Tau Function of the Dispersionless Toda Hierarchy and planar limit of the Free Energy of the Hermitean Two-Matrix Model

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Outline

•) Definition of the Two-Matrix Model

•) Free Energy and $\frac{1}{N}$ expansion

•) Planar Limit and integration of the Loop Equations

•) Residue Formulas

•) Connection with the Tau function of domains and the NMM

•) Whitham Hierarchy and the String Equation.

Two-Matrix Models

Statistical model on the space of pairs of Hermitean Matrices of size $N \times N$

$$M_i = M_i^\dagger \ , \quad \mathrm{d}\mu(M_1, M_2) :=$$
$$(\mathcal{Z}_N)^{-1} \exp\left[-\frac{1}{\hbar}\mathrm{Tr}\,(V_1(M_1) + V_2(M_2) - M_1M_2)\right] \mathrm{d}M_1\mathrm{d}M_2 \ .$$

$$\mathcal{Z}_N(V_1, V_2) := \iint \mathrm{d}M_1\mathrm{d}M_2\, \mathrm{e}^{-\frac{1}{\hbar}\mathrm{Tr}(V_1(M_1)+V_2(M_2)-M_1M_2)} \ .$$

The *potentials* here are polynomials

$$V_1(x) = \frac{u_{d_1+1}}{d_1+1}x^{d_1+1} + \sum_{J=1}^{d_1}\frac{u_J}{J}x^J \, ,$$
$$\deg(V_1) = d_1{+}1$$

$$V_2(y) = \frac{v_{d_2+1}}{d_2+1}y^{d_2+1} + \sum_{J=1}^{d_1}\frac{v_K}{K}y^K \, ,$$
$$\deg(V_2) = d_2{+}1$$

It is known [BHE1,BHE2] that the quasipolynomials satisfy a finite rank set of Frobenius compatible PDEs, amongst which the polynomial ODEs

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi_{N-d_2}\\ \vdots \\ \psi_N\end{pmatrix} = G_N(x)\begin{pmatrix}\psi_{N-d_2}\\ \vdots \\ \psi_N\end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix}\phi_{N-d_1}\\ \vdots \\ \phi_N\end{pmatrix} = H_N(y)\begin{pmatrix}\phi_{N-d_1}\\ \vdots \\ \phi_N\end{pmatrix}$$

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\pi_n(x)\mathrm{e}^{-V_1(x)}\ ,\ \phi_n(y) = \frac{1}{\sqrt{h_n}}\sigma_n(y)\mathrm{e}^{-V_2(y)}$$

$$\int\int \pi_n(x)\sigma_m(y)\mathrm{e}^{-N(V_1(x)+V_2(y)-xy)} = \delta_{nm}h_n$$

These ODE's have the same spectral curve

$$v_{d_2+1}\det(y\mathbf{1}-G_N(x)) = u_{d_1+1}\det(x\mathbf{1}-H_N(y)) \quad (1)$$

This spectral curve should tend to the spectral curve defining the spectral densities.

Free Energy and $\frac{1}{N}$ expansion

The (renormalized) free energy has (should have) a $\frac{1}{N^2}$ expansion

$$\mathcal{F} = \frac{1}{N^2} \ln \mathcal{Z}_N = \mathcal{F}^{(0)} + \frac{1}{N^2}\mathcal{F}^{(1)} + \dots \tag{2}$$

The expansion is determined by the spectral curve of the model, which depends on an external assumption (subject to consistency) of the number of cuts (i.e. its **genus**). We will describe the *planar limit* $\mathcal{F}^{(0)}$ (the "genus one" correction is in D. Korotkin's talk).

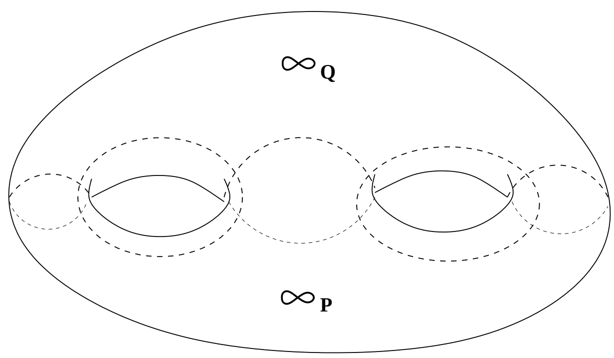
The spectral curve $\mathcal{C}$ is the (limit of) the spectral curve of the associated ODE but can be deduced from the *loop equations* (i.e. reparametrization invariance).

The p.l. $\mathcal{F}^{(0)}$ also depends from other parameters: the **filling fractions** ϵ_j, the fraction of eigenvalues of the two matrices in each connected component of the limiting density. The number of filling fractions is the genus of the spectral curve.

The loop equations and the stability yield the PDE's of $\mathcal{F}^{(0)}$; a (smooth) curve $\mathcal{C}_g$ of genus g is assigned with two marked points ∞_Q, ∞_P. On the curve we are given two functions P and Q which have the following pole structure;

1. The function Q has a simple pole at ∞_Q and a pole of degree d_2 at ∞_P.

2. The function P has a simple pole at ∞_P and a pole of degree d_1 at ∞_Q.



$$P = V_1'(Q) - \frac{t}{Q} + \mathcal{O}(Q^{-2}) \ , \quad \text{near } \infty_Q \tag{3}$$

$$Q = V_2'(P) - \frac{t}{P} + \mathcal{O}(P^{-2}) \ , \quad \text{near } \infty_P. \tag{4}$$

$$u_K = -\operatorname*{res}_{\infty_Q} P Q^{-K} \mathrm{d}Q \ , \quad v_J = -\operatorname*{res}_{\infty_P} Q P^{-J} \mathrm{d}P \ , \tag{5}$$

$$t = \operatorname*{res}_{\infty_Q} P \mathrm{d}Q = \operatorname*{res}_{\infty_P} Q \mathrm{d}P \ . \tag{6}$$

$$\epsilon_i := \frac{1}{2i\pi} \oint_{a_i} P \mathrm{d}Q \ , \quad i = 1, \dots g. \tag{7}$$

$$\partial_{u_K}\mathcal{F}_g = \frac{1}{K}\mathop{\mathrm{res}}_{\infty_Q} PQ^K \mathrm{d}Q \ , \tag{8}$$

$$\partial_{v_J}\mathcal{F}_g = \frac{1}{J}\mathop{\mathrm{res}}_{\infty_P} QP^J \mathrm{d}P \ , \tag{9}$$

$$\partial_{\epsilon_i}\mathcal{F}_g = \oint_{b_j} P\mathrm{d}Q =: \Gamma_i. \tag{10}$$

Theorem 1 The free energy over the curve $\mathcal{C}_g$ is given by

$$2\mathcal{F}_g = \mathop{\mathrm{res}}_{\infty_Q} P\Phi_1\mathrm{d}Q + \mathop{\mathrm{res}}_{\infty_P} Q\Phi_2\mathrm{d}P + \frac{1}{2}\mathop{\mathrm{res}}_{\infty_Q} P^2Q\mathrm{d}Q + t\mu + \sum_{i=1}^{g}\epsilon_i\Gamma_i$$

$$\begin{aligned}
\Phi_1 &= -\frac{1}{2i\pi}\oint \ln\left(1-\frac{\tilde{Q}}{Q}\right)\tilde{P}\mathrm{d}\tilde{Q} \\
&= -V_1(Q) + t\ln(Q) + \int_{X_q} P\mathrm{d}Q = \mathcal{O}(Q^{-1}) \ , \quad \text{near } \infty_Q \\
\Phi_2 &= -\frac{1}{2i\pi}\oint \ln\left(1-\frac{\tilde{P}}{P}\right)\tilde{Q}\mathrm{d}\tilde{P} \\
&= -V_2(P) + t\ln(P) + \int_{X_p} Q\mathrm{d}P = \mathcal{O}(P^{-1}) \ , \quad \text{near } \infty_P.
\end{aligned}$$

$$\mu := \mathop{\mathrm{res}}_{\infty_Q}\left[V_1(Q) - t\ln(Q/\lambda)\right]\mathrm{d}S + \tag{11}$$

$$-\mathop{\mathrm{res}}_{\infty_P}\left[V_2(P) - t\ln(P\lambda)\right]\mathrm{d}S + \tag{12}$$

$$-\mathop{\mathrm{res}}_{\infty_Q} PQ\mathrm{d}S + \sum_{i=1}^{g}\epsilon_i\oint_{b_i}\mathrm{d}S \ , \tag{13}$$

where dS is the normalized differential of the third kind with poles at $\infty_{P,Q}$ and residues ± 1 and the function λ is the following function (defined up to a multiplicative constant) on the universal covering of the curve with a simple zero at ∞_Q and a simple pole at ∞_P

$$\lambda := \exp\left(\int \mathrm{d}S\right) \ . \tag{14}$$

In the literature on the dToda hierarchy [TT] they take implicitly the convenient choice

$$\lambda = \exp\left(\fint_{\infty_Q} \mathrm{d}S\right)$$

so that $Q \sim \infty_Q \lambda$.

Residue Formulas

Residue formulas express the derivatives of the free energy: in short there are two key ingredients
(1) The **Two-point correlators** kernel Ω (= **Bergman kernel**), $\left\langle \mathrm{Tr}(x-M_1)^{-1}\mathrm{Tr}(y-M_2)^{-1}\right\rangle$, etc.
(2) The **Three-point correlators** like
$\left\langle \mathrm{Tr}(x-M_1)^{-1}\mathrm{Tr}(x'-M_1)^{-1}\mathrm{Tr}(x''-M_1)^{-1}\right\rangle$, etc.

Two-point correlator

Determined by a canonical object of the curve (*universality*) $\Omega(\zeta,\zeta') = \Omega(z,z')\mathrm{d}z\mathrm{d}z'$ (z, z' local coordinates near ζ,ζ') with the properties

1. **Simmetry** $\Omega(\zeta,\zeta') = \Omega(\zeta',\zeta)$
2. **Normalization** $\oint_{a_j}\Omega \equiv 0$ and $\oint_{b_j}\Omega = 2i\pi\omega_j$
3. **Singularity** $\Omega(z,z')\mathrm{d}z\mathrm{d}z' \underset{z\to z'}{\sim} \left(\frac{1}{(z-z')^2} + \mathcal{O}(1)\right)\mathrm{d}z\mathrm{d}z'$

All second derivatives of $\mathcal{F}^{(0)}$ are expressed in terms of Ω, e.g.

$$\partial_{u_J}\partial_{v_K}\mathcal{F}^{(0)} = \frac{1}{2i\pi K\, 2i\pi J}\oint_{\infty_Q}\oint_{\infty_P} P^K Q^J \Omega$$

$$\partial_{\epsilon_j}\partial_{\epsilon_k}\mathcal{F}^{(0)} = \oint_{b_j}\oint_{b_k}\Omega = 2i\pi\tau_{jk} \ .$$

$$\partial_t^2\mathcal{F}^{(0)} = \int\!\!\!\!\!-_{\infty_Q}^{\infty_P}\int_{\infty_Q}^{\infty_P}\Omega$$

In general there is a correspondence

$$\partial_{\epsilon_j} \ \longrightarrow\ \frac{1}{2i\pi}\oint_{b_j}$$

$$\partial_{u_K} \ \longrightarrow\ \frac{1}{2i\pi}\oint_{\infty_Q}\frac{Q^K}{K} \ ; \ \partial_{v_J} \ \longrightarrow\ \frac{1}{2i\pi}\oint_{\infty_P}\frac{P^J}{J}$$

$$\partial_t \ \longrightarrow\ \int_{\infty_Q}^{\infty_P}$$

For example the two-point correlator (on the *physical sheet* of the map $Q : \mathcal{C} \mapsto \mathbb{C}$) is

$$\left\langle \mathrm{Tr}(q-M_1)^{-1}\mathrm{Tr}(q'-M_1)^{-1}\right\rangle = \frac{\Omega(q,q')}{\mathrm{d}q\mathrm{d}q'}$$

Three-point correlator

The third derivatives depend "explicitly" on the potentials (i.e. the functions P, Q – not universal) and are given by (e.g.)

$$\partial_{u_K}\partial_{u_J}\partial_{u_L}\mathcal{F}^{(0)} = \oint \frac{Q^K}{K} \oint \frac{Q^J}{J} \oint \frac{Q^L}{L} \sum \operatorname*{res}_{\mathrm{d}Q=0} \frac{\Omega\Omega\Omega}{\mathrm{d}P\mathrm{d}Q}$$

$$\partial_{\epsilon_j}\partial_{\epsilon_k}\partial_{\epsilon_\ell}\mathcal{F}^{(0)} = \oint_{b_j} \oint_{b_k} \oint_{b_\ell} \sum \operatorname*{res}_{\mathrm{d}Q=0} \frac{\Omega\Omega\Omega}{\mathrm{d}P\mathrm{d}Q}$$

All formulas depend on the quadrikernel

$$\Omega^{(3,1)}(\zeta,\zeta',\zeta'';\xi) := \frac{\Omega(\zeta,\xi)\Omega(\zeta',\xi)\Omega(\zeta'',\xi)}{\mathrm{d}P(\xi)\mathrm{d}Q(\xi)}$$

The formula is proven using deformation theory and Beltrami differentials [see Be2].

E.g. we have

$$\left\langle \mathrm{Tr}\frac{1}{q-M_1}\mathrm{Tr}\frac{1}{q'-M_1}\mathrm{Tr}\frac{1}{q''-M_1}\right\rangle = \sum \operatorname*{res}_{\mathrm{d}Q=0} \frac{\Omega(q,\xi)\Omega(q',\xi)\Omega(q'',}{\mathrm{d}q\mathrm{d}q'\mathrm{d}q''\mathrm{d}Q(\xi)}$$

The key point is the variational formula for the Bergman kernel following a variation of the conformal structure of the curve. Note that Krichever [K94] has similar formulas for the universal Whitham Hierarchy and Seiberg-Witten model (without proof).

Connection with the Tau function of domains and NMM

Given a domain $\mathcal{D}$ in $\mathbb{C}$ the tau function is defined by

$$\ln(\tau_{\mathcal{D}}) := -\frac{1}{2\pi}\iint_{\mathcal{D}} \mathrm{d}^2 z \mathrm{d}^2 z' \ln\left|\frac{1}{z} - \frac{1}{z'}\right|^2$$

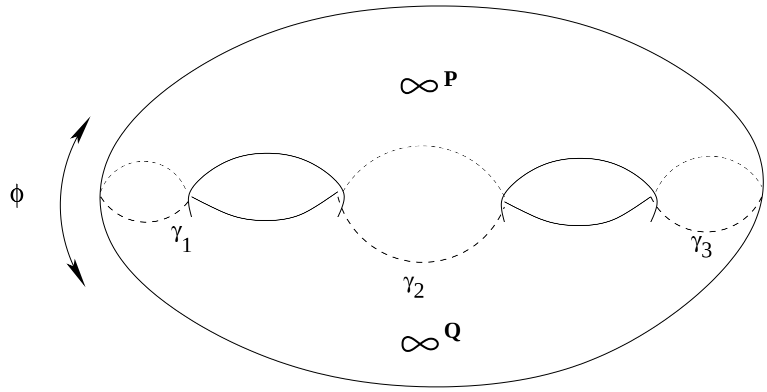
This is a function of the *exterior harmonic moments*

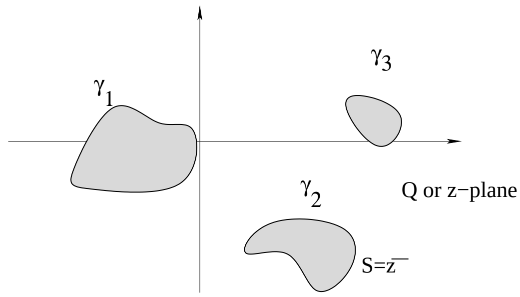
$$t_0 = Area(\mathcal{D}) \ , \ t_k = \iint_{\mathbb{C}\setminus\mathcal{D}} \mathrm{d}^2 z z^{-k} \ .$$

$$\partial_{t_0}\tau_{\mathcal{D}} = \frac{1}{\pi}\iint \mathrm{d}^2 z \ln|z|^2 \ , \quad \partial_{t_k}\tau_{\mathcal{D}} = \frac{1}{\pi k}\iint_{\mathcal{D}} \mathrm{d}^2 z\, z^k =: w_k$$

We recover Wiegmann-Zabrodin-Krichever et al. setting if we impose the further constraint that $\mathcal{C}_g$ is a **M-curve** (Harnack's sense); there is an antiholomorphic involution ϕ which exchanges $P(=S)$ and $Q(=z)$ and has $g+1$ cycles of fixed points.

$$S(\zeta) = \overline{z(\phi(\zeta))}$$





The simplest example of genus 0 curves (i.e. simply connected domains) is

$$z, S: \quad \mathcal{C}_0 \simeq \mathbb{C}P^1 \quad \rightarrow \quad \mathbb{C}P^1$$

$$\lambda \quad \mapsto \quad \begin{array}{l} z := \gamma\lambda + \sum \alpha_j \lambda^{-j} \qquad , \gamma \in \mathbb{R}_+ \\ S := \frac{\gamma}{\lambda} + \sum \overline{\alpha_j} \lambda^j \end{array}$$

$$\phi(\lambda) := \overline{\lambda}^{-1} \; ; \; S(\lambda) = \overline{z}(\lambda) \; , \text{ for } \lambda = \phi(\lambda) \;\Leftrightarrow\; |\lambda| = 1$$

S= Schwarz function of the curve $z : S^1 \hookrightarrow \mathbb{C}$

Using the formula for $\mathcal{F}^{(0)}$ restricted to this case one finds

$$\frac{1}{\pi}\int\!\!\int \mathrm{d}^2 z \ln|z|^2 = 2\Re(V'(z(\lambda)) - S(\lambda)z(\lambda))_0 - t\ln\gamma^2$$

The Tau function is thus given by (one of many equivalent formulas that can be written)

$$\begin{aligned} 4\mathcal{F}^{(0)} &= -{t_0}^2 + 2t_0\Big(2\Re(V'(z(\lambda)) - S(\lambda)z(\lambda))_0 - t_0\ln\gamma^2\Big) + \\ &\quad + 2\Re\sum_k (2-k)t_k w_k \end{aligned}$$

Note that everything can be written in terms of residues of the Riemann map $z(\lambda)$, in particular also the logarithmic moment.

dToda and the String equation

The extension of the dToda hierarchy to higher genus curves is the loop space with target the moduli space $(\mathcal{C}_g, \mathcal{L}, \widetilde{\mathcal{L}}, d_1, d_2)$ of curves of genus g with two marked points and meromorphic functions $\mathcal{L}, \widetilde{\mathcal{L}}$ with certain poles at the marked points. One can define consistently a set of commuting flows on this loop space

$$\partial_{t_K}\mathcal{L} = \left\{\mathcal{L}, \oint \mathcal{L}^K \Omega\right\} \ , \quad \partial_{\tilde{t}_K}\mathcal{L} = \left\{\mathcal{L}, \oint \widetilde{\mathcal{L}}^K \Omega\right\}$$

$$\partial_{\in_j}\mathcal{L} = \left\{\mathcal{L}, \oint_{b_j} \Omega\right\}$$

(and similar equations for $\widetilde{\mathcal{L}}$ with a minus sign), where the "Poisson brackets" are

$$\{A, B\} = \partial_s A \partial_{\mathfrak{t}} B - \partial_{\mathfrak{t}} A \partial_s B$$
$$s := \int \mathrm{d}S \ , \mathfrak{t} = \text{ loop parameter.}$$

The times have in general no geometrical meaning and the two Lax functions do not satisfy the string equation. However

Theorem 2 Any solution of the dToda Hierarchy satisfying the string(s) equation(s) is a (shifted) solution of the generalized free-energy problem

$$\{\mathcal{L}, \widetilde{\mathcal{L}}\} = 1 \;\; \Rightarrow \begin{cases} t_k = u_K + u_K^{(0)}, \\ \tilde{t}_J = v_J + v_J^{(0)} \\ \in_j = \epsilon_j + \epsilon_j^{(0)} \\ \mathfrak{t} = t + t^{(0)} \;\; . \end{cases}$$

In the setting of conformal maps this means that a solution of dToda with reality condition and string equation comes from a conformal map.

It can be shown that the string equation is the symplectic reduction of a rank-two Poisson structure inherited from the embedding of the universal curve in the symplectic structure of the universal Jacobian. Existence of the Tau function is a statement of the integrability of a certain lattice of extended-pseudo periods of meromorphic differentials.

Conclusion

• The formula in terms of residues applies also to one-matrix models providing alternative expressions for the free energy.
• The geometry of these moduli spaces is under investigation and has interesting interpretations in terms of extended Jacobians (work in collaboration with J. Harnad and J. Hurtubise).
• Much of the theory of tau functions for conformal maps is included (and connection with the Normal Matrix Model)

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