

## Definable subsets of free groups

Mark Feighn

`feighn@newark.rutgers.edu`

*Department of Mathematics & Computer Science*

*Rutgers, The State University of New Jersey*

*101 Warren Street*

*Newark, NJ 07102*

*USA*

### Abstract

Inspired by Zlil Sela's work on the Tarski problem, we are interested in the structure of *definable subsets of a free group*  $\mathbb{F}$ , i.e. sets of the form

$$\{p \in \mathbb{F} \mid S(p)\}$$

where  $S(p)$  is an open sentence in the first order theory of  $\mathbb{F}$ . For example, if  $\mathbb{F} = \langle a, b \rangle$  we might take  $S(p)$  to be the sentence:

$$\exists x \in \mathbb{F} \text{ such that } p = [a, x].$$

This particular sentence has one quantifier, a constant (namely  $a \in \mathbb{F}$ ), and an equality. Inequalities are also allowed as well as the usual operations of logic (such as  $\vee$  and  $\neg$ ).

A definable subset of  $\mathbb{F}$  is a *two-quantifier set* if it can be defined using an  $S(p)$  with at most two quantifiers. Sela has shown that the collection of definable sets coincides with the Boolean algebra generated by two-quantifier sets.

We first consider the structure of zero- and one-quantifier subsets. As a sample result, we follow an idea of Razborov and show that there is a two-quantifier set that is not in the Boolean algebra generated by one-quantifier sets. We also find restrictions on definable subsets of  $\mathbb{F}$  forced by restrictions on the defining sentences. For example, we might ask what definable subsets of  $\mathbb{F}$  look like if  $S(p)$  is required to have no constants or no inequalities.

Secondly, we want to understand definable subsets of  $\mathbb{F}$  up to *negligible sets* where we say that a subset  $D$  of  $\mathbb{F}$  is *negligible* if there is an integer  $K > 0$  such that for every  $\epsilon > 0$  all but finitely many elements

$p \in D$ , thought of as reduced words, admit  $K$  pairs of isomorphic (but not equal) subwords that cover  $(1 - \epsilon)|p|$  of its length  $|p|$ .  $D$  is *co-negligible* if  $D^c$  is negligible. The isomorphism between the subwords is allowed to reverse orientation. For example, if  $\mathbb{F} = \langle a, b \rangle$  the word  $aabbAbaaBBaBB$  is partially covered by 2 pairs of isomorphic subwords, namely  $(aa, aa)$  and  $(bbAb, BaBB)$ , with only one letter uncovered. (Here  $A$  and  $B$  denote the inverses of  $a$  and  $b$ , respectively.) In general, these subwords are allowed to overlap. An example of a negligible subset of  $\mathbb{F}$  (with  $K = 1$  and isomorphic pairs  $(a^{n-1}, a^{n-1})$ ) is  $\{ba^n \mid n \in \mathbb{Z}_+\}$ .

Our main goal (which we are in the process of writing up) is to show that modulo negligible sets there are only two definable subsets of  $\mathbb{F}$ , namely the empty set and all of  $\mathbb{F}$ . More precisely, a definable subset of  $\mathbb{F}$  is either negligible or co-negligible.

Corollaries answer some questions posed by Rips and Sela: a definable subgroup of  $\mathbb{F}$  is either cyclic or all of  $\mathbb{F}$ ; the set of primitive elements of  $\mathbb{F}$  (i.e. those belonging to some basis) is not definable if the rank of  $\mathbb{F}$  is greater than two.

Though hardly apparent from the statements, as with Sela our techniques are geometric. To get into a geometric situation, statements are reinterpreted in terms of sets of maps from finitely presented groups to  $\mathbb{F}$ . These maps give actions of our groups on real trees that are in turn studied via dual laminations on finite 2-complexes.

*This is joint work with Mladen Bestvina.*