# Fourier-Mukai transforms for coherent systems on elliptic curves

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# Fourier-Mukai and Nahm transforms on curves

- Fourier-Mukai and Nahm transforms have proved to be a very useful tool for studying moduli spaces of vector bundles on a wide range of situations:
  - Abelian varieties (Mukai, Braam, van Baal, Maciocia, Bridgeland, . . . )
  - K3 surfaces (Bartocci, Bruzzo, Hernández Ruipérez, Yoshioka . . . )
  - Elliptic fibrations (Bartocci, Bruzzo, Hernández Ruipérez, Muñoz Porras, Bridgeland . . . )
- The applications to moduli spaces of augmented vector bundles (i.e vector bundles + additional data) have been initiated recently:
  - Higgs bundles, Bartocci, Biswas '01, Bonsdorff '06, Jardim, Frejlich '07
  - Holomorphic triples, García-Prada, Hernández Ruipérez, Pioli, TP '05
  - In this talk we deal with a different type of augmented structure: coherent systems on elliptic curves.

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# Coherent systems

## Definitions

$X$ complex projective smooth curve.

### Definition

A coherent system of type $(r, d, k)$ on $X$ is a pair $(E, V)$, where $E$ is a rank $r$ and degree $d$ v.b. on $X$ and $V$ a $k$-dimensional subspace of $H^0(X, E)$. $(E', V')$ is a coherent subsystem of $(E, V)$ if $E' \subset E$, $V' \subset V$.

- Introduced by Le Potier '93, Raghavendra, Vishwanath '94. Stability depending on a parameter $\alpha$.

### Definition

Fix $\alpha \in \mathbb{R}$, the $\alpha$-slope of $(E, V)$ is: $\mu_\alpha(E, V) = \frac{d}{r} + \alpha \frac{k}{r}$.
$(E, V)$ is $\alpha$-stable ( $\alpha$-ss) if

$$\mu_\alpha(E', V') < \mu_\alpha(E, V) \ \ (\mu_\alpha(E', V') \leq \mu_\alpha(E, V))$$

for every proper coherent subsystem $(E', V')$ of $(E, V)$.

# Coherent systems

- Le Potier '93, Raghavendra, Vishwanath '94, King, Newstead '95. Schmitt '00. Construction of moduli spaces. GIT.
- These moduli spaces have been studied by several authors:
  - Brambilla-Paz, Grzegorczyk, Newstead '97, Bradlow, García-Prada '02, Bradlow, García-Prada, Muñoz, Newstead '03. Brill-Noether theory.
  - Bradlow, García-Prada, Mercat, Muñoz, Newstead '03, '07. Geometry of moduli spaces.
  - Lange, Newstead '05. Moduli spaces on elliptic curves.

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# Orthonormal vortex equations

- Bradlow, Daskalopoulos, García-Prada, Wentworth '95 introduced the orthonormal vortex equations.
- If $(E, V)$ is a coherent system, these are equations for a metric on $E$ and a basis of $V$.

## Definition

Fix a Kähler metric on $X$. If $H$ is a hermitian metric on $E$ and $\{\phi_1, \ldots, \phi_k\}$ is a basis of $V$, the orthonormal vortex equations are

$$i\Lambda F_H + \Sigma_{i=1}^{k} \phi_i \otimes \phi_i^* = \tau Id_E$$
$$< \phi_i, \phi_j >_{L^2} = \alpha \delta_{ij}$$

where $F_H$ is the curvature of the Chern connection, $\phi_i^*$ is the adjoint with respect to the metric and $\tau$ is a real parameter related to $\alpha$ by $d + \alpha k = r\tau$.

# Hitchin-Kobayashi correspondence

- The Hitchin-Kobayashi corrrespondence for coherent systems is a correspondence between the property $\alpha$-stability, and the existence of solutions to the orthonormal vortex equations.
- Bradlow, Daskalopoulos, García-Prada, Wentworth '95 proved the easy half of this correspondence.

**Theorem**

*If a coherent system $(E, V)$ admits a solution to the orthonormal vortex equations, then $(E, V)$ is $\alpha$-polystable.*

- Bradlow, García-Prada '96 completed the correspondence.

**Theorem**

*Fix $\alpha > 0$. If $(E, V)$ is an $\alpha$-stable coherent system, then it supports a unique smooth solution to the orthonormal vortex equations.*

# Fourier-Mukai and Nahm transforms for coherent systems

- Let us suppose that we have a Fourier-Mukai transform which preserves $\alpha$-stability of coherent systems.
- That is we have a map

$$\{\alpha\text{-stable coherent systems}\} \xrightarrow{FM} \{\alpha\text{-stable coherent systems}\}$$

- The Hitchin-Kobayashi correspondence gives the following diagram

$$\begin{array}{ccc}
\{\alpha\text{-stable coherent systems}\} & \xrightarrow{FM} & \{\alpha\text{-stable coherent systems}\} \\
HK \downarrow \wr & & \wr \downarrow HK \\
\left\{ \begin{array}{c} \text{Solutions to the orthonormal} \\ \text{vortex equations} \end{array} \right\} & \overset{N}{\dashrightarrow} & \left\{ \begin{array}{c} \text{Solutions to the orthonormal} \\ \text{vortex equations} \end{array} \right\}
\end{array}$$

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# Integral functors

## Fourier-Mukai transforms

$X$ projective variety.

$D(X) \equiv$ bounded derived category of coherent sheaves.

- $X$ and $Y$ smooth varieties, $\mathcal{K}^\bullet \in D(X \times Y)$ a kernel
- Projections

$$\begin{array}{ccc} & X \times Y & \\ {}^{\pi_X}\swarrow & & \searrow^{\pi_Y} \\ X & & Y \end{array}$$

- It defines an integral functor

$$\Phi^{\mathcal{K}^\bullet}_{X \to Y} \colon D(X) \to D(Y)$$

$$\mathcal{E}^\bullet \mapsto \Phi^{\mathcal{K}^\bullet}_{X \to Y} = \mathbf{R}\pi_{Y,*}(\pi_X^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet)$$

# Fully faithful integral functors and equivalences

From now on we assume that $\mathcal{K}^\bullet$ is a single locally free sheaf $\mathcal{P}$.

## Definition

$\mathcal{P}$ is strongly simple over $X$ if $\mathcal{P}_x = \mathcal{P}_{|\{x\}\times Y}$ is simple for every $x$ and for $x_1 \neq x_2$ one has $\operatorname{Ext}^i(\mathcal{P}_{x_1}, \mathcal{P}_{x_2}) = 0$ for every $i$.

## Theorem (Bondal-Orlov-Bridgeland)

1. $\Phi^{\mathcal{P}}_{X\to Y}$ *fully faithful* $\iff$ $\mathcal{P}$ *strongly simple over* $X$.
2. $\Phi^{\mathcal{P}}_{X\to Y}$ *equivalence* $\iff$ $\mathcal{P}$ *strongly simple over* $X$ *and* $\mathcal{P}_x \otimes \omega_Y \simeq \mathcal{P}_x$ *for every* $x \in X$ $\iff$ $\mathcal{P}$ *strongly simple over both factors.*

If $\Phi^{\mathcal{P}}_{X\to Y}$ is an equivalence we call it a *Fourier-Mukai transform*.

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# WIT and IT conditions.

We write $\Phi = \Phi^{\mathcal{P}}_{X\to Y}$, and $\Phi^i(\mathcal{E}^\bullet) = \mathcal{H}^i(\Phi(\mathcal{E}^\bullet))$.

## Definition

A sheaf $E$ on $X$ is $\Phi$-WIT$_i$ if $\Phi^j(E) = 0$ for $j \neq i$. $E$ is $\Phi$-WIT if it is $\Phi$-WIT$_i$ for some $i$. We write $\hat{E}$ for $\Phi^i(E)$.
A sheaf $E$ on $X$ is $\Phi$-IT$_i$ if $H^j(X, E \otimes \mathcal{P}_y) = 0$ for every $j \neq i$ and every $y \in Y$.

## Theorem

*$E$ $\Phi$-IT$_i$ $\iff$ $\Phi$-WIT$_i$ and $\hat{E} = \Phi^i(E)$ is locally-free.*
*Moreover $\hat{E} \otimes \mathcal{O}_y \simeq H^i(X, E \otimes \mathcal{P}_y)$ for every $i$.*

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# Chern characters and Poincaré bundles.

- $X$ an elliptic curve $X$.
- $\mathcal{E}^\bullet \in D(X)$. $\mathrm{ch}(\mathcal{E}^\bullet) = \sum(-1)^i \mathrm{ch}(\mathcal{H}^i(\mathcal{E}^\bullet))$.
- We can write $\mathrm{ch}(\mathcal{E}^\bullet) = (r(\mathcal{E}^\bullet), d(\mathcal{E}^\bullet))$, rank and degree.
- Let $\alpha$ and $\beta$ be coprime integers, $\beta > 0$. $Y = M(\beta, \alpha)$ the moduli space of stable bundles on $X$ with Chern character $(\beta, \alpha)$.
- Actually $Y \simeq X$ (Atiyah). We preserve the distinction for clarity.
- $\mathcal{P}$ be a universal bundle on $X \times Y$. Defined up to twists by pull-backs of line bundles on $Y$.

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# Equivalences. Fourier-Mukai transforms

- $\mathcal{P}$ strongly simple over both factors $\implies$ $\Phi = \Phi^{\mathcal{P}}_{X\to Y}\colon D(X) \to D(Y)$ is an equivalence of (triangulated) categories.
- The following diagram commute

$$\begin{array}{ccc} D(X) & \xrightarrow{\quad\Phi\quad} & D(Y) \\ \Big\downarrow{\scriptstyle \mathrm{ch}} & & \Big\downarrow{\scriptstyle \mathrm{ch}} \\ H^{even}(X,\mathbb{Z}) & \xrightarrow{\ \Phi_*\ } & H^{even}(Y,\mathbb{Z}) \end{array}$$

Here $\Phi_*(v) = \pi_{Y*}(\pi_X^* v \cdot \mathrm{ch}(\mathcal{P}) \cdot \sqrt{\mathrm{td}(X)}) = \pi_{Y*}(\pi_X^* v \cdot \mathrm{ch}(\mathcal{P}))$.

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# Classification of Fourier-Mukai transforms I

## Theorem (Bridgeland)

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}), \quad \beta > 0$$

*There exist a vector bundle $\mathcal{P}$ on $X \times X$ such that:*

1. *$\mathcal{P}$ is strongly simple over both factors. Then $\Phi = \Phi^{\mathcal{P}}_{X \to Y}$ is an equivalence (a FM transform).*
2. $\mathrm{ch}(\mathcal{P}_x) = (\beta, \delta)$, $\mathrm{ch}(\mathcal{P}_y) = (\beta, \alpha)$ *for every $x \in X$, $y \in Y$.*
3. $\begin{pmatrix} r(\Phi(\mathcal{E}^\bullet)) \\ d(\Phi(\mathcal{E}^\bullet)) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} r(\mathcal{E}^\bullet) \\ d(\mathcal{E}^\bullet) \end{pmatrix}$, *for all $\mathcal{E}^\bullet$.*

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# Classification of Fourier-Mukai transforms II

## Theorem (Orlov)

*$X$, $Y$, projective smooth. Any equivalence $\Phi\colon D(X) \to D(Y)$ is given by a Fourier-Mukai functor.*

## Theorem (Orlov, Hille, van den Bergh)

*$X$ elliptic curve. The group of autoequivalences $\mathrm{Aut}(X)$ sits in the exact sequence*

$$0 \to 2\,\mathbb{Z} \times \mathrm{Aut}(X) \ltimes \mathrm{Pic}^0(X) \to \mathrm{Aut}(D(X)) \xrightarrow{\mathrm{ch}} \mathrm{SL}(2,\mathbb{Z}) \to 0$$

*$n \in \mathbb{Z}$ acts by means of the shift functor $[n]$, $(f, L) \in \mathrm{Aut}(X) \ltimes \mathrm{Pic}^0(X)$ sends $\mathcal{E}$ to $f_*(L \otimes \mathcal{E})$ and for any $\Phi \in \mathrm{Aut}(D(X))$ we have $\mathrm{ch}(\Phi) = \Phi_*$.*

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# WIT conditions on elliptic curves

- $\Phi\colon D(X) \simeq D(X)$ FM transform on an elliptic curve $X$ with $\Phi_* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $(\beta > 0)$
- $E$ be a semistable (stable) v.b. on $X$ with $\mathrm{ch}(E) = (r, d)$.

## Theorem

1. $\alpha\, r + \beta\, d > 0 \implies$ *$E$ $\Phi$-IT$_0$ and $\widehat{E}$ ss (stable).*
2. $\alpha\, r + \beta\, d = 0 \implies$ *$E$ $\Phi$-WIT$_1$ and $\widehat{E}$ is a torsion sheaf.*
3. $\alpha\, r + \beta\, d < 0 \implies$ *$E$ $\Phi$-IT$_1$ and $\widehat{E}$ is also ss (stable).*
4. *Finally, if $E$ is $\Phi$-WIT$_i$ then* $\mathrm{ch}(\widehat{E}) = (-1)^i\, \Phi_*(\mathrm{ch}(E))$.

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# WIT conditions on elliptic curves, II

Similar properties are true for the quasi-inverse FM-transform.

- Write $\hat{\Phi} = \Phi^{\mathcal{P}^*}_{X\to X} \colon D(X) \to D(X)$
- Then $\hat{\Phi}[1]$ is a quasi-inverse of $\Phi = \Phi^{\mathcal{P}}_{X\to X}$ (i.e $\hat{\Phi}[1] \circ \Phi = \mathrm{Id}_{D(X)}$).

**Theorem**

1. $-\delta\, r + \beta\, d > 0 \implies E$ $\hat{\Phi}$-IT$_0$ *and* $\widehat{E}$ *ss (stable).*
2. $-\delta\, r + \beta\, d = 0 \implies E$ $\hat{\Phi}$-WIT$_1$ *and* $\widehat{E}$ *is a torsion sheaf.*
3. $-\delta\, r + \beta\, d < 0 \implies E$ $\hat{\Phi}$-IT$_1$ *and* $\widehat{E}$ *is also ss (stable).*
4. *Finally, if $E$ is* $\hat{\Phi}$-WIT$_i$ *then* $\mathrm{ch}(\widehat{E}) = (-1)^{i+1}\,(\Phi_*)^{-1}(\mathrm{ch}(E))$.

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# The evaluation map

- $(E, V)$ coherent system of type $(r, d, k)$. It has an evaluation map $V \otimes \mathcal{O}_X \to E$.
- $\mathcal{C}(X)$ abelian category with objects:
  - sheaf morphisms $\varphi\colon V \otimes \mathcal{O}_X \to \mathcal{E}$, where $V$ is a finite dimensional v.s. and $\mathcal{E}$ is a coherent sheaf. Morphisms are defined in the obvious way.
- The category of coherent systems $S(X)$ is the full additive subcategory of $\mathcal{C}(X)$ formed by the morphisms $\varphi\colon V \otimes \mathcal{O}_X \to \mathcal{E}$ such that $H^0(X, \operatorname{Ker}\varphi) = 0$.
- $\alpha$-(semi)stability in $S(X)$ can be extended to $\mathcal{C}(X)$.
- $\varphi \in \mathcal{C}(X)$ is $\alpha$-(semi)stable $\iff$ it is an $\alpha$-(semi)stable coherent system. (King, Newstead)

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# Fourier-Mukai transforms on $\mathcal{C}(X)$

## Theorem

*$X$ elliptic curve, $0 < a \in \mathbb{Z}$. There exists a unique FM transform $\Phi_a \colon D(X) \to D(X)$, up to composition with automorphisms of $X$, such that*

- *$\mathcal{O}_X$ is $\Phi_a$-$\mathrm{IT}_0$ and $\Phi_a^0(\mathcal{O}_X) = \mathcal{O}_X$*
- $(\Phi_a)_* = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.

If $\hat{\Phi}_a[1]$ is the quasi-inverse of $\Phi_a$, the above theorem $\implies$

- $\mathcal{O}_X$ is $\hat{\Phi}_a$-$\mathrm{IT}_1$ and $\hat{\Phi}_a^1(\mathcal{O}_X) = \mathcal{O}_X$
- $(\hat{\Phi}_a)_* = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$.

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# Transforming coherent systems

- $X$ elliptic curve. $\varphi\colon V \otimes \mathcal{O}_X \to E$ coherent system $\implies$
- $\Phi^0_a(\varphi)\colon V \otimes \mathcal{O}_X \to \Phi^0_a(E)$ is a coherent system
- If $E$ is $\hat{\Phi}_a - \mathrm{IT}_1$, $\hat{\Phi}^1_a(\varphi)\colon V \otimes \mathcal{O}_X \to \hat{\Phi}^1_a(E)$ is a coherent system.

Now we have to see when these FM transforms for coherent systems pass to the moduli spaces, that is when they preserve stability and semistability.

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# Moduli spaces of coherent systems

- $X$ a projective smooth curve
- $G(\alpha; r, d, k) \equiv$ moduli space of $\alpha$-stable coherent systems of type $(r, d, k)$ on $X$.
- $G(\alpha; r, d, k)$ is quasi-projective. $G(\alpha; r, d, k) = \emptyset$ for $\alpha < 0$ (Bradlow, García-Prada, Muñoz and Newstead).

Here we are interested in the case of $X$ an elliptic curve. In this case the moduli spaces of coherent systems have been studied by Lange and Newstead. They proved the following result:

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# Moduli spaces of coherent systems on elliptic curves

### Theorem (Lange, Newstead)

*$X$ an elliptic curve, $r \geq 1$, $k \geq 0$. Then*

- *if $G(\alpha; r, d, k) \neq \emptyset$, it is smooth and irreducible of dimension $\beta(d, k) = k(d - k) + 1$.*
- *$G(\alpha; r, d, 0) \simeq M(r, d)$ for all $\alpha$; then $G(\alpha; r, d, 0) \neq \emptyset \iff \gcd(r, d) = 1$.*
- *for $\alpha > 0$ and $k \geq 1$, $G(\alpha; 1, d, k)$ is independent of $\alpha$. $G(\alpha; 1, d, k) \neq \emptyset \iff$ either $d = 0$, $k = 1$ or $k \leq d$;*
- *for $\alpha > 0$, $r \geq 2$ and $k \geq 1$, $G(\alpha; r, d, k) \neq \emptyset \iff (r - k)\alpha < d$ and either $k < d$ or $k = d$ and $\gcd(r, d) = 1$.*

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# Dependence on the parameter

$X$ smooth projective curve. The $\alpha$-range is divided into open intervals determined by a finite number of critical values

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L$$

such that the moduli spaces for any two values of $\alpha$ in the interval $(\alpha_i, \alpha_{i+1})$ coincide; if $k < r$ this is also true for the interval $(\alpha_L, \infty)$ (Bradlow, García-Prada, Muñoz and Newstead).

- We write $G_0(r, d, k) = G(\alpha; r, d, k)$ for small $\alpha$, $0 < \alpha < \alpha_1$.
- In the rest of the talk we assume that $d \neq 0$, $k > 0$.
- If $X$ elliptic and $(E, V)$ is $\alpha$-stable, every indecomposable direct summand of $E$ has positive degree (Lange, Newstead),

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# Preservation of stability for small $\alpha$, I

Coherent systems in $G_0(r, d, k)$ are characterised by this result

**Theorem (Raghavendra, Vishwanath)**

*A coherent system $(E, V)$ of type $(r, d, k)$ is $\alpha$-stable, with $0 < \alpha < \alpha_1$, $\iff$ $E$ is semistable and one has $\frac{k'}{r'} < \frac{k}{r}$, for all coherent subsystems $(E', V')$ of type $(r', d', k')$ with $0 \neq E' \neq E$ and $\mu(E') = \mu(E)$.*

- $\varphi\colon V \otimes \mathcal{O}_X \to E \in G_0(r, d, k) \implies E$ ss of positive degree.
- Then $r + ad > 0$, so that $E$ is $\Phi_a - \mathrm{IT}_0$ and $\Phi_a^0(E)$ is ss.
- This is not enough to prove that $\Phi_a^0(\varphi)\colon V \otimes \mathcal{O}_X \to \Phi_a^0(E)$ is in $G_0(r, d, k)$
- Using the above theorem one proves that this is indeed the case.

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# Preservation of stability for small $\alpha$, II

Using similar techniques one proves that

- if $\varphi\colon V\otimes\mathcal{O}_X\to E\in G_0(r+ad,d,k)$, then
- $E$ is $\hat{\Phi}_a-\mathrm{IT}_1$ and $\hat{\Phi}^1_a(\varphi)\colon V\otimes\mathcal{O}_X\to\hat{\Phi}^1_a(E)\in G_0(r,d,k)$.

**Theorem**

*The FM transform $\Phi_a$ induces an isomorphism of moduli spaces*

$$\Phi^0_a\colon G_0(r,d,k)\simeq G_0(r+ad,d,k)$$

*Therefore, the isomorphism type of $G_0(r,d,k)$ depends only on the class $[r]\in\mathbb{Z}/d\,\mathbb{Z}$.*

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# Moduli spaces for large $\alpha$

- Here we assume $0 < k < r$. One has
- $G(\alpha; r, d, k) \neq \emptyset \iff 0 < \alpha < \frac{d}{r-k}$.
- In this case $\alpha_L < \frac{d}{r-k}$
- We write $G_L(r, d, k)$ the moduli space of $\alpha$-stable coherent systems of type $(r, d, k)$ with $\alpha_L < \alpha < \frac{d}{r-k}$

The spaces $G_L(r, d, k)$ have been determined by Bradlow and García-Prada in terms of "Brambila-Grzegorczyk-Newstead-extensions", or BGN-extensions, with semistable quotient.

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# BGN extensions

## Definition

$X$ smooth projective. A BGN extension of type $(r, d, k)$ is an extension of vector bundles

$$0 \to \mathcal{O}_X^k \to E \to F \to 0\,,$$

where

1. $\operatorname{rk} E = r > k$ and $\deg E = d > 0$.
2. $H^0(X, F^*) = 0$
3. If $(e_1, \ldots, e_k) \in \operatorname{Ext}^1_X(F, \mathcal{O}_X^k) \simeq H^1(X, F^*)^{\oplus k}$ denotes the class of the extension, then $e_1, \ldots, e_k$ are linearly independent as vectors in $H^1(X, F^*)$.

# BGN extensions and coherent systems I

**Theorem (Bradlow, García-Prada)**

*$(E, V)$ $\alpha$-semistable with $\alpha_L < \alpha < \frac{d}{r-k}$. The evaluation map defines a BGN extension of type $(r, d, k)$ $0 \to \mathcal{O}_X^k \to E \to F \to 0$ with $F$ semistable. Conversely, a BGN extension of type $(r, d, k)$ with $F$ stable yields an $\alpha$-stable coherent system, with $\alpha_L < \alpha < \frac{d}{r-k}$.*

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# BGN extensions and coherent systems II

One can also characterise BGN extensions inducing $\alpha$-stable coherent systems

## Theorem

*A BGN extension of type* $(r, d, k)$ $0 \to \mathcal{O}_X^k \to E \to F \to 0$ *defines an* $\alpha$*-stable coherent system, with* $\alpha_L < \alpha < \frac{d}{r-k}$*,* $\iff$ $F$ *is semistable and one has* $\frac{k'}{r'} > \frac{k}{r}$*, for all subextensions* $0 \to \mathcal{O}_X^{k'} \to E' \to F' \to 0$ *of type* $(r', d', k')$ *with* $\mu(F') = \mu(F)$*.*

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# BGN extensions and coherent systems III

We denote

- $\mathcal{BGN}(r, d, k) \equiv$ family of BGN extension classes of type $(r, d, k)$ with quotient semistable.
- $\mathcal{BGN}^s(r, d, k) \equiv$ family of BGN extension classes of type $(r, d, k)$ with quotient stable.

By the above Bradlow & García-Prada's results one has

$$\mathcal{BGN}^s(r, d, k) \hookrightarrow G_L(r, d, k) \hookrightarrow \mathcal{BGN}(r, d, k).$$

# Fourier-Mukai transform for BGN extensions and coherent systems I

Using the characterisation of the BGN extensions that induce an $\alpha$-stable coherent system, we get

## Theorem

*For $X$ elliptic, the FM transform $\Phi_a$ induces an isomorphism*

$$\Phi_a^0 \colon \mathcal{BGN}(r, d, k) \simeq \mathcal{BGN}(r + ad, d, k)$$

*by sending a BGN extension $0 \to \mathcal{O}_X^k \to E \to F \to 0$ to*
$0 \to \mathcal{O}_X^k \to \Phi_a^0(E) \to \Phi_a^0(F) \to 0$.
*The inverse isomorphism is given by $\hat{\Phi}_a^1$.*

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# Fourier-Mukai transform for BGN extensions and coherent systems II

Moreover both $\Phi_a^0$ and $\hat{\Phi}_a^1$ preserve stability and induce isomorphisms

$$\begin{array}{ccccc}
\mathcal{BGN}^s(r,d,k) & \hookrightarrow & G_L(r,d,k) & \hookrightarrow & \mathcal{BGN}(r,d,k) \\
\wr\downarrow \Phi_a^0 & & \wr\downarrow \Phi_a^0 & & \wr\downarrow \Phi_a^0 \\
\mathcal{BGN}^s(r+ad,d,k) & \hookrightarrow & G_L(r+ad,d,k) & \hookrightarrow & \mathcal{BGN}(r+ad,d,k)
\end{array}$$

$\Longrightarrow$ the isomorphism type of $G_L(r,d,k)$ depends only on the class $[r] \in \mathbb{Z}/d\,\mathbb{Z}$.

# Birational type of the moduli spaces of coherent systems I

**Theorem (Lange, Newstead)**

*Let $X$ be an elliptic curve. The birational type of $G(\alpha; r, d, k)$ is independent of $\alpha \in (\alpha_0, \alpha_{L+1})$.*

- $\Longrightarrow$ the common birational type of the moduli spaces $G(\alpha; r, d, k)$ can be computed by considering any of them.
- We then study $G_0(r, d, k)$.

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# Birational type of the moduli spaces of coherent systems II

Our final result is

**Theorem**

*Let $X$ be an elliptic curve, and $a$ a positive integer. The birational types of $G(\alpha; r, d, k)$ and $G(\alpha; r + ad, d, k)$ are the same. Therefore, the birational type of $G(\alpha; r, d, k)$ is independent of $\alpha$ and depends only on the class $[r] \in \mathbb{Z}/d\,\mathbb{Z}$.*

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