Curves in moduli stacks

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http://www.uni-due.de/~mat903/montreal07.pdf

$f : X \to Y$ a non-isotrivial semistable family of n-folds over a curve, smooth over $U = Y \setminus S$.

Theorem. (V.-Z.) For all $\nu \geq 1$ with $f_* \omega^\nu_{X/Y} \neq 0$

$$(0 \leq) \frac{\deg(f_* \omega^\nu_{X/Y})}{\mathrm{rk}(f_* \omega^\nu_{X/Y})} \leq \frac{n \cdot \nu}{2} \cdot \deg(\Omega^1_Y(\log S)).$$

Assume the general fibre F is either canonically polarized or with $\omega^\nu_F = \mathcal{O}_F$. So f induces a map φ from U to a moduli stack $\mathcal{M}_h$ of polarized minimal manifolds with a certain Hilbert polynomial h.

Theorem. There exists a projective compactification $\bar{M}_h$ of the moduli scheme M_h and an invertible ($\mathbb{Q}$-) sheaf λ_ν such that $\varphi^* \lambda_\nu = \det(f_* \omega^\nu_{X/Y})$,

with λ_ν nef and ample w.r.t. M_h.

$f : X \to Y$ a non-isotrivial semistable family of n-folds over a curve, smooth over $U = Y \setminus S$.

Theorem. (V.-Z.) For all $\nu \geq 1$ with $f_*\omega^\nu_{X/Y} \neq 0$

$$h(f) = \frac{\deg(f_*\omega^\nu_{X/Y})}{\mathrm{rk}(f_*\omega^\nu_{X/Y})} \leq \frac{n \cdot \nu}{2} \cdot \deg(\Omega^1_Y(\log S)).$$

Assume from now on: $\mathcal{M}_h$ is the moduli scheme of canonically polarized manifolds or of polarized minimal models of Kodaira dimension zero.

- *Interpretation* 1. $h(f)$ is a height function on the moduli stack.
- $h(f) > 0$ if f is non-isotrivial and it is bounded from above in terms of $g(Y)$ and $\#S$.
- *Interpretation* 2. $\mathcal{M}_h$ is algebraically hyperbolic, i.e. it does not contain $\mathbb{C}^*$ or an elliptic curve E.

Conjectures. (Y projective, $U = Y \setminus S$ for a NCD S, $U \to M_h$ generically finite, induced by a family $f : V \to U$).

1. The moduli stack $\mathcal{M}_h$ is Brody hyperbolic, i.e.:

   There is no holomorphic curve $\gamma : \mathbb{C} \to U$ with dense image.
2. The moduli stack $\mathcal{M}_h$ is of log-general type. Or stronger: $\kappa(\omega_Y(S)) = \dim(Y)$, for all Y as above.
3. The sheaf $\Omega^1_Y(\log S)$ is weakly positive.

   i.e. For $\mathcal{H}$ ample invertible, for all $\alpha > 0$ and for $\beta \gg 1$ the sheaf

$$S^{\alpha\beta}(\Omega^1_Y(\log S)) \otimes \mathcal{H}^\beta$$

   is globally generated over some open set.

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3. The sheaf $\Omega^1_Y(\log S)$ is weakly positive.

   Known: 1) for moduli of canonically polarized manifolds.

   Known: 3) $\Longrightarrow$ 2).

   Known: (Kebekus, Kovacs) 2) for moduli of canonically polarized manifolds if either $\dim(Y) = 2$
   or if $S = \emptyset$ assuming the MMP.

   Remark: 1), 2), 3) hold for period domains, hence if one has local Torelli Theorems for $\mathcal{M}_h$ as well.

Main Lemma:

Consider for $\nu \geq 1$ an ample invertible subsheaf $\mathcal{H}$ of $f_*\omega^{\nu}_{X/Y}$.

Then there is a non-trivial morphism $\mathcal{H} \to S^{\ell}(\Omega^1_Y(\log S))$ for some $\ell > 0$.

Idea of proof:

Replacing Y by a covering one may assume that $\mathcal{H} = \mathcal{H}'^{\nu}$.

Let $\tau \in H^0(X, \mathcal{H}'^{-\nu} \otimes \omega^{\nu}_{X/Y})$ be the induced section with zero divisor B.

Study the VHS for the cyclic coverings obtained by taking the ν-th root out of τ. First term is $F^{n,0} = f'_*\omega_{X'/Y}$.

It contains $f_*\big(\omega_{X/Y} \otimes \mathcal{H}' \otimes \omega^{-1}_{X/Y}\big) = \mathcal{H}'$.

The iterated Higgs field of the covering X' gives maps

$$F^{n,0} \to F^{n-1,1} \otimes \Omega^1_Y(\log S + T) \to \cdots$$
$$\cdots \to F^{n-\ell,\ell} \otimes S^\ell(\Omega^1_Y(\log S + T))$$

where T corresponds to the new singular fibres.

Since Kernels of Higgs fields are negative one obtains:

$$\mathcal{H}' \subset f'_*\omega_{X'/Y} \to \cdots \to \mathrm{negative} \otimes S^\ell(\Omega^1_Y(\log S + T)),$$

So one gets $\{negative\}^\vee \otimes \mathcal{H}' \xrightarrow{\neq 0} S^\ell(\Omega^1_Y(\log S + T))$.

Since a quotient of $\{negative\}^\vee$ has a non-negative degree $S^\ell(\Omega^1_Y(\log S + T))$ contains an ample invertible sheaf $\mathcal{H}''$.

An easy calculation of the new singular fibres shows that $\mathcal{H}''$ lies in $S^\ell(\Omega^1_Y(\log S))$.

$\mathcal{H}''$ is larger than $\mathcal{H}^{\frac{1}{\nu}}$ and $\ell \leq n$. So

For some $\ell \leq n \cdot \nu$ the sheaf $S^{\ell}(\Omega^1_Y(\log S))$ contains $\mathcal{H}$.

First Applications:

- $\Omega_Y(\log S)$ w.p. $\Longrightarrow$ 2): Apply this to the $r = \mathrm{rk}(f_*\omega^{\nu}_{X/Y})$ fold selfproduct of the family $\Longrightarrow S^{\ell}(\Omega^1_Y(\log S))$ contains an ample subsheaf, $+$ w.p. $\Longrightarrow \Omega^1_Y(\log S)$ is big.

- Y, S can not be:

  Y an Abelian variety and $S = \emptyset$.

  $Y = \mathbb{P}^m$ and S a normal crossing divisor with less than m irreducible components.

For Brody hyperbolicity one needs:

For $\gamma : \mathbb{C} \to U$ and $\sigma \in H^0(Y, S^\ell(\Omega^1_Y(\log S)) \otimes \mathcal{H}^{-1})$ the pullback $\gamma^*(\sigma) \in H^0(\mathbb{C}, S^\ell(\Omega^1_{\mathbb{C}}) \otimes \gamma^*\mathcal{H}^{-1})$ is non zero.

Method: Construct the covering on U "transversal to γ".

Then replace "negative" or "positive" by curvature estimates.

However, to controll the new singular fibres one needs (at the moment) "canonically polarized".

Applying the main Lemma to the $\mathrm{rk}(f_*\omega^\nu_{X/Y})$-th selfproduct of f one gets the inequality

$$\frac{\deg(f_*\omega^\nu_{X/Y})}{\mathrm{rk}(f_*\omega^\nu_{X/Y})} \leq n \cdot \nu \cdot \deg(\Omega^1_Y(\log S)).$$

Up to a factor $\frac{1}{2}$ this is what I claimed.

"Winning $\frac{1}{2}$ in the Main Lemma": for $\dim(Y) = 1$.

Consider for $\nu \geq 1$ an ample invertible subsheaf $\mathcal{H}$ of $f_* \omega^{\nu}_{X/Y}$.

Then $d = \deg(\mathcal{H}) \leq \frac{n \cdot \nu}{2} \cdot \deg(\Omega^1_Y(\log S))$.

The argument used before only gives $n \cdot \nu$.

To get an additional $\frac{1}{2}$ one has to use the negativity of the degree of the sub Higgs field generated by $\mathcal{H}'$ instead of the negativity of kernels of Higgs fields.

In this form, the Main Lemma is of interest, even if the local Torelli holds, hence if 1), 2), and 3) are true.

(And even for families of curves:).

Theorem. (MVZ) There is no Shimura curve in $\mathcal{M}_g$, except for $g = 1$ or $g = 3$.

Recall: $f : X \to Y$ with $\dim(Y) = 1$, smooth on $V = F^{-1}(U)$ and semi-stable. Write $S = Y \setminus U$, $\Delta = f^{-1}(S)$ and $n = \dim(F)$ for a general fibre F.

Consider: $\mathbb{V} \subset R^n f_* \mathbb{C}_V$ a non-unitary local subsystem.

$\rightsquigarrow$ Deligne extension $\mathcal{H}$ of $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_U$ with $F^\bullet$ filtration.

$\rightsquigarrow$ logarithmic Higgs bundle $E = \mathfrak{gr}_{F^\bullet}(\mathcal{H}) = E^{n,0} \oplus \cdots \oplus E^{0,n}$ with Higgs field $\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y(\log S)$

(induced by the Gauss-Manin connection).

Here $E^{p,q} \subset R^q f_* \Omega^p_{X/Y}(\log \Delta)$.

For example $E^{n,0} \subset f_* \omega_{X/Y} = f_* \Omega^1_{X/Y}(\log \Delta)$.

Definition. $E^{n,0}$ is non-isotrivial iff the image of X in $\mathbb{P}(E^{n,0})$ is non isotrivial.

For $\nu = 1$ the Main Lemma implies:

$$\frac{\deg(E^{n,0})}{\mathrm{rk}(E^{n,0})} \leq \frac{n}{2} \cdot \deg(\Omega^1_Y(\log S)).$$

Addendum: "$=$" $\Longrightarrow E^{n,0}$ is a semistable vector bundle.

Proposition. $E^{n,0}$ is non-isotrivial $\Longrightarrow$

$$0 < \frac{\deg(E^{n,0})}{\mathrm{rk}(E^{n,0})} < \frac{n}{2} \cdot \deg(\Omega^1_Y(\log S)).$$

Reason: If $=$, then the image of $S^\nu(E^{n,0}) \to f_*\omega^\nu_{X/Y}$ has to have degree $\geq \frac{n\cdot\nu}{2} \cdot \deg(\Omega^1_Y(\log S))$
and "non-isotrivial" $\Longrightarrow\ > \frac{n\cdot\nu}{2} \cdot \deg(\Omega^1_Y(\log S))$, contradicting the Main Lemma.

Addendum. For families of curves ($n = 1$) same holds if $\mathrm{rk}(E^{1,0}) > 1$ and the Higgs field without kernel.

This implies the non-existence of Shimura curves in M_g, $g \neq 1, 3$.

Proof uses the Theory of Teichmüller curves, starting from:

Theorem. (M. Möller) $f : V \to U$ is a Teichmüller curve, iff there exists some $\mathbb{V}$ of rank two, with $\deg(E^{1,0}) = \frac{1}{2} \cdot \deg(\Omega^1_Y(\log S))$.

And in addition:

Theorem. (VZ) If $f : V \to U$ is a Shimura curve, then the non-unitary part of the Higgs bundle $E^{1,0} \oplus E^{0,1}$ satisfies $\frac{\deg(E^{1,0})}{\mathrm{rk}(E^{1,0})} = \frac{1}{2} \cdot \deg(\Omega^1_Y(\log S))$.

Idea of proof of Addendum and Theorem (MVZ):

If $\mathrm{rk}(E^{1,0}) > 1$, and if $E^{1,0}$ is isotrivial, then it is easy to see that $U \neq Y$.

$\Longrightarrow$ $\mathbb{V}$ decomposes as a direct sum of rank 2 subsystems

$\Longrightarrow$ Teichmüller curve. Then use:

Theorem. (C. McMullen) For a Teichmüller curve there exists at most one $\mathbb{V}$ of rank two, with $\frac{\deg(E^{1,0})}{\mathrm{rk}(E^{1,0})} = \frac{1}{2} \cdot \deg(\Omega^1_Y(\log S))$.

Finally, for $\mathrm{rk}(\mathbb{V}) = 2$ it is easy to see that $g \leq 7$, and finally one can exclude all g except $g = 1,\ 3$.

So one should expect few extremal (or Shimura curves) in moduli schemes of canonically polarized manifolds.

What about $\kappa = 0$, for example Calabi-Yau manifolds of dimension n?

Known: For all n there exists extremal curves in $\mathcal{M}_h$, i.e. curves $U \in \mathcal{M}_h$ such that $E^{n,0}$ has degree $\frac{1}{2} \cdot \deg(\Omega^1_Y(\log S))$.

For $n = 2$ there exist compact curves with this property.

For n odd there are no compact extremal curves.

Open Problems.

1. $\exists$ compact extremal curves for n even?
2. $\exists$ Brody curves?

What about higher dimensional bases? Example: Families of Abelian varieties over Y, $\dim(Y) = m$:

So consider a non-unitary irreducible weight one VHS $\mathbb{V} \subset R^1 f_* \mathbb{C}_V$ with Higgs bundle $E^{1,0} \oplus E^{0,1}$.

Assume $\Omega^1_Y(\log S)$ is nef, and $\omega_Y(S)$ ample with respect to U.

One has an inequality

$$\mu(\mathbb{V}) := \mu(E^{1,0}) - \mu(E^{0,1}) \leq \mu(\Omega^1_Y(\log S)),$$

where $\mu(\mathcal{F}) = \frac{\mathrm{c}_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} . \mathrm{c}_1(\omega_Y(S))^{m-1}$.

And $=$ implies that $E^{1,0}$ and $E^{0,1}$ are semistable.

$= +$ second numerical condition $\stackrel{?}{\Longleftrightarrow}$ U Shimura.