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Matching polytopes, toric geometry, and the non-negative part of the Grassmannian

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Abstract

We use toric geometry to investigate the topology of the totally non-negative part of the Grassmannian $(\mathrm{Gr}_{kn})_{\geq 0}$. $(\mathrm{Gr}_{kn})_{\geq 0}$ is a cell complex whose cells Δ_G can be parameterized in terms of the combinatorics of bicolored planar graphs G. To each cell Δ_G we associate a complete fan $\mathcal{F}_G$ which is normal to a certain polytope $P(G)$. We then use $P(G)$ to define an associated toric variety X_G, and extend our parameterization of the cell Δ_G to a rational map from X_G. This allows us to prove that the cell decomposition of $(\mathrm{Gr}_{kn})_{\geq 0}$ is in fact a CW complex, and furthermore, that the Euler characteristic of the closure of each cell of $(\mathrm{Gr}_{kn})_{\geq 0}$ is 1.

The polytopes $P(G)$ possess some remarkable properties. The combinatorial structure of the polytopes $P(G)$ is reminiscent of the well-known Birkhoff polytopes, and we describe their face lattices in terms of matchings and unions of matchings of G. We also define a surjective map Ψ from $P(G)$ to the corresponding matroid polytope. Finally, we explore the connection between the $P(G)$ and the cluster algebra structure on cells of $(\mathrm{Gr}_{kn})_{\geq 0}$, showing that fibers of Ψ are in fact Newton polytopes of the cluster variables.

This is joint work with Alex Postnikov and David Speyer.