16x_1 + 2x_2 + 24y_1 + y_2 \geq 38$$

$\alpha(3) = \min\{ \underbrace{1}_{\text{arc }(3,1)}, \underbrace{6}_{\text{arc }(1,3)}, \underbrace{5}_{\text{arc }(3,2)}, \underbrace{4}_{\text{arc }(2,3)} \} = 1$

$\beta(3) = \min\{ \underbrace{4}_{\text{arc }(1,3)}, \underbrace{3}_{\text{arc }(3,1)}, \underbrace{5}_{\text{arc }(2,3)}, \underbrace{1}_{\text{arc }(3,2)} \} = 1$

The extended-circuit-inequalities associated with

$$s + 16x_1 + 2x_2 + 24y_1 + y_2 \geq 38$$

are:

$$s + 15x_1 + 2x_2 + x_3 + 24y_1 + y_2 \geq 39$$

$$s + 16x_1 + 2x_2 + 23y_1 + y_2 + y_3 \geq 39$$

$$s + 15x_1 + 2x_2 + x_3 + 23y_1 + y_2 + y_3 \geq 40$$

Theorem: *The trivial inequalities together with the extended-circuit inequalities suffice to describe* P.

References

[1] Balas, E. (1998) *Disjuntive programming: Properties of the convex hull of feasible points*, Discrete Applied Mathematics **89**, pp 3–44.