Logarithmic Surfaces and Hyperbolicity

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Notations:

$\bar{X}$: (compact) algebraic surface,
$D \subset \bar{X}$: divisor with normal crossings,
$(\bar{X}, D)$: log surface.

$T^*_{\bar{X}}$: cotangent bundle,
$\bar{T}^*_X$: log cotangent bundle.

$q_{\bar{X}} = \dim_{\mathbf{C}} H^0(\bar{X}, T^*_{\bar{X}})$: irregularity,
$\bar{q}_X = \dim_{\mathbf{C}} H^0(\bar{X}, \bar{T}^*_X)$: log irregularity,
$\bar{K}_X = \bigwedge^2 \bar{T}^*_X$: log canonical bundle,
$\bar{\kappa}_X = \kappa(\bar{X}, \bar{K}_X)$: log Kodaira dimension.

$\bar{\kappa}_X = -\infty, 0, 1, 2$. If $\bar{\kappa}_X = 2$, the log surface $(\bar{X}, D)$ is called of log general type.

$\alpha_X : X \to \mathcal{A}_X$ quasi-Albanese map. It is a holomorphic map which extends to a rational map $\bar{\alpha}_X : \bar{X} \to \bar{\mathcal{A}}_X$ (Iitaka '76).

Theorem 1 (Noguchi '81) *For any log surface s.th. $\bar{q}_X > 2$, any entire holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerated.*

Theorem 2 *Let $(\bar{X}, D)$ a log surface of log general type with log irregularity $\bar{q}_X = 2$. Then any entire holomorphic Brody curve $f : \mathbf{C} \to X$ is algebraically degenerated.*

Corollary 3 *Let $(\bar{X}, D)$ be a log surface with log irregularity $\bar{q}_X = 2$ and log Kodaira dimension $\bar{\kappa}_X > 0$. If X doesn't contain any non-hyperbolic algebraic curve and if D is hyperbolically stratified, then X is complete hyperbolic and hyperbolically imbedded.*

Corollary 4 *Let $D \subset \mathbf{P}_2$ be a normal crossing curve of degree at least 4 consisting of at least 3 components. Then every Brody curve $f : \mathbf{C} \to \mathbf{P}_2 \setminus D$ is algebraically degenerated.*

Theorem 2 does not hold in general for $\bar{\kappa}_X \leq 1$:

Proposition 5 *(Counterexample for $\bar{\kappa}_X = 1$) Let E be an elliptic curve and $p : E \times E \to E$ the projection to the first factor. Let $P_1, P_2 \in E$ be two distinct points, and $Q_i \in p^{-1}(P_i)$, $i = 1, 2$ two points. Let $b : \bar{X} \to E \times E$ be the blow up of $E \times E$ in the points Q_1, Q_2. Let D the union of the proper transforms of $p^{-1}(P_i)$, $i = 1, 2$ in $\bar{X}$, and $X := \bar{X} \setminus D$. Then $\bar{q}_X = 2$ and $\bar{\kappa}_X = 1$, but X admits entire curves $f : \mathbf{C} \to X$ which are not algebraically degenerate.*

So what kind of additional condition is needed ?

Let $(\bar{X}, D)$ be a log surface with log irregularity $\bar{q}_X = 2$. Let $\bar{\alpha}_X : \bar{X} - \to \bar{\mathcal{A}}_X$ be the compactified Albanese map, I its finite set of points of indeterminacy and $\bar{\alpha}_0 = \bar{\alpha}_X|_{\bar{X} \setminus I}$. In the case of dominant $\bar{\alpha}_X$, we consider the following condition:

(*) For all $z \in \mathcal{A}_X$ and $\bar{E}$ a connected component of the Zariski closure of $\bar{\alpha}_0^{-1}(z)$ with $\bar{E} \cap X \neq \emptyset$, any connected component of D intersecting $\bar{E}$ is contained in $\bar{E}$ (i.e. $\bar{E}$ is a connected component of $\bar{E} \cup D$).

We remark that condition (*) can be expressed intrinsically, is equivalent to a condition on the Stein factorization of a desingularization of $\bar{\alpha}_X$, and is implied by the condition that all the fibers of $\alpha_X : X \to \mathcal{A}_X$ are compact. In particular, this condition is much weaker than the properness of α_X.

In the case $\bar{\kappa}_X = 1$ we have the following result:

Theorem 6 *Let* $(\bar{X}, D)$ *be a log surface with log irregularity* $\bar{q}_X = 2$ *and with log Kodaira dimension* $\bar{\kappa}_X = 1$. *Assume condition* (*) *in the case of dominant* $\bar{\alpha}_X$. *Then every entire curve* $f : \mathbf{C} \to X$ *is algebraically degenerate.*

The following result generalizes Kawamata's theorem for normal surfaces finite over a semi-abelian surface to surfaces with log irregularity 2 by relating it to entire holomorphic curves.

Theorem 7 *Let* $(\bar{X}, D)$ *be a log surface with log irregularity* $\bar{q}_X = 2$ *. In the case of dominant* $\bar{\alpha}_X$*, assume condition (*). Then the following are equivalent.*

(1) *There is an entire curve* $f : \mathbf{C} \to X$ *such that* $f^* w \equiv 0$ *for some* $w \in H^0(\bar{T}_X^*)$ *and* f *is not algebraically degenerate.*

(2) $\bar{\kappa}_X = 0$.

(3) $\bar{\alpha}_X$ *is birational and* $\mathcal{A}_X \setminus \alpha_X(X)$ *is finite.*

Remark 8 *The construction of Proposition 5 also shows that without condition (*) condition (1) is neither equivalent to condition (2) nor to condition (3).*

Remark 9 *Much is known already about the hyperbolicity of log surfaces having ample irreducible components D_i of D and small Neron-Severi groups (Noguchi-Winkelmann '02).*

The case where $\bar{X}$ is the projective plane (e.g. Demailly-ElGoul) or a torus (e.g. Noguchi et al.) are particularily well understood.

In the case of proper quasi-Albanese map α_X, the result was recently proved by Noguchi-Winkelmann-Yamnoi '05 even in dimension n. However, our result do not require any condition in the case of log general type and our condition (*), which we use in the general case, is still much weaker than the condition of properness of the quasi-Albanese map:

Let X be the complement of a smooth ample divisor D in a abelian surface $\bar{X}$. Then $\bar{K}_X = D$ and so $\bar{\kappa}_X = 2$. Also $\bar{q}_X = q_{\bar{X}} = 2$. This means that the compactified quasi-Albanese map is the identity map and so the Albanese map is not proper.

Basic Proof Ideas of Theorem 2:

I:)

We will reduce Theorem 2 by a result of McQuillen and ElGoul and by log-Bloch's theorem to the following:

Theorem 10 *Let* $(\bar{X}, D)$ *be a log surface with* $\bar{q}_X = 2$ *and dominant* $\bar{\alpha}_X$ *and with log Kodaira dimension* $\bar{\kappa}_X = 2$*. Let* $f : \mathbf{C} \to X$ *be a Brody curve. Assume that* f *is not algebraically degenerate. Then* $\alpha_X \circ f : \mathbf{C} \to \mathcal{A}_X$ *is a translate of a complex one parameter subgroup of* $\mathcal{A}_X$*.*

The result of McQuillan and ElGoul we need is the following:

Theorem 11 (ElGoul '03) *Let $(\bar{X}, D)$ be a log surface of log general type. Let $f : \mathbf{C} \to X$ be an entire curve. Suppose that there exists a foliation $\mathcal{F}$ on $\bar{X}$ such that f is (contained in) a leaf of $\mathcal{F}$. Then f is algebraically degenerate in $\bar{X}$.*

What we will need for the proof of Theorem 2 is only a corollary of a very special case of Theorem 11:

Proposition 12 *Let $(\bar{X}, D)$ be a log surface of log general type. Let*
$\Psi : X \to \mathcal{A}_X$ be a dominant morphism which extends to a rational map $\bar{\Psi} : \bar{X} - \to \bar{\mathcal{A}}_X$. Let $f : \mathbf{C} \to X$ be an entire curve. Assume that the map $\Psi \circ f : \mathbf{C} \to \mathcal{A}_X$ is linearly degenerate with respect to the universal cover $\mathbf{C}^2 \to \mathcal{A}_X$. Then f is algebraically degenerate.

II:)

- In the case $q_{\bar{X}} = 2$, the compactified quasi-Albanese map $\bar{\alpha}_X$ is a morphism and so this theorem is trivial.

- If $q_{\bar{X}} < 2$, $\bar{\alpha}_X$ can have points of indeterminacy so that Brody curves are not preserved by $\bar{\alpha}_X$ in general.

- From value distribution theory, the order of growth of a holomorphic curve is preserved under rational maps and Brody curves are of order at most two.

- The key analysis in this proof consists of a detailed study of the geometry of the quasi-Albanese map (in particular at its points of indeterminacy) with respect to f to reduce the order of $\bar{\alpha}_X \circ f$ to one or less. Then it is a translate of a one parameter subgroup.

- We do this componentwise where in the case $q_{\bar{X}} = 1$, we use the fact from Noguchi-Winkelmann-Yamanoi '02 that one can choose a metric on $\bar{\mathcal{A}}_X$ which lifts to the product metric on the universal cover $\mathbf{C} \times \mathbf{P}_1$ of $\bar{\mathcal{A}}_X$.

- In the case $q_{\bar{X}} = 0$ we take rational monomials of the components of $\bar{\alpha}_X$ motivated by arranging residues in a way that allows us to control the points of indeterminacy of the resulting map with respect to f.

Basic Proof Ideas of Theorems 6 and 7:

I:)
We first prove a generalization of the structure theorem of Kawamata for open subsets of finite branched covers of semi-abelian varieties:

Theorem 13 (Kawamata '81) *Let X be a normal algebraic variety, $\mathcal{A}$ a semi-abelian variety and let $f : X \to \mathcal{A}$ be a finite morphism. Then $\bar{\kappa}_X \geq 0$ and there exist a connected complex algebraic subgroup $\mathcal{B} \subset \mathcal{A}$, étale covers $\tilde{X}$ of X and $\tilde{\mathcal{B}}$ of $\mathcal{B}$, and a normal algebraic variety $\tilde{Y}$ such that*

(i) $\tilde{Y}$ is finite over $\mathcal{A}/\mathcal{B}$.

(ii) $\tilde{X}$ is a fiber bundle over $\tilde{Y}$ with fiber $\tilde{\mathcal{B}}$ and translations by $\tilde{\mathcal{B}}$ as structure group.

(iii) $\bar{\kappa}_{\tilde{Y}} = \dim \tilde{Y} = \bar{\kappa}_X$.

If further $\bar{\kappa}_X = 0$ and f is surjective, then f is an étale morphism.

Theorem 14 *Let X be a normal algebraic surface, $\mathcal{A}$ a semi-abelian surface and let $f : X \to \mathcal{A}$ be a finite morphism. Let $X_0 \subset X$ be an open algebraic subvariety.*

(1) *In the case $\bar{\kappa}_{X_0} = 1$, let $\Phi : X^* \to Y^*$ be the logarithmic Iitaka fibration of X_0, $\Psi : X^* \to X_0$ the birational morphism relating X_0 to X^*, and for $y \in Y^*$, let $X^*_y = \Phi^{-1}(y)$, $X_y = \Psi(X^*_y) \subset X_0 \subset X$ and $B_y = f(X_y)$. Then for generic $y \in Y^*$, $X_y \subset X$ is a closed subvariety and B_y is a translate of a complex one parameter algebraic subgroup $\mathcal{B}$ of $\mathcal{A}$. Moreover, there are étale covers $\tilde{X}$ of X and $\tilde{\mathcal{B}}$ of $\mathcal{B}$, and a smooth algebraic curve $\tilde{Y}$ such that*

(i) $\tilde{Y}$ *is finite over* $\mathcal{A}/\mathcal{B}$*.*

(ii) $\tilde{X}$ *is a fiber bundle over* $\tilde{Y}$ *with fiber* $\tilde{\mathcal{B}}$ *and translations by* $\tilde{\mathcal{B}}$ *as structure group. In particular,* X *and* $\tilde{X}$ *are smooth.*

(iii) $\dim\tilde{Y} = \bar{\kappa}_X = 1$*.*

In particular, for generic $y \in Y^*$*,* $X_y \subset X$ *is equal to the image of a suitable fiber of the fiber bundle* $\tilde{X}$ *over* $\tilde{Y}$*, and the image of a generic fiber of this fiber bundle is of the form* X_y *for* $y \in Y^*$*.*

(2) *If* $\bar{\kappa}_{X_0} = 0$*, then* f *is an étale morphism and* $X \setminus X_0$ *is finite.*

We follow essentially the original ideas of Kawamata but with several new ingredients:

- In the case $\bar{\kappa}_{X_0} = 1$, one needs to observe that even though the quasi-Albanese map to the semi-abelian variety is not proper, the restriction to the generic fiber is.

- In the case $\bar{\kappa}_{X_0} = 0$, we need to observe that a complement of a (singular) curve in a semi-abelian variety is of log general type unless the curve is a translate of a algebraic subgroup. We reduce (the proof of) this observation by the addition theorem of Kawamata below to the case where the semi-abelian variety is a simple abelian variety.

Theorem 15 (Kawamata '77) *Let $f : V \to B$ be a fibered variety for a nonsingular algebraic surface V and a nonsingular algebraic curve B. Let F be a general fiber of f. Then*

$$\bar{\kappa}_V \geq \bar{\kappa}_F + \bar{\kappa}_B.$$

II:)

For Theorem 6, the main observation is that condition (*) is equivalent to a condition on the Stein factorization of a desingularization of the quasi-Albanese map and that this allows us to use Theorem 14 to conclude that the base of the Iitaka fibration is hyperbolic.

III:) Proof of Theorem 7

- (1) $\Rightarrow$ (2):

  Theorem 6 and Theorem 11.

- (2) $\Rightarrow$ (3):

  Theorem 15 and Theorem 14.

- (3) $\Rightarrow$ (1):

  Elementary methods.