Risk and conditional risk measures in an agent-object insurance market

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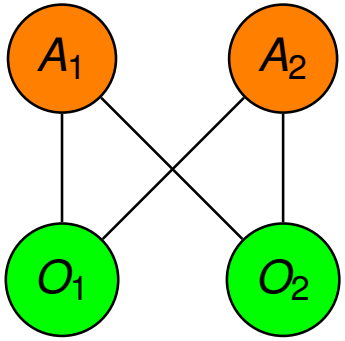
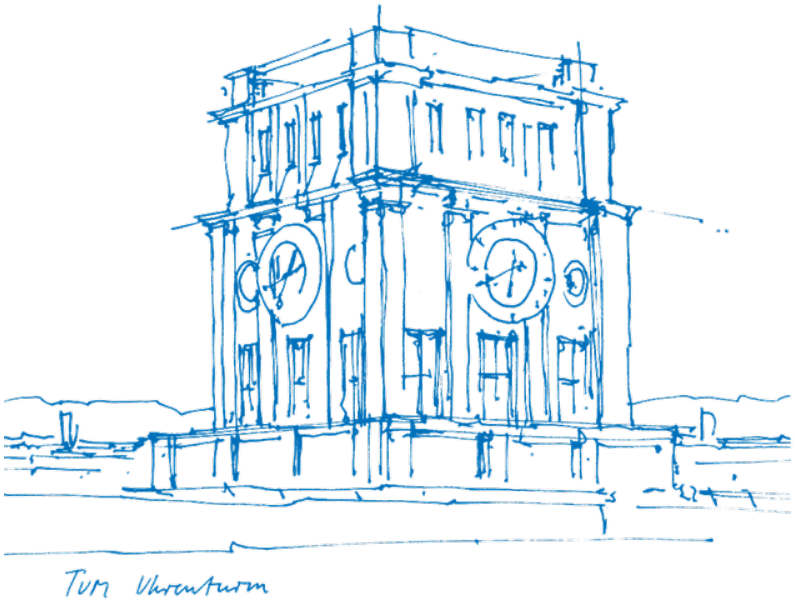




Outline

1. Modelling risk: the reinsurance market with bipartite graph structure
2. Measuring risk: Value-at-Risk and Conditional Tail Expectation
3. Findings: risk exposures of single company and market
4. Network scenarios
5. Conditional systemic risk measures
6. What comes next

Support is acknowledged from the Isaac Newton Institute, Cambridge:
"Systemic Risk: Mathematical Modelling and Interdisciplinary Approaches"

Kley, O., Klüppelberg, C., and Reinert G. (2016)[1]
Risk in a large claims insurance market with bipartite graph structure.
Operations Research **64**(5), 1159-1176 Lloyd's Science of Risk Prize 2016

[1] **"Lloyd's Science of Risk Prize 2016"**

Systemic risk in reinsurance markets

Financial agents are often related through an interwoven network of business relationships. [2] [3] [4]

This interconnected world generates interdependencies that need to be understood to ensure that systemic risks do not aggregate to a point where the system itself is unable to sustainably manage risks.

Understanding how the global reinsurance system functions as a system will allow informed action by all stakeholders within the system.

[2] Boss, Elsinger, Summer and Thurner (2004)
[3] Haldane and May (2011)
[4] Cont, Moussa and Santos (2013)

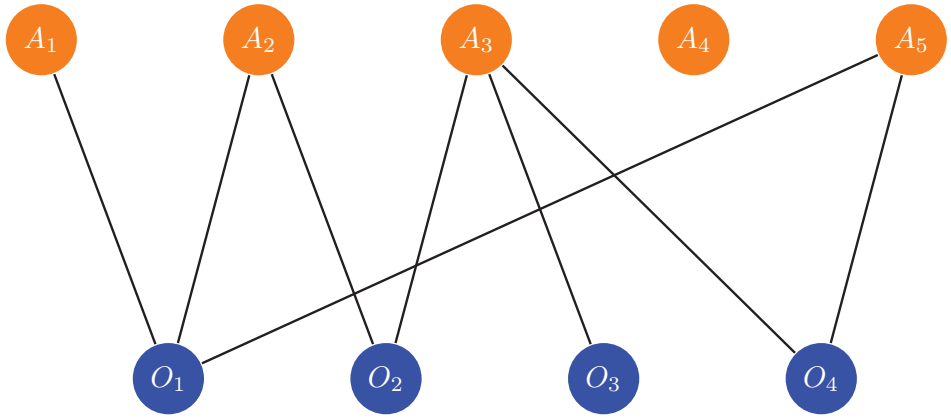
Why a network approach?

In reinsurance markets exogeneous risks play the major role; correlation/dependence at the level of companies occurs due to risk sharing.

To reflect the interdependencies of risks and reinsurance companies we model a reinsurance market using a network approach, drawing edges between companies and the risks which they insure.

We study the effect of risk sharing (also referred to as diversification) for reinsurance companies in different market situations.

The model



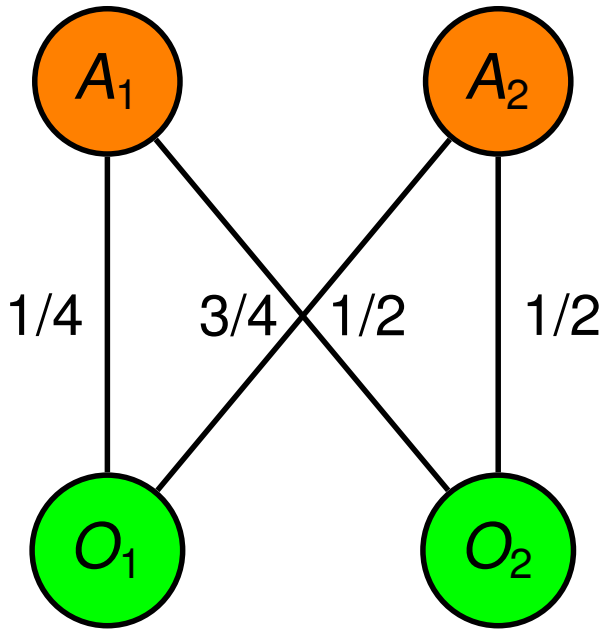
- Companies and risks are represented as nodes in a network; there are q companies and d risks. A reinsurance company insures a risk with a probability which depends on the company and on the risk, independently. This choice is represented by an edge between the nodes.

- Claims V_j for risks $j = 1, \ldots, d$ are independent with (heavy) Pareto-tails $\mathbb{P}(V_j > t) \sim K_j t^{-\alpha}$ as $t \to \infty$.

- The risk exposure vector is $F = AV$; A is the weighted adjacency matrix of the network; V is the claims vector.

Example: A bipartite network

The bipartite graph creates dependence between exposures.



If a claim in Risk 1 occurs, 1/4 of it would be covered by Company 1, and 3/4 would be covered by Company 2.

If a claim in Risk 2 occurs, it would be split evenly between the two companies.

Industry standard for a single institution [5]

Value-at-Risk ($\mathrm{VaR}_{1-\gamma}(F_i)$) at level $\gamma \in (0,1)$, indeed γ near 0:

$$\mathbb{P}(F_i > \mathrm{VaR}_{1-\gamma}(F_i)) \leq \gamma$$

- $\mathrm{VaR}_{1-\gamma}(F_i)$: losses larger than this number should be allowed to happen with very small probability $\gamma = 0.01, \gamma = 0.0001$
- There have been debates about this risk measure:
  - $\mathrm{VaR}_{1-\gamma}(F_i)$ not always subadditive
  - $\mathrm{VaR}_{1-\gamma}(F_i)$ may create moral hazard:
    amounts of losses beyond $\mathrm{VaR}_{1-\gamma}(F_i)$ are ignored

Conditional Tail Expectation

$$\mathrm{CoTE}_{1-\gamma}(F_i) = \mathbb{E}[F_i \mid F_i > \mathrm{VaR}_{1-\gamma}(F_i)]$$

[5] McNeil, Frey and Embrechts (2006) Quantitative Risk Management.

Systemic risk regulation

- A regulator needs to decide on the aggregated risk in the market.
- Axiomatic approaches for single institutions have to be amended so that the regulator can decide on the aggregated risk in the market.
- **Suggestion:** [6] Take an aggregation function h with certain properties (e.g. homogeneous, linear, convex, concave, ...)
- **Example:** $h(F) := \|F\|_r = (\sum_{i=1}^q F_i^r)^{1/r}$ for $r > 0$,

  $\|F\|_1 = \sum_{i=1}^q F_i$ and $\|F\|_\infty := \max\{F_1, \ldots, F_q\}$.

  In this talk: $h(F) = \sum_{i=1}^q F_i$ (the sum of all exposures).

[6] Chen, Iyengar and Maollemi (2012)

Diversification benefit

Based on the Value-at-Risk, **diversification benefit** (or the benefit of risk sharing) [7] is measured via the behaviour for small γ of

$$D = 1 - \frac{\mathrm{VaR}_{1-\gamma}(\text{market exposure})}{\sum_{i=1}^{q} \mathrm{VaR}_{1-\gamma}(\text{exposure of company } i)} = 1 - \frac{\mathrm{VaR}_{1-\gamma}(\sum_{i=1}^{q} F_i)}{\sum_{i=1}^{q} \mathrm{VaR}_{1-\gamma}(F_i)}$$

If $D > 0$, then diversification is judged as being beneficial to the market.

[7] Embrechts, Lambrigger and Wüthrich (2009)

Risk exposure of single company and market

For F_i the **exposure of company** i and $\sum_{i=1}^{q} F_i$ the **market exposure** we find that for small γ the risk measures are described by only two constants, C_i and C_S:

- for the individual risk measure $\mathrm{VaR}_{1-\gamma}(F_i) = C_i \gamma^{-1/\alpha}$
- for the systemic risk measure $\mathrm{VaR}_{1-\gamma}(\sum_{i=1}^{q} F_i) = C_S \gamma^{-1/\alpha}$
- and for the diversification benefit $D = 1 - \frac{C_S}{\sum_{i=1}^{q} C_i}$

These constants depend on the network as well as on the distribution of the risks.

Small constants correspond to small risk!

Risk exposure of the market

The **degree of claim** j is the number of companies which insure it:

$$\deg(j) = \sum_{i=1}^{q} \mathbb{1}(i \sim j).$$

Assume **for simplicity** that a claim is **evenly distributed** among all companies who insure the risk.

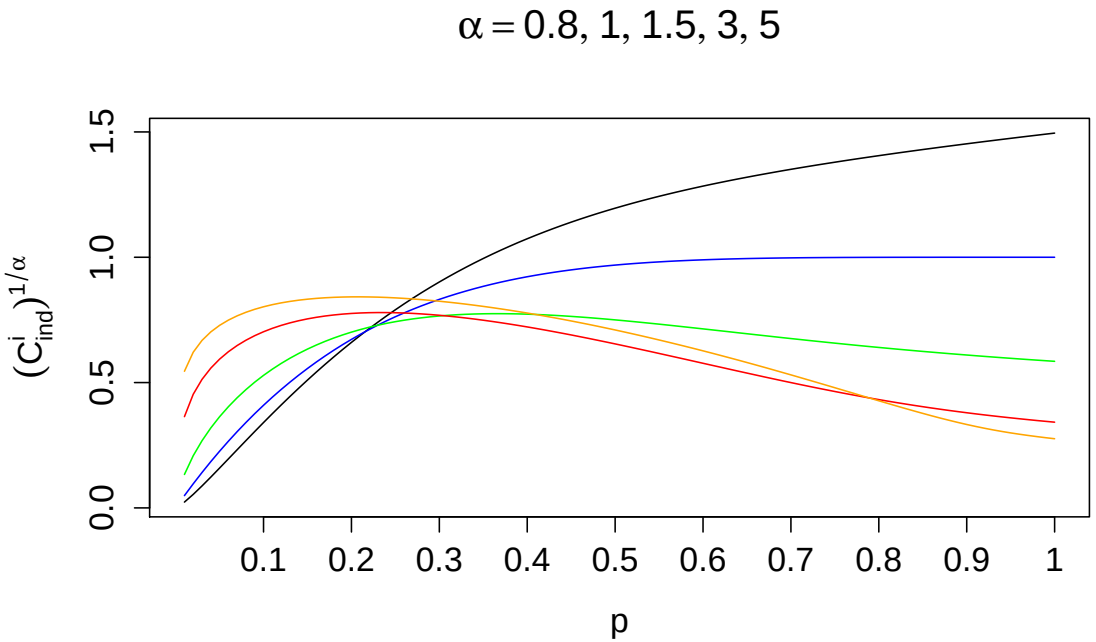
Example: Define the **random** weighted (5×4)-adjacency matrix

$$A_{ij} = \frac{\mathbb{1}(i \sim j)}{\deg(j)} \qquad \textbf{One realisation:} \qquad A = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Then the **vector of exposures of all companies:**

$$(F_1, \ldots, F_q)^\top = F = AV = A(V_1, \ldots, V_d)^\top.$$

If all edge probabilities are equal ....

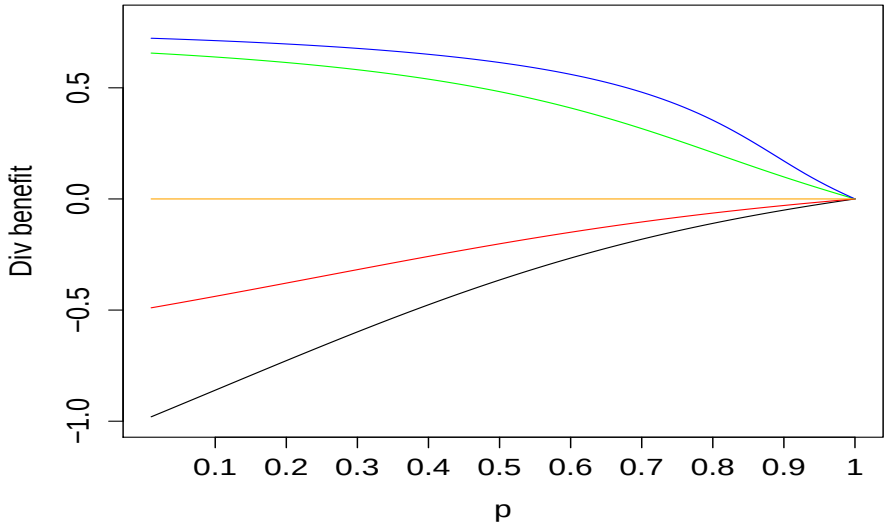


The individual risk constant C_i, when the Pareto exponent varies:
$\alpha = 0.8$ (black), $\alpha = 1$ (blue), $\alpha = 1.5$ (green), $\alpha = 3$ (red), $\alpha = 5$ (orange).

The behavour in p is non-monotone for $\alpha > 1$. Increasing p:

- increases the market diversification;
- increases the number of insured risks;
- may increase or may decrease the individual risk.

If all edge probabilities are equal ...



Diversification benefit when the Pareto exponent varies:

$\alpha = 1$ (orange), $\alpha = 5$ (blue), $\alpha = 3$ (green), $\alpha = 0.8$ (red), $\alpha = 0.7$ (black).

- The diversification benefit is negative for $\alpha < 1$; positive for $\alpha > 1$;
- when risks have infinite mean, diversification is not beneficial. [8]

[8] Mainik and Rüschendorf (2010) for the non-network situation

If companies have different risk aversion ...

Assume that the probability of a specific company insuring a specific risk is given by edge probabilities p_{ij}.

$p_{ij} =$ (risk proneness of company i)×(attractiveness of risk j) $= \beta_i \delta_j$

Assume that all risks are equally attractive: $p_{ij} = \beta_i \delta$

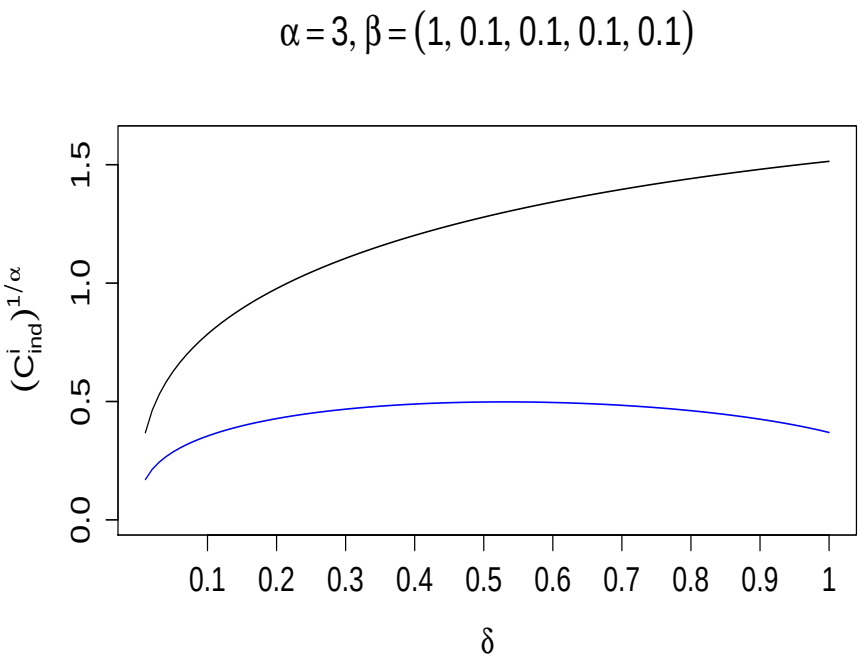
The larger δ, the higher the diversification.
5 companies, 5 claims; two different risk types of companies:

(1) $\beta = (1, 0.1, 0.1, 0.1, 0.1)$:
Company 1 is more risk-prone than the others;

(2) $\beta = (0.1, 1, 1, 1, 1)$:
Company 1 is more risk-averse than the others.

If companies have different risk aversion ...

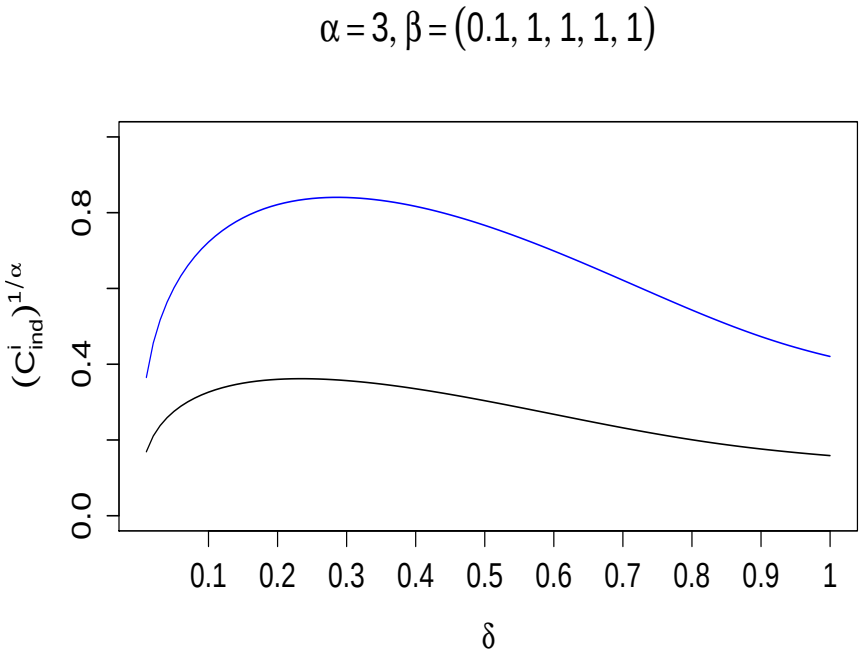
One risk-prone company: $\beta = (1, 0.1, 0.1, 0.1, 0.1)$



$C_i(\delta)$ for different companies: $i = 1$ (black), $i = 2$ (blue).
Company 1 does not benefit from the diversification.
The other companies experience risk reduction for large attractiveness of risk, δ.

If companies have different risk aversion ...

One risk-averse company: $\beta = (0.1, 1, 1, 1, 1)$



$C_i(\delta)$ for different companies: $i = 1$ (black), $i = 2$ (blue).
All companies benefit from diversification for large attractiveness of risk, δ.

What we have learnt so far

- The individual and the market risk depend crucially on the network linking companies and risks.
- In the examples the individual risk has a non-monotone dependence on the edge probabilities p_{ij}.
- Increasing the edge probabilities increases the market diversification and increases the number of insured risks, but does not necessarily decrease the individual risk.
- In the examples, when $\alpha < 1$, the diversification benefit is negative but increases with increasing edge probabilities; when $\alpha > 1$, the diversification benefit is positive but decreases with increasing edge probabilities.
- For $p = 1$; i.e. maximal degree of connectivity (the complete network), the diversification benefit is 0.

Conditional risk measures [9]

marginal	agent | system	system | agent	agent | agent
VaR	ICoVaR	SCoVaR	MCoVaR
CoTE	ICoTE	SCoTE	MCoTE

I = individual: risk of a company given high market risk;
S = system: risk of the system given high risk of an company;
M = mutual: risk of one company given high risk of another company.

[9] Danielsson, James, Valenzuela, and Zer (2014)

Conditional systemic risk measures

Let $F = (F_1, \ldots, F_q)^\top = A(V_1, \ldots, V_d)^\top$ be the exposure vector of the companies, the claims V_j have Pareto exponent $\alpha > 1$, market exposure $h(F) = \sum_{i=1}^q F_i$.
Then for small γ the conditional risk measures are described by constants, $C^{i|S}$, $C^{S|i}$ and $C^{i|k}$:

Individual Conditional Tail Expectation:

$$\mathrm{ICoTE}_{1-\gamma}(F_i \mid h(F)) := \mathbb{E}[F_i \mid h(F) > \mathrm{VaR}_{1-\gamma}(h(F))] = C^{i|S}\gamma^{-1/\alpha}$$

Systemic Conditional Tail Expectation:

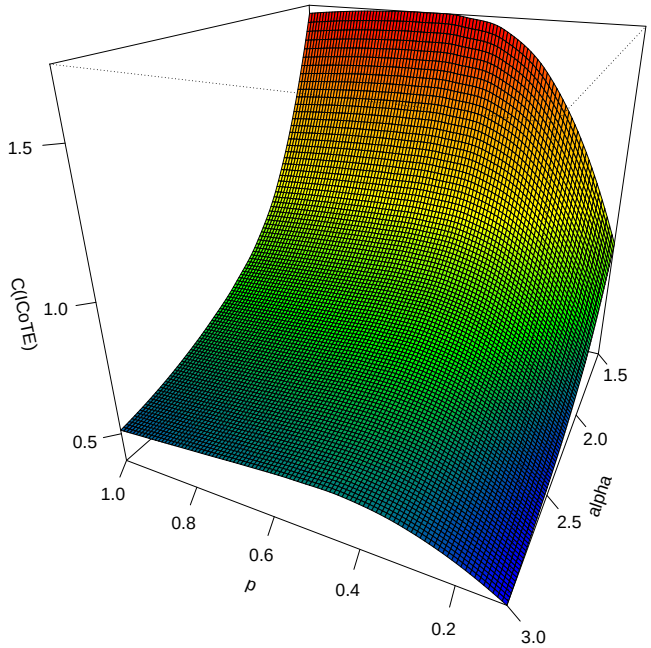
$$\mathrm{SCoTE}_{1-\gamma}(h(F) \mid F_i) := \mathbb{E}[h(F) \mid F_i > \mathrm{VaR}_{1-\gamma}(F_i)] = C^{S|i}\gamma^{-1/\alpha}$$

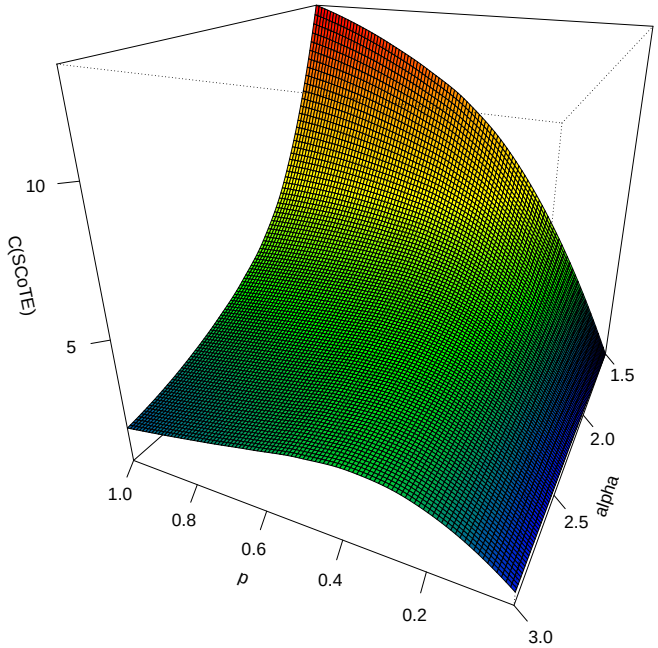
Mutual Conditional Tail Expectation:

$$\mathrm{MCoTE}_{1-\gamma}(F_i \mid F_k) := \mathbb{E}[F_i \mid F_k > \mathrm{VaR}_{1-\gamma}(F_k)] = C^{i|k}\gamma^{-1/\alpha}$$

If all edge probabilities are equal ...

Risk of company i, given the market is under stress (left), $C^{i|S}$; and risk of the market given company i is under stress, $C^{S|i}$ (right)





Left plot: $C^{i|S}$ Right plot: $C^{S|i}$

For tail index $\alpha \in [1.5, 3]$ and market activity $p \in (0.01, 1.0]$.

If all edge probabilities are equal ...

- All companies are exchangeable.
- **Risk of a company given the market is under stress, $C^{i|S}$:**
  For fixed Pareto exponent α, the conditional risk increases first in p and then decreases, for larger p there is a positive diversification effects. This effect is stronger for larger α (lighter tails).
- **Risk of the market given a company is under stress, $C^{S|i}$:**
  - For large Pareto exponent α (lighter tails), the conditional risk increases first in p and then decreases, for larger p there is a positive diversification effects.
  - For smaller α (heavier tails) the conditional risk increases for all p.
  Consequently, an increased market diversification does not always lower the corresponding conditional risk in the system.

If companies have different risk aversion ...

Recall that the probability of a specific company insuring a specific risk is given by edge probabilities p_{ij}.

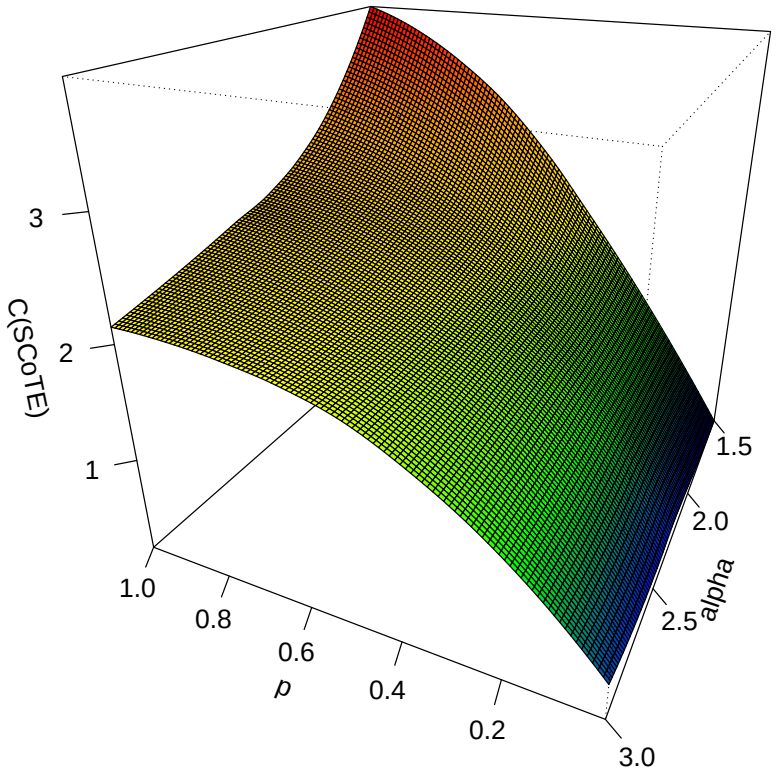
Recall also from the unconditional risk measures:
$p_{ij} =$ (risk proneness of company i)$\times$(attractiveness of risk j) $= \beta_i \delta_j$.

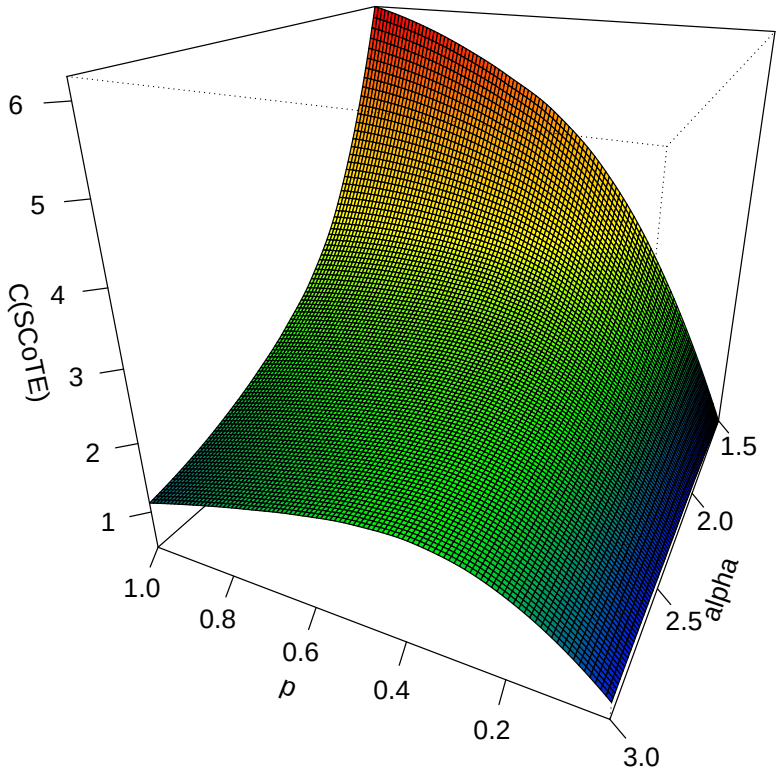
Assume the following scenario:
1 risk averse company and 4 risk-prone companies, and
4 attractive risks and one less attractive risk:
$\beta = (0.2, 1, 1, 1, 1)$, $\delta = (0.9, 0.9, 0.9, 0.9, 0.2)$.

$p_{ij} = p\beta_i\delta_j$, where p stands for the **market activity.**

Scenario : $\beta = (0.2, 1, 1, 1, 1)$, $\delta = (0.9, 0.9, 0.9, 0.9, 0.2)$

Risk of the market given company i is under stress, $C^{S|i}$.





Left: risk averse company. Right: risk prone company.

$C^{S|i}$ for tail index $\alpha \in [1.5, 3]$ and market activity $p \in (0.01, 1.0]$.

If companies have different risk aversion ...

- There is a profound difference when the risk-averse company is under stress compared to when the risk-prone company is under stress.
- Risk averse company under stress: risk of the system increases for increasing edge probabitlities given by p and decreasing Pareto exponent α (heavier tails).
- Risk prone company under stress: risk of the system exhibits a positive effect of risk diversification for not too small α, when p increases.
- This diversification effect can compensate in the system the stress of a risk-prone company better than the stress of the risk-averse company: the risk for lighter tailed risk and large p is smaller.

Some ideas of where to go from here ...

- Fitting the model to real data sets (possibly amend the model) and study a range of scenarios through extensive simulation.
- Optimisation of the risk portfolio of a company (including premium considerations).
- Adapt the model for Operational Risk modelling and apply to real data; in preparation.

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