A Remark on a Theorem of Bogomolov

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Dedicated to Professor Heisuke Hironaka on his 77th Birthday

Abstract

Let X be a minimal projective surface of general type defined over the complex numbers and let $C \subset X$ be an irreducible curve of geometric genus g. Assume that K_X^2 is greater than the topological Euler number $c_2(X)$. Then we prove that the "canonical degree" CK_X of C is uniformly bounded in terms of the given invariants g, K_X^2 and $c_2(X)$, thus giving an effective version of a theorem of Bogomolov on the boundedness of the curves of fixed genus in X.

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1. Introduction

In 1977, F. Bogomolov showed that, if a minimal complex projective surface X of general type satifies $K_X^2 > c_2(X)$, then the curves of fixed geometric genus g on X form a bounded family. In particular, such X contains only finitely many rational/elliptic curves. Bogomolov's proof (see [2]) involves beautiful ideas, eventually leading to M. McQuillan's partial solution [7] of the Green-Griffiths conjecture [3] about the algebraicity and finiteness of entire holomorphic curves (*i.e.,* non-constant holomorphic images of $\mathbf{C}$) on a surface of general type.

Unfortunately, the result of Bogomolov-McQuillan is not effective. For instance, when the ambient surface X deforms in an analytic family, their theory cannot rule out the possibility that the number of rational/elliptic curves therein tends to infinity. In this note we make the Bogomolov-McQuillan theorem effective, by showing that the *canonical degree* of an irreducible curve of genus g on X is bounded from above by an explicitly given function of g, K_X^2 and $c_2(X)$:

Theorem 1.1(Uniform bound of the canonical degree). *Let X be a minimal projective surface of general type defined over the complex numbers. Let $C \subset X$ be an irreducible curve of geometric genus g. Put $x = CK_X/K_X^2$, $\sigma = c_2(X)/K_X^2$, $\gamma = (g-1)/K_X^2$. If $x > 3\gamma + 3\sigma - 1$, then the inequality*

$$(1-\sigma)x^2 - (3\sigma - 1 + 4\gamma)x + 2\gamma(3\sigma - 1 + 3\gamma) \leq 0 \tag{1}$$

holds. Consequently, when $\sigma < 1$ (i.e., $c_2(X) < K_X^2$), the canonical degree CK_X of C is bounded by a function in g, K_X^2 and $c_2(X)$.

Corollary 1.2 (Uniformity of the number of rational/elliptic curves). *Let X be a minimal projective surface of general type over* **C**. *If $K_X^2 > c_2(X)$, then the number of irreducible curves of genus ≤ 1 on X is bounded by a function of K_X^2 and $c_2(X)$.*

Remarks. A. In connection with S. Kobayashi's complex hyperbolic geometry and P. Vojta's value distribution theory, S. Lang [5] conjectured that the union of all the entire holomorphic curves (rational/elliptic curves, for example) on a variety of general type should be contained in a certain proper algebraic closed subset. L. Caporaso, J. Harris and B. Mazur [1] pointed out that Lang's conjecture in arbitrary dimension would entail Corollary 1.2, *i.e.,* uniform finiteness of rational/elliptic curves on surfaces, even when the assumption $K^2 > c_2$ is dropped (their argument does not give any *explicit* bound, though). It is yet to be seen whether the assumption is really necessary or redundant. It might be any way noteworthy that Bogomolov-Macquillan's proof and ours alike require the very same condition $K^2 > c_2$ in quite different contexts (see Remark **D** below).

B. In case C is nonsingular of genus g, the Miyaoka-Yau-Sakai inequality [11][9] for the open surface $X \setminus C$ gives

$$CK_X \leq 4g - 4 + 3c_2(X) - K_X^2, \tag{2}$$

regardless of the ratio $\sigma = c_2(X)/K_X^2$. This smoothness hypothesis on C was marginally relaxed in [6]. We still have a similar bound for CK_X under the assumption that C contains neither ordinary double points nor ordinary triple points. Strangely, complicated singularities of high multiplicity do not cause any serious problem to get an estimate of the canonical degree. Curves with many ordinary double points are technically the hardest to deal with.

C. Consider a sufficiently ample divisor H on a surface X. In the complete linear system $|H|$ of dimension $(H^2 - HK_X + 2\chi(X, \mathcal{O}_X) - 2)/2$, the curves with m ordinary double points form a locally closed subset of codimension $\leq m$. Hence we have a crude estimate of the minimum g_H of the geometric genera of the members of $|H|$: $g_H \leq HK_X + O(1)$. To put it another way, the supremum $\mathfrak{d}(X,g) = \sup\{CK_X \mid C \subset X, g(C) = g\} \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ of the canonical degrees of curves of genus g is expected to satisfy:

$$\mathfrak{d}(X,g) \geq g + O(1). \tag{3}$$

On the other hand, our quadratic inequality (1) gives the asymptotically linear estimate

$$\mathfrak{d}(X,g) \leq \frac{2+\sqrt{6\sigma-2}}{1-\sigma} g + o(g) \tag{4}$$

as g grows. If compared with the expected lower bound (3), the estimate (4) is not so bad. In fact, it asserts something new even for nonsingular curves. When σ is sufficiently small ($1/3 \leq \sigma < 3/8$), the bound (4) is asymptotically sharper than (2). In the extremal case $\sigma = 1/3$, the left-hand side of (1) is $(2/3)(x-3\gamma)^2$ and Theorem 1.1 boils down to the simple linear inequality $\mathfrak{d}(X,g) \leq 3g-3$. In this case X is an unramified quotient of the complex unit ball $B^2 \subset \mathbf{C}^2$, a hermitian symmetric domain with constant holomorphic sectional curvature [12], and our upper bound $3g-3$ of the canonical degree is indeed attained if the curve $C \subset X$ in question lifts to a totally geodesic holomorphic curve in B^2 [4].

D. The basic idea of Bogomolov-McQuillan is to look at (a) the projective bundle $\pi : \mathbf{P}(\Omega^1_X) \to X$ with the tautological divisor $\mathbf{1}$, (b) an effective divisor $F \in |n\mathbf{1} - m\pi^* K_X|$, and (c) the rational map $\sigma \colon C \to \mathbf{P}(\Omega^1_X)$ which is induced by the natural homomorphism $\Omega_X|_C \to \omega_C$ defined on the non-singular locus of C. Unless $\sigma(C) \subset F$ (*i.e.*, C is a leaf of the multi-valued foliation induced by F), the intersection number $\sigma(C)F$ is non-negative, whence follows the inequality $mCK_X \leq n(2g-2)$ (certain extra techniques take care of the exceptional case $\sigma(C) \subset F$). The hypothesis $K_X^2 > c_2(X)$ appears as a guarantee of the existence of $F \in |n\mathbf{1} - m\pi^* K_X|$ for $n \gg m > 0$.

Our proof of Theorem 1.1 does not depend on the existence of an effective divisor F as above, but relies on old inequalities of [8][9]. What is new in the present note is more or less of technical nature: the systematic use of *orbibundles* (though [9] already used the notion implicitly), along with an elementary reduction process ("*G-nef reduction*") for vector bundles of special type (Lemma 2.3). Theorem 1.1 derives from the Miyaoka-Yau-Sakai inequality [9] applied to a family of orbibundles $\widetilde{\mathcal{E}}_\alpha$ with parameter $\alpha \in [0,1] \cap \mathbf{Q}$. The precise construction of $\widetilde{\mathcal{E}}_\alpha$ via G-nef reduction process will be given in Section 3 below.

Conventions. In this note we work in the category of complex algebraic varieties. Surfaces will be non-singular and projective unless otherwise mentioned. Chern classes of coherent sheaves are regarded as elements in the real Betti cohomology ring $H^*(\cdot, \mathbf{R})$. By the (geometric) genus of an irreducible curve, we mean the genus of its normalization. Effective divisors are often identified with closed subschemes of pure dimension one via the correspondence $A \mapsto \operatorname{Spec}(\mathcal{O}/\mathcal{O}(-A))$.

2. G-nef reductions of certain vector bundles

In this section, we formulate a couple of technical but elementary results, which allow us to modify vector bundles of certain type into much simpler ones, without changing their second Chern classes. Let Z be a surface. A finite sum $G = \sum G_i$ of irreducible curves $G_i \subset Z$ is said to be a *negative definite cycle* if the intersection matrix $(G_i G_j)_{ij}$ is negative definite (from the definition it follows that $G_i \neq G_i$ for $i \neq i$, so that G is reduced). Given a surjective morphism $f\colon Z \to Y$ onto another surface, the f-exceptional locus (the union of the curves which f contracts to points) is a typical example of negative definite cycles.

Proposition 2.1 (The Zariski decomposition with support in a negative definite cycle). *Let $G = \sum G_i \subset Z$ be a negative definite cycle and let A be an effective $\mathbf{Q}$-divisor on Z. Then there exists a unique decomposition $A = P + N$ into $\mathbf{Q}$-divisors which satisfies the following four conditions:*

(a) *Both P and N are effective: $P \geq 0$, $N \geq 0$.*

(b) *N is supported by a subset of G, i.e., $N = \sum \nu_i G_i$, $\nu_i \geq 0$.*

(c) *P is nef on G, i.e., $PG_i \geq 0$ for every i.*

(d) *P and N are mutually orthogonal, i.e., $PN = 0$ (hence $A^2 = P^2 + N^2$ and, in view of* (c)*, P is numerically trivial on N, i.e., $PG_i = 0$ for each $G_i \subset \operatorname{supp} N$).*

Furthermore, P is the largest effective $\mathbf{Q}$-divisor $\leq A$ that is nef on G:

(e) *If a $\mathbf{Q}$-divisor B with $0 \leq B \leq A$ is nef on G, then $B \leq P$.*

Definition. The unique decomposition $A = P + N$ as above is said to be the *Zariski decomposition with support in G*. We call the $\mathbf{Q}$-divisors P and N the *G-nef part* and the *G-negative part* of A, respectively.

This definition generalizes classical Zariski decompositions [13]. Indeed, the classical decomposition $A = \overline{P} + \overline{N}$ is the decomposition with support in $\overline{N}$. Among the Zariski decompositions of a fixed divisor A with support in various negative cycles, the classical one is characterized as the one which has the largest negative part and the smallest positive part (see Corollary 2.2 (iii) below). In order to avoid unnecessary confusions, we hereafter call the classical Zariski decomposition the *absolute* Zariski decomposition.

Proposition 2.1 is proved in exactly the same manner (essentially an exercise of linear algebra) as in the case of absolute Zariski decompositions [13][10].

Corollary 2.2. (i) Take two effective $\mathbf{Q}$-divisors A, A' and let $A = P + N$, $A' = P' + Q'$ be the decompositions into the G-nef parts and the G-negative parts. If $A \le A'$, then $P \le P'$.

(ii) *Let G and $\widehat{G}$ be two negative definite cycles and let $A = P + N = \widehat{P} + \widehat{N}$ be the Zariski decompositions with supports in G and $\widehat{G}$. If $G \le \widehat{G}$, then $N \le \widehat{N}$, $\quad P \ge \widehat{P}$. Furthermore, we have the inequalities between self-intersection numbers: $0 \ge N^2 \ge \widehat{N}^2$, $P^2 \le \widehat{P}^2$.*

(iii) *Let $A = P + N$ be the Zariski decomposition with support in G and let $A = \overline{P} + \overline{N}$ denote the absolute Zariski decomposition. Then we have $N \le \overline{N}$, $P \ge \overline{P}$, $0 \ge N^2 \ge \overline{N}^2$, $P^2 \le \overline{P}^2$.*

Proof. (i) is a direct consequence of the property (e) of P'. Let us prove (ii). By definition, $\widehat{P}$ is nef on $\widehat{G}$ and hence on G, so that $\widehat{P} \le P$ and $\widehat{N} \ge N$. In particular, supp $\widehat{N} \ge$ supp N. Let $\widehat{V}$ [resp. V] denote the subspace of $H^2(Z, \mathbf{R})$ generated by the curves in $\operatorname{supp} \widehat{N}$ [resp. $\operatorname{supp} N$]. Then $\widehat{P}$ [resp. P] sits in the orthogonal complement $\widehat{V}^\perp$ [resp. $V^\perp$], while $R = P - \widehat{P} = \widehat{N} - N \in V^\perp \cap \widehat{V}$. Hence $P = \widehat{P} + R \in \widehat{V}^\perp \oplus (V^\perp \cap \widehat{V})$ is an orthogonal decomposition in $V^\perp$ and so is $\widehat{N} = N + R \in V \oplus (V^\perp \cap \widehat{V})$ in $\widehat{V}$. Noting that $R^2 \le 0$, we get (ii). The proof of (iii) is quite similar to that of (ii). $\square$

Take a surjective morphsm $f\colon Z \to Y$ between nonsingular projective surfaces and denote by $G \subset Z$ the f-exceptional locus (which is of course a divisor of simple normal crossings). Let $\Delta \subset Z$ be an effective, reduced, normal crossing divisor which contains G.

Pick up an effective reduced divisor Γ on Y such that $G \subset f^{-1}(\Gamma) \subset \Delta$. The image $f(G)$ of G in Y is a finite subset of Γ and therefore we can find an affine open neighborhood $U \subset Y$ of $f(G)$ on which Γ is defined by a single equation φ. Since the inverse image $f^{-1}(\Gamma)$ is supported by a normal crossing divisor $\subset \Delta$, the pull back $f^*\varphi$ of its defining equation is of the form (unit)$z_1^a z_2^b$, where $a \geq 0, b \geq 0, a+b > 0$ and z_1, z_2 are local coordinates of Z around a point $q \in G \subset f^{-1}(\Gamma)$. Thus the logarithmic 1-form

$$f^* d\log\varphi = a\frac{dz_1}{z_1} + b\frac{dz_2}{z_2} + (\text{regular 1-form}), \qquad (a,b) \neq (0,0)$$

is a nowhere-vanishing section of $\Omega^1_Z(\log\Delta)|_{f^{-1}(U)}$ near G.

Lemma 2.3. *Let $Y, Z, f, G, \Gamma, \Delta, d\log\varphi$ be as above. Assume that a vector bundle $\mathcal{E}$ of rank two on Z satisfies the following four conditions:*

(a) $\mathcal{E} \subset \Omega^1_Z(\log\Delta)$.

(b) *The determinant divisor $D = \det\mathcal{E}$ is effective* (as a $\mathbf{Q}$-divisor).

(c) *The G-nef part P and the G-negative part N of D are integral divisors.*

(d) *$\mathcal{E}$ contains $f^*d\log\varphi \in \Omega^1_Z(\log\Delta)$ on a certain neighborhood V of G.*

Then, after shrinking V to a smaller neighborhood if necessary, we have an exact sequence $0 \to \mathcal{O}_V \to \mathcal{E}|_V \to \mathcal{O}_V(\det\mathcal{E}) \to 0$ and we can construct a new vector bundle $\widetilde{\mathcal{E}}$ on Z which satisfies the following three conditions:

i) $\widetilde{\mathcal{E}} \subset \mathcal{E}$ and $\widetilde{\mathcal{E}} = \mathcal{E}$ outside G;

ii) $\det\widetilde{\mathcal{E}} = P$;

iii) $c_2(\widetilde{\mathcal{E}}) = c_2(\mathcal{E})$;

Proof. The logarithmic 1-form $f^*d\log\varphi$ gives a nowhere vanishing global section of $\mathcal{E}|_V$ and hence an injection $\mathcal{O}_V \to \mathcal{E}|_V$, of which the cokernel is locally free and isomorphic to $\det\mathcal{E}|_V$.

We regard N as the subscheme determined by the ideal $\mathcal{O}_Z(-N)$. Consider the composite of the natural projections $\mathcal{E} \to \mathcal{E}|_V \to \mathcal{O}_V(\det\mathcal{E}) \to \mathcal{O}_N(\det\mathcal{E})$ and define $\widetilde{\mathcal{E}}$ to be the kernel of this composite map. We have the equality $c(\widetilde{\mathcal{E}})c(\mathcal{O}_N(\det\mathcal{E})) = c(\mathcal{E})$ in $H^*(Z, \mathbf{R})$. Since $\det\mathcal{E} = P + N$, P being numerically trivial on N, we get

$$c(\mathcal{O}_N(\det\mathcal{E})) = c(\mathcal{O}_N(N)) = c(\mathcal{O}_Z(N))\,c(\mathcal{O}_Z)^{-1} = 1 + N.$$

Hence

$$\begin{aligned} P + N &= \det\mathcal{E} = c_1(\mathcal{E}) = c_1(\widetilde{\mathcal{E}}) + N = \det\widetilde{\mathcal{E}} + N, \\ c_2(\mathcal{E}) &= c_2(\widetilde{\mathcal{E}}) + c_1(\widetilde{\mathcal{E}})N = c_2(\widetilde{\mathcal{E}}) + PN = c_2(\widetilde{\mathcal{E}}). \quad \square \end{aligned}$$

Definition. The new vector bundle $\widetilde{\mathcal{E}} \subset \mathcal{E}$ obtained in Lemma 2.3 is called the *G-nef reduction* of $\mathcal{E}$.

3. Proof of Theorem 1.1

In this section, we define an orbibundle $\mathcal{E}_\alpha$ determined by the triple (α, X, C), where $\alpha \in [0,1] \cap \mathbf{Q}$, X is a minimal surface of general type and C is an irreducible curve on it. Theorem 1.1 follows from the Miyaoka-Yau-Sakai inequality applied to the nef reduction $\widetilde{\mathcal{E}}_\alpha$ of $\mathcal{E}_\alpha$.

Throughout the section, we use the following symbols:

α	a parameter $\in [0,1] \cap \mathbf{Q}$,
X	a minimal surface of general type,
C	an irreducible curve on X,
e	the topological Euler number $c_2(X)$ of X,
g	geometric genus of C,
s	the number of the singular points of C,
$\mu : Y \to X$	the blowing up at the s singular points of C,
$E_1, \dots, E_s$	the exceptional curves on Y,
Γ	the exceptional locus $E_1 + \cdots + E_s$ of μ.

Since $\mu^{-1}(C)$ may not be a divisor of simple normal crossings, we choose a log-resolution π: $(\widetilde{Y}, \overline{C}) \to \big(Y, \mu^{-1}(C)\big)$. Namely,

(a) $\pi : \widetilde{Y} \to Y$ is the composite $\widetilde{Y} = Y_r \xrightarrow{\pi_r} Y_{r-1} \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_1} Y_0 = Y$, where π_i is the blowing up at a singular point of $\pi_{i-1}^{-1} \cdots \pi_1^{-1} \mu^{-1}(C)$;

(b) $\overline{C}$, the inverse image $\pi^{-1}\mu^{-1}(C)$ with reduced structure, is a divisor of simple normal crossings. (In particular, the strict transorm $\tilde{C} \subset \tilde{Y}$ of $C \subset X$ is nonsingular. $\overline{C}$ is a sum of $\tilde{C}$ and a disjoint union of s connected trees of $\mathbf{P}^1$'s.)

We introduce further symbols associated with this log resolution:

- $E_{s+i} \subset Y_i$ is the (-1)-curve attached to $\pi_i : Y_i \to Y_{i-1}$ $\quad (i = 1, \dots, r)$.
- $\overline{E}_i \in \mathrm{Div}(\widetilde{Y})$ is the total transform of E_i $\quad (i = 1, \dots, s, s+1, \dots, s+r)$, and $\overline{E}$ is the sum $\overline{E}_1 + \cdots + \overline{E}_{s+r}$ (hence $K_{\widetilde{Y}} = \pi^*\mu^*K_X + \overline{E}$).
- $F_1, \dots, F_s$ [resp. $G_1, \dots, G_r$] are the strict transforms on $\widetilde{Y}$ of $E_1, \dots, E_s$ [resp. of $E_{s+1}, \dots, E_{s+r}$], and $F = F_1 + \cdots + F_s$ $\quad$ (the strict transform of $\Gamma = E_1 + \cdots + E_s \subset Y$).

Thus the exceptional locus G of π is $G_1 + \cdots + G_r$, while the inverse image $\pi^{-1}\mu^{-1}(C)$ is $\overline{C} = \widetilde{C} + F + G$.

For a given parameter $\alpha \in [0,1] \cap \mathbf{Q}$, we define the *orbibundles* $\mathcal{E}_\alpha$ to be the kernel of the homomorphism $\rho\colon \Omega^1_{\widetilde{Y}}(\log \overline{C}) \to \mathcal{O}_{(1-\alpha)\widetilde{C}}$ induced by the natural residue map $\Omega^1_{\widetilde{Y}}(\log \overline{C}) \to \mathcal{O}_X/\mathcal{O}_X(-\widetilde{C})$. To be more precise, choose any branched Galois covering $f\colon Z \to \widetilde{Y}$ such that $A_\alpha = (1-\alpha)f^*\widetilde{C} \leq f^*\widetilde{C}$ is an integral divisor. Then let $\mathcal{E}_\alpha$ denote $\operatorname{Ker}\left(f^*\Omega^1_{\widetilde{Y}}(\log \overline{C}) \to \mathcal{O}_Z/\mathcal{O}_Z(-A_\alpha)\right)$, which is a well defined vector bundle on Z.[1] Its total Chern class $c(\mathcal{E}_\alpha)$ is computed by

$$\begin{aligned} c(\mathcal{E}_\alpha) &= c\left(f^*\Omega^1_{\widetilde{Y}}(\log \overline{C})\right) c\left(\mathcal{O}_{A_\alpha}\right)^{-1} \\ &= c\left(f^*\Omega^1_{\widetilde{Y}}(\log \overline{C})\right)\left(1-(1-\alpha)f^*\widetilde{C}\right). \end{aligned}$$

Denoting by d the mapping degree $[\mathbf{C}(Z):\mathbf{C}(\widetilde{Y})] = [\mathbf{C}(Z):\mathbf{C}(X)]$ of f, we have:

$$\begin{aligned} c_1\left(\mathcal{E}_\alpha\right) &= f^*\left(K_{\widetilde{Y}} + F + G + \alpha\widetilde{C}\right) = f^*\left(\pi^*\mu^*K_X + \overline{E} + F + G + \alpha\widetilde{C}\right), \\ c_2\left(\mathcal{E}_\alpha\right) &= c_2\left(f^*\Omega^1_{\widetilde{Y}}\left(\log(F+G)\right)\right) + c_2\left(\mathcal{O}_{\alpha f^*\widetilde{C}}(-(1-\alpha)f^*\widetilde{C})\right) \\ &\qquad + c_1\left(f^*\Omega^1_{\widetilde{Y}}\left(\log(F+G)\right)\right) c_1\left(\mathcal{O}_{\alpha f^*\widetilde{C}}(-(1-\alpha)f^*\widetilde{C})\right) \\ &= d\left(e - s + \alpha(K_{\widetilde{Y}} + F + G)\widetilde{C} + \alpha\widetilde{C}^2\right) \\ &= d\left(e - s + \alpha(2g-2) + \alpha\widetilde{C}(F+G)\right) \in H^4(Z,\mathbf{Z}) \simeq \mathbf{Z}. \end{aligned}$$

Put $D_\alpha = \pi^*\mu^*K_X + \overline{E} + F + G + \alpha\widetilde{C}$. The $\mathbf{Q}$-divisor D_α and the second Chern class of $\mathcal{E}_\alpha$ can be computed in terms of resolution data. Write

$$\widetilde{C} = \pi^*C - \sum_{i=1}^{s+r} m_i\overline{E}_i, \qquad F + G = \sum_{i=1}^{s}\overline{E}_i + \sum_{i=s+1}^{s+r}\varepsilon_i\overline{E}_i,$$

where

$$\begin{cases} m_i \geq 2 & \text{for } i = 1,\ldots,s \\ m_i \geq 1, \varepsilon_i \leq 0 & \text{for } i = s+1,\ldots,s+r. \end{cases}$$

[1] Rigorously speaking, our choice of the covering $f\colon Z \to \widetilde{Y}$ depends on the (denominator of the) rational number α and we should write $f_\alpha\colon Z_\alpha \to \widetilde{Y}$; however, our subsequent argument is little affected by the choice and we may always replace Z by its good ramified covers of sufficiently large degree. In this sense, the natural framework for our purpose is the projective limit of good coverings of $\widetilde{Y}$.

Then

$$D_\alpha = \pi^*\mu^*(K_X + \alpha C) + \sum_{i=1}^{s}(2 - \alpha m_i)\overline{E}_i + \sum_{j=s+1}^{s+r}(1 + \varepsilon_j - \alpha m_j)\overline{E}_j,$$
$$\frac{c_2(\mathcal{E}_\alpha)}{d} = \frac{c_2(\widetilde{\mathcal{E}}_\alpha)}{d} = e + \alpha(2g-2) + \sum_{i=1}^{s}(\alpha m_i - 1) + \sum_{i=s+1}^{r}\alpha\varepsilon_i m_i.$$

As the expression above shows, the $\mathbf{Q}$-divisor D_α and hence $c_1(\mathcal{E}_\alpha)$ are in general not nef on G or on $f^{-1}(G)$. Let $D_\alpha = P_\alpha + N_\alpha$ be the Zariski decomposition into the G-nef part and the G-negative part (thanks to the uniqueness property, the Zariski decomposition of $c_1(\tilde{\mathcal{E}}_\alpha)$ with support in $f^{-1}(G)$ is given by $f^*D_\alpha = f^*P_\alpha + f^*N_\alpha$). We write

$$N_\alpha = \sum_{j=s+1}^{s+r} b_j\overline{E}_j,$$

$b_j = b_j(\alpha)$ being rational numbers. If $j = s+1, \ldots, s+r$, then the effective divisor $\overline{E}_j$ is supported by a subset of G so that $0 \le P_\alpha\overline{E}_j = (D_\alpha - N_\alpha)\overline{E}_j = -1 - \varepsilon_j + \alpha m_j + b_i$, thus proving

Lemma 3.1. *$b_j \ge 1 + \varepsilon_j - \alpha m_j$ for $j = s+1, \ldots, s+r$, and therefore*

$$-N_\alpha^2 \ge \sum_{j=s+1}^{s+r}\left(\max\{1 + \varepsilon_j - \alpha m_j, 0\}\right)^2. \tag{5}$$

Noting the inclusion relations $(\pi f)^*\Omega^1_Y(\log\Gamma) \subset \mathcal{E}_\alpha \subset \Omega^1_Z(\log f^{-1}(\overline{C}))$, we can apply Lemma 2.3 to $\mathcal{E} = \mathcal{E}_\alpha$, with minor modifications (we change f, G, Δ, P, N *etc.* to $\pi\circ f, f^{-1}(G), f^{-1}(\overline{C}), f^*P_\alpha, f^*N_\alpha$ *etc.* and, if necessary, we replace Z by its suitable ramified cover in order to make f^*P_α, f^*N_α integral divisors). Let $\widetilde{\mathcal{E}}_\alpha$ be the $f^{-1}(G)$-nef reduction of $\mathcal{E}_\alpha$ with $c_1(\widetilde{\mathcal{E}}_\alpha) = f^*P_\alpha$, $c_2(\widetilde{\mathcal{E}}_\alpha) = c_2(\mathcal{E}_\alpha)$.

Proposition 3.2. We have the formula:

$$\begin{aligned}\frac{3c_2(\widetilde{\mathcal{E}}_\alpha) - c_1^2(\widetilde{\mathcal{E}}_\alpha)}{d} &= \frac{3c_2(\mathcal{E}_\alpha) - c_1^2(\mathcal{E}_\alpha) + (f^*N_\alpha)^2}{d}\\ &= 3e - K_X^2 - 2\alpha(CK - 3g + 3) - \alpha^2C^2\\ &\quad + \sum_{i=1}^{s}\left(1 - \alpha m_i + \alpha^2 m_i^2\right) + \sum_{j=s+1}^{t}\left(3\alpha\varepsilon_j + (1 + \varepsilon_j - \alpha m_j)^2 - b_j^2\right).\end{aligned}$$

The proof is immediate by simple calculation.

In this formula, the right-hand side is a sum of three terms: the first is independent of the singularity of C, the second involves certain data coming from the first s blowups and the third is concerned with further infinitely near singularities of C. As for the third term, we observe

Lemma 3.3. *For $j = s+1, \dots, s+r$, we have the estimate*

$$3\alpha\varepsilon_j m_j + (1 + \varepsilon_j - \alpha m_j)^2 - b_j^2 \leq \frac{3\alpha^2 m_j(m_j - 1)}{2}.$$

Proof. If $1+\varepsilon_j-\alpha m_j > 0$, then the assertion follows from the two inequalities $b_j \geq 1 + \varepsilon_j - \alpha m_j$ and $\varepsilon_j \leq 0$. Assume that $1 + \varepsilon_j \leq \alpha m_j$. Then

$$\begin{aligned}
&3\alpha\varepsilon_j m_j + (1 + \varepsilon_j - \alpha m_j)^2 = \alpha^2 m_j^2 - \alpha(2 - \varepsilon_j)m_j + (1 + \varepsilon_j)^2 \\
&\quad \leq \alpha^2 m_j^2 - \alpha(2 - \varepsilon_j)m_j + \alpha(1 + \varepsilon_j)m_j = \alpha^2 m_j^2 - \alpha(1 - 2\varepsilon_j)m_j \\
&\quad \leq \alpha^2 m_j^2 - \alpha m_j \leq \alpha^2 m_j^2 - \alpha^2 m_j \leq (3/2)\alpha^2 m_j(m_j - 1). \quad \square
\end{aligned}$$

The G-nef divisor P_α may not be nef on $F+G$. Let $P_\alpha = \widehat{P}_\alpha + \widehat{N}_\alpha$ [resp. $P_\alpha = \overline{P}_\alpha + \overline{N}_\alpha$] be the Zariski decomposition with support in $F + G$ [resp. the absolute Zariski decomposition].

Lemma 3.4 (*Cf.* Lemma 3.1, Equation (5)). In the notation above,

$$\overline{N}_\alpha^2 \leq \widehat{N}_\alpha^2 \leq -\sum_{i=1}^{s} \left(\max\{2 - \alpha m_i, 0\}\right)^2.$$

Proof. We showed that $\overline{N}_\alpha^2 \leq \widehat{N}_\alpha^2$ in Corollary 2.2. Put $\widehat{N}_\alpha = \sum_{i=1}^{s+r} \widehat{b}_i \overline{E}_i$. Then the coefficient $2 - \alpha m_i - \widehat{b}_i$ of $\overline{E_i}$ in $\widehat{P}_\alpha$ $(i \leq s)$ is non-positive by the same reason as in Lemma 3.1. Therefore $|\widehat{b}_i| \geq \max\{2 - \alpha m_i, 0\}$ for $i \leq s$, whence follows $\widehat{N}_\alpha^2 \leq -\sum_{i=1}^{s}(\max\{2 - \alpha m_i, 0\})^2$. $\square$

Proposition 3.5. *We have*

$$\begin{aligned}
&\frac{1}{d}\left(3c_2(\widetilde{\mathcal{E}}_\alpha) - c_1^2(\widetilde{\mathcal{E}}_\alpha) + \frac{(f^*\overline{N}_\alpha)^2}{4}\right) \\
&\qquad \leq 3e - K_X^2 - 2\alpha(CK_X - 3g + 3) - \alpha^2 C^2 + \frac{3\alpha^2}{2}\sum_{i=1}^{s+r} m_i(m_i - 1) \\
&\qquad = 3e - K_X^2 - 2\alpha(CK_X - 3g + 3) + \frac{\alpha^2}{2}\left(C^2 + 3CK_X - 6g + 6\right).
\end{aligned}$$

Proof. The second equality readily follows from the adjunction formula $C^2 + CK_X + \sum m_i(m_i - 1) = 2g - 2$ for $\widetilde{C} \subset \widetilde{Y}$. In view of Proposition 3.2, Lemma 3.3 and Lemma 3.4, the first inequality reduces to the estimate

$$4\left(1 - \alpha m_i + \alpha^2 m_i\right) - (\max\{2 - \alpha m_i, 0\})^2 \leq 6\alpha^2 m_i(m_i - 1)$$

for $i = 1, \ldots, s$. In case $\alpha m_i \leq 2$, the left-hand side is $3\alpha^2 m_i^2$, which satisfies $\leq 6\alpha^2 m_i(m_i - 1)$ because $m_i \geq 2$. If $\alpha m_i \geq 2$, then

$$\begin{aligned} &4 - 4\alpha m_i + 4\alpha^2 m_i^2 - 6\alpha^2 m_i(m_i - 1) \\ &\quad \leq 2\alpha m_i - 4\alpha m_i + 4\alpha^2 m_i^2 - 6\alpha^2 m_i(m_i - 1) \\ &\quad = 2\alpha m_i(3\alpha - 1 - \alpha m_i) \leq 2\alpha m_i(2 - \alpha m_i) \leq 0. \qquad \square \end{aligned}$$

Proof of Theorem 1.1. Put $x = CK_X/K_X^2$, $\sigma = e/K_X^2$, $\gamma = (g-1)/K_X^2$. We view the polynomial

$$Q(\alpha) = 6\sigma - 2 - 4\alpha(x - 3\gamma) + \alpha^2\left(x^2 + 3x - 6\gamma\right)$$

as a function of α with parameters x, σ, γ. Then $Q(\alpha)$ must be non-negative on the interval $[0, 1]$ by virtue of

(a) Proposition 3.5,

(b) the Hodge index theorem $C^2/K_X^2 \leq x^2$ and

(c) the Miyaoka-Yau-Sakai inequality $3c_2(\widetilde{\mathcal{E}}_\alpha) - c_1^2(\widetilde{\mathcal{E}}_\alpha) + (1/4)(f^*\overline{N}_\alpha)^2 \geq 0$ ([9] Theorem 1.1).

Specifically the values $Q(0)$ and $Q(1)$ give $3\sigma - 1 \geq 0$ (the Miyaoka-Yau inequality [8]) and $x^2 \geq x - 6\gamma - 6\sigma + 2$, respectively. Assume that $x > 3\gamma + 3\sigma - 1$ $(\geq (6\gamma + 3\sigma - 1)/2)$. The coefficient $x^2 + 3x - 6\gamma$ $(\geq 2(2x - 6\gamma - 3\sigma + 1))$ of α^2 is then positive and $Q(\alpha)$ attains the minimum value

$$Q(\alpha_0) = 6\sigma - 2 - \frac{4(x - 3\gamma)^2}{x^2 + 3x - 6\gamma}$$

at $\alpha_0 = \dfrac{2(x - 3\gamma)}{x^2 + 3x - 6\gamma} \leq \dfrac{x - 3\gamma}{2x - 6\gamma - 3\sigma + 1}$. Since $x > 3\gamma + 3\sigma - 1 \geq 3\gamma$, we have $0 < \alpha_0 < 1$. Thus $Q(\alpha_0) \geq 0$ or, equivalently,

$$2(x - 3\gamma)^2 - (3\sigma - 1)(x^2 + 3x - 6\gamma) \leq 0,$$

which turns out to be identical with the desired inequality (1). $\square$

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