

# The polyhedral description of a mixed integer set

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## Model:

$$X = \{(s, x, y) \in \mathbb{R} \times \mathbb{Z}^n \times \mathbb{Z}^m : s + a_1x_i \geq b_i, \forall i \in N_1, \\ s + a_2y_j \geq d_j, \forall j \in N_2\}$$

where  $N_1 = \{1, \dots, n\}$ ,  $N_2 = \{1, \dots, m\}$  and  $a_1, a_2 \in \mathbb{Z}_+$ .

**Notation:**  $P$  denotes the convex hull of  $X$

$$g = \gcd(a_1, a_2) \quad (\text{assume } g > 1)$$

$$g_1 = a_1/g$$

$$g_2 = a_2/g$$

$$\vec{r}_{ij} = g - (d_j - b_i) \bmod(g)$$

$$\vec{p}_{ij} = \left\lfloor \frac{d_j - b_i}{g} \right\rfloor$$

$$\overleftarrow{r}_{ji} = g - (b_i - d_j) \bmod(g)$$

$$\overleftarrow{p}_{ji} = \left\lfloor \frac{b_i - d_j}{g} \right\rfloor$$

For  $i \in N_1, j \in N_2$  let

$$\vec{P}^{ij} = \{(s, x, y) :$$

$$s = b_i - a_1x_i$$

$$-g_1x_i + g_2y_j \geq \vec{p}_{ij}$$

$$x_t = x_i, t \in N_1, t < i \quad x_t = x_i + 1, t \in N_1, t > i$$

$$y_t = y_j, t \in N_2, t < j \quad y_t = y_j + 1, t \in N_2, t > j\}$$

$$\overleftarrow{P}^{ji} = \{(s, x, y) :$$

$$s = d_j - a_2y_j$$

$$g_1x_i - g_2y_j \geq \overleftarrow{p}_{ji}$$

$$x_t = x_i, t \in N_1, t < i \quad x_t = x_i + 1, t \in N_1, t > i$$

$$y_t = y_j, t \in N_2, t < j \quad y_t = y_j + 1, t \in N_2, t > j\}$$

and let

$$\bar{P} = \left( \bigcup_{i \in N_1, j \in N_2} \vec{P}^{ij} \right) \cup \left( \bigcup_{i \in N_1, j \in N_2} \overleftarrow{P}^{ji} \right)$$

$$\bar{X} = \bar{P} \cap \mathbb{R} \times \mathbb{Z}^{n+m}$$

Then

$$X = \bar{X} + \mathbb{R}_+ \times \mathbb{Z}_+^{n+m} \quad \text{and} \quad P = \bar{P} + \mathbb{R}_+^{n+m+1}$$

Using this relation between  $P$  and  $\bar{P}$  and a result of Balas (1998) on the projection of union of polyhedra it follows that:

**Proposition:** *The inequality*

$$s + \sum_{j \in N_1} v_j x_j + \sum_{j \in N_2} w_j y_j \geq \alpha$$

is valid for  $X$  if and only if there exists  $v_t \geq 0, t \in N_1, w_t \geq 0, t \in N_2, u \geq 0$  such that

$$\sum_{t \in N_1} v_t = a_1 - g_1u \quad (1)$$

$$\sum_{t \in N_2} w_t = a_2 - g_2(g - u) \quad (2)$$

$$\alpha \leq b_i + \vec{p}_{ij}u + \sum_{t \in N_1, t > i} v_t + \sum_{t \in N_2, t > j} w_t, \quad i \in N_1, j \in N_2 \quad (3)$$

$$\alpha \leq d_j + \overleftarrow{p}_{ji}(g - u) + \sum_{t \in N_1, t > i} v_t + \sum_{t \in N_2, t > j} w_t, \quad i \in N_1, j \in N_2 \quad (4)$$

$$u \leq g \quad (5)$$

Let  $Q = \{(\alpha, v, w, u) \in \mathbb{R}_+^{1+n+m+1} : (1), (2), (3), (4), (5)\}$ .

**Separation:** Given a point  $p^* = (s^*, x^*, y^*)$  then  $p^*$  is in  $P$  if and only if

$$\min \{s^* + \sum_{j \in N_1} v_j x_j^* + \sum_{j \in N_2} w_j y_j^* - \alpha : (\alpha, v, w, u) \in Q\}$$

has nonnegative optimal value

Define the bipartite graph  $G = (N_1, N_2, A)$  where  $N_1 = \{1, \dots, n\}$ ,  $N_2 = \{1, \dots, m\}$  and  $A$  is the set of all arcs  $A = \vec{A} \cup \overleftarrow{A}$  where  $\vec{A} = \{(i, j) : i \in N_1, j \in N_2\}$  and  $\overleftarrow{A} = \{(j, i) : i \in N_1, j \in N_2\}$ . Associate with each arc  $(i, j) \in \vec{A}$  the weight  $\vec{r}_{ij}$ , and with  $(j, i) \in \overleftarrow{A}$  the weight  $\overleftarrow{r}_{ji}$  (see Figure 1).

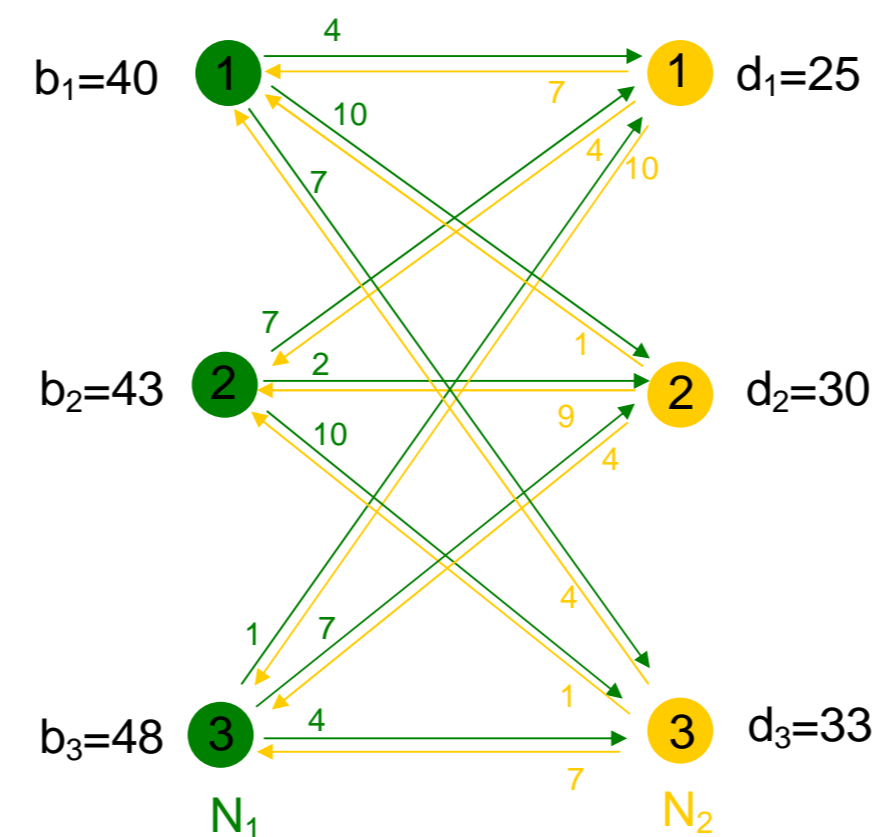


Figure 1. Bipartite graph  $G = (N_1, N_2, A)$  for Example 1 (with  $g = 11$ ).

**Theorem:** Consider the bipartite graph  $G = (N_1, N_2, A)$  and a circuit  $C$  in  $G$ , where  $C = \vec{C} \cup \overleftarrow{C}$ ,  $\vec{C} \subset \vec{A}$ ,  $\overleftarrow{C} \subset \overleftarrow{A}$ , with length  $\sum_{(i,j) \in \vec{C}} \vec{r}_{ij} + \sum_{(j,i) \in \overleftarrow{C}} \overleftarrow{r}_{ji} = g$ . Then the **circuit-inequality**

$$s + \sum_{j \in S_1} v_j x_j + \sum_{j \in S_2} w_j y_j \geq \alpha$$

where  $(\alpha, v, w, u) \in Q$  satisfies

$$\alpha = b_i + \vec{p}_{ij}u + \sum_{t \in N_1, t > i} v_t + \sum_{t \in N_2, t > j} w_t, \forall (i, j) \in \vec{C}$$

$$\alpha = d_j + \overleftarrow{p}_{ji}(g - u) + \sum_{t \in N_1, t > i} v_t + \sum_{t \in N_2, t > j} w_t, \forall (j, i) \in \overleftarrow{C}$$

and  $v_t = 0, t \notin S_1, w_t = 0, t \notin S_2$  where  $S_1$  and  $S_2$  are, respectively, the subsets of  $N_1$  and  $N_2$  in  $C$ , is valid for  $P$ .

**Example 1.** Consider

$$X = \{(s, x, y) : s + 33x_1 \geq 40, s + 33x_2 \geq 43, s + 33x_3 \geq 48, \\ s + 55y_1 \geq 25, s + 55y_2 \geq 30, s + 55y_3 \geq 33\}$$

The cycles with length  $g = 11$  and the corresponding inequalities are:

inequality	circuit
$s + 12x_1 + 35y_1 \geq 33$	$\{(1, 1), (1, 1)\}$
$s + 30x_1 + 5y_2 \geq 40$	$\{(1, 2), (2, 1)\}$
$s + 21x_1 + 20y_3 \geq 40$	$\{(1, 3), (3, 1)\}$
$s + 21x_2 + 20y_1 \geq 39$	$\{(2, 1), (1, 2)\}$
$s + 6x_2 + 45y_2 \geq 34$	$\{(2, 2), (2, 2)\}$
$s + 30x_2 + 5y_3 \geq 43$	$\{(2, 3), (3, 2)\}$
$s + 3x_3 + 50y_1 \geq 28$	$\{(3, 1), (1, 3)\}$
$s + 21x_3 + 20y_2 \geq 44$	$\{(3, 2), (2, 3)\}$
$s + 12x_3 + 35y_3 \geq 41$	$\{(3, 3), (3, 3)\}$
$s + 16x_1 + 2x_2 + 24y_1 + y_2 \geq 38$	$\{(1, 1), (1, 2), (2, 2), (2, 1)\}$
$s + 25x_1 + 2x_2 + 9y_2 + y_3 \geq 42$	$\{(1, 3), (3, 2), (2, 2), (2, 1)\}$
$s + 14x_2 + 4x_3 + 24y_2 + y_3 \geq 43$	$\{(2, 2), (2, 3), (3, 3), (3, 2)\}$

Considering the circuit  $\{(1, 1), (1, 2), (2, 2), (2, 1)\}$  then

$$v_3 = w_3 = 0$$

$$v_1 + v_2 = 33 - 3u$$

$$w_1 + w_2 = 55 - 5(11 - u)$$

$$\alpha = 40 - u + v_2 + w_2 \quad \text{arc } (1, 1) \in \vec{A}$$

$$\alpha = 25 + 2(11 - u) + w_2 \quad \text{arc } (1, 2) \in \vec{A}$$

$$\alpha = 43 - u \quad \text{arc } (2, 2) \in \vec{A}$$

$$\alpha = 30 + (11 - u) + v_2 \quad \text{arc } (2, 1) \in \vec{A}$$

which implies  $u = 5, v_1 = 16, v_2 = 2, w_1 = 24, w_2 = 1$ . Hence, the circuit-inequality is

$$s + 16x_1 + 2x_2 + 24y_1 + y_2 \geq 38$$

**Theorem:** Consider the circuit-inequality associated with the circuit  $C = \{(i_1 = 1, j_1 = 1), \dots, (i_k, j_k), (j_k, i_1)\}$  with length  $g$  and  $\vec{r}_{ij} > 0$  for all  $(i, j) \in \vec{C}$  and  $\overleftarrow{r}_{ji} > 0$  for all  $(j, i) \in \overleftarrow{C}$ :

$$s + \sum_{i \in V_1} v_i x_i + \sum_{j \in V_2} w_j y_j \geq \alpha$$

where  $(V_1, V_2)$  denote the set of vertices in  $C$ . For each subset  $(S_1, S_2)$  of  $(N_1, N_2)$  such that  $V_1 \subseteq S_1, V_2 \subseteq S_2$  the following **extended-circuit inequality** is valid for  $P$

$$s + \sum_{i \in S_1} v_i x_i + \sum_{j \in S_2} w_j y_j \geq \alpha + \sum_{i \in S_1 \setminus V_1, i > i_k} \alpha(i) + \sum_{j \in S_2 \setminus V_2, j > j_k} \beta(j)$$

where

$$v_i' = \begin{cases} \alpha(i) - \sum_{t \in S_1, p(i) < t < i} \alpha(t) & \text{if } i \in S_1 \setminus V_1 \\ v_1 - \sum_{t \in S_1, t > i_k} v_t' & \text{if } i \in V_1, i = 1 \\ v_i - \sum_{t \in S_1, p(i) < t < i} v_t' & \text{if } i \in V_1, 1 < i < i_k \\ v_i - \sum_{t \in S_1, p(i) < t} v_t' & \text{if } i \in V_1, 1 < i = i_k \end{cases}$$

$$w_j' = \begin{cases} \beta(j) - \sum_{t \in S_2, p(j) < t < j} \beta(t) & \text{if } j \in S_2 \setminus V_2 \\ w_1 - \sum_{t \in S_2, t > j_k} w_t' & \text{if } j \in V_2, j = 1 \\ w_j - \sum_{t \in S_2, p(j) < t < j} w_t' & \text{if } j \in V_2, 1 < j < j_k \\ w_j - \sum_{t \in S_2, p(j) < t} w_t' & \text{if } j \in V_2, 1 < j = j_k \end{cases}$$

and

$$p(i) = \min\{t \in V_1 : t < i\}, i \in N_1 \setminus \{1\}$$

$$p(j) = \min\{t \in V_2 : t < j\}, j \in N_2 \setminus \{1\}$$

$$\alpha(i) = \min_{j \in V_2} \{ \min\{b_i + \vec{p}_{ij}u, d_j + \overleftarrow{p}_{ji}(g - u)\} \\ + \sum_{t \in V_1, t > i} v_t + \sum_{t \in V_2, t > j} w_t - \alpha \}, \forall i \in S_1 \setminus V_1$$

$$\beta(j) = \min_{i \in V_1} \{ \{d_j + \overleftarrow{p}_{ji}(g - u), b_i + \vec{p}_{ij}u\} \\ + \sum_{t \in V_1, t > i} v_t + \sum_{t \in V_2, t > j} w_t - \alpha \}, \forall j \in S_2 \setminus V_2.$$

**Example 1 (cont.)** Consider the circuit-inequality

$$s + 16x_1 + 2x_2 + 24y_1 + y_2 \geq 38$$

$$\alpha(3) = \min \left\{ \frac{1}{\text{arc}(3,1)}, \frac{6}{\text{arc}(1,3)}, \frac{5}{\text{arc}(3,2)}, \frac{4}{\text{arc}(2,3)} \right\} = 1$$

$$\beta(3) = \min \left\{ \frac{4}{\text{arc}(1,3)}, \frac{3}{\text{arc}(3,1)}, \frac{5}{\text{arc}(2,3)}, \frac{1}{\text{arc}(3,2)} \right\} = 1$$

The extended-circuit-inequalities associated with

$$s + 16x_1 + 2x_2 + 24y_1 + y_2 \geq 38$$

are:

$$s + 15x_1 + 2x_2 + x_3 + 24y_1 + y_2 \geq 39$$

$$s + 16x_1 + 2x_2 + 23y_1 + y_2 + y_3 \geq 39$$

$$s + 15x_1 + 2x_2 + x_3 + 23y_1 + y_2 + y_3 \geq 40$$

**Theorem:** The trivial inequalities together with the extended-circuit inequalities suffice to describe  $P$ .

## References

[1] BALAS, E. (1998) *Disjunctive programming: Properties of the convex hull of feasible points*, Discrete Applied Mathematics 89, pp 3–44.