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Definable subsets of free groups

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Abstract

Inspired by Zlil Sela's work on the Tarski problem, we are interested in the structure of *definable subsets of a free group* $\mathbb{F}$, i.e. sets of the form

$$\{p \in \mathbb{F} \mid S(p)\}$$

where $S(p)$ is an open sentence in the first order theory of $\mathbb{F}$. For example, if $\mathbb{F} = \langle a, b\rangle$ we might take $S(p)$ to be the sentence:

$$\exists x \in \mathbb{F} \text{ such that } p = [a, x].$$

This particular sentence has one quantifier, a constant (namely $a \in \mathbb{F}$), and an equality. Inequalities are also allowed as well as the usual operations of logic (such as $\vee$ and $\neg$).

A definable subset of $\mathbb{F}$ is a *two-quantifier set* if it can be defined using an $S(p)$ with at most two quantifiers. Sela has shown that the collection of definable sets coincides with the Boolean algebra generated by two-quantifier sets.

We first consider the structure of zero- and one-quantifier subsets. As a sample result, we follow an idea of Razborov and show that there is a two-quantifier set that is not in the Boolean algebra generated by one-quantifier sets. We also find restrictions on definable subsets of $\mathbb{F}$ forced by restrictions on the defining sentences. For example, we might ask what definable subsets of $\mathbb{F}$ look like if $S(p)$ is required to have no constants or no inequalities.

Secondly, we want to understand definable subsets of $\mathbb{F}$ up to *negligible sets* where we say that a subset D of $\mathbb{F}$ is *negligible* if there is an integer $K > 0$ such that for every $\epsilon > 0$ all but finitely many elements

$p \in D$, thought of as reduced words, admit K pairs of isomorphic (but not equal) subwords that cover $(1-\epsilon)|p|$ of its length $|p|$. D is *co-negligible* if D^c is negligible. The isomorphism between the subwords is allowed to reverse orientation. For example, if $\mathbb{F} = \langle a, b\rangle$ the word $aabbAbaaBBaBB$ is partially covered by 2 pairs of isomorphic subwords, namely (aa, aa) and $(bbAb, BaBB)$, with only one letter uncovered. (Here A and B denote the inverses of a and b, respectively.) In general, these subwords are allowed to overlap. An example of a negligible subset of $\mathbb{F}$ (with $K = 1$ and isomorphic pairs (a^{n-1}, a^{n-1})) is $\{ba^n \mid n \in \mathbb{Z}_+\}$.

Our main goal (which we are in the process of writing up) is to show that modulo negligible sets there are only two definable subsets of $\mathbb{F}$, namely the empty set and all of $\mathbb{F}$. More precisely, a definable subset of $\mathbb{F}$ is either negligible or co-negligible.

Corollaries answer some questions posed by Rips and Sela: a definable subgroup of $\mathbb{F}$ is either cyclic or all of $\mathbb{F}$; the set of primitive elements of $\mathbb{F}$ (i.e. those belonging to some basis) is not definable if the rank of $\mathbb{F}$ is greater than two.

Though hardly apparent from the statements, as with Sela our techniques are geometric. To get into a geometric situation, statements are reinterpreted in terms of sets of maps from finitely presented groups to $\mathbb{F}$. These maps give actions of our groups on real trees that are in turn studied via dual laminations on finite 2-complexes.

This is joint work with Mladen Bestvina.