Non-Liouvillian exact solutions for linear ODEs

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Abstract

Four different approaches for computing exact Non-Liouvillian solutions for second-order linear ODEs, covering a relatively wide variety of problems, are discussed. Implementations of these algorithms are at the base of the current Maple ability for solving this type of ODE problem.

The first approach consists of the systematic solving (when a solution exists) of the equivalence problem between an arbitrary linear ODE and any of the ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ or ${}_0\mathrm{F}_1$ hypergeometric equations, by means of transformations of the form

$$x \rightarrow \frac{p x^n + q}{r x^n + s}, \quad y(x) \rightarrow P(x) y(x)$$

where $\{p, q, r, s, n\}$ are arbitrary constants, and $P(x)$ is an arbitrary function.

The second approach consists of formulating a sequence of invariants for the corresponding Riccati equation, similar to what was done by Liouville with respect to Abel type equations. Then an equivalence problem is formulated and specialized numerical routines are used to tackle the otherwise untractable problem of compositions of resultants.

The third approach explores some simple restrictions to the general form of the (Lie) symmetry generator $\xi d/dx + \eta d/dy$ for Riccati

equations, leading to the systematic solving of a number of ODE families. This is about cases where the solving of the whole ODE family can be reduced to only algebraic manipulations after analyzing the problem using differential algebra techniques.

The fourth approach explores non-standard forms of (Lie) infinitesimals $[\xi, \eta]$ for linear ODEs, where $\xi(x, y)$ different from zero and $\eta(x, y)$ is not a solution of the ODE being solved. This approach is particularly useful in tackling linear ODEs with doubly-periodic coefficients, typically expressed in terms of elliptic functions.