$*$-Quantization of Fourier-Mukai transforms

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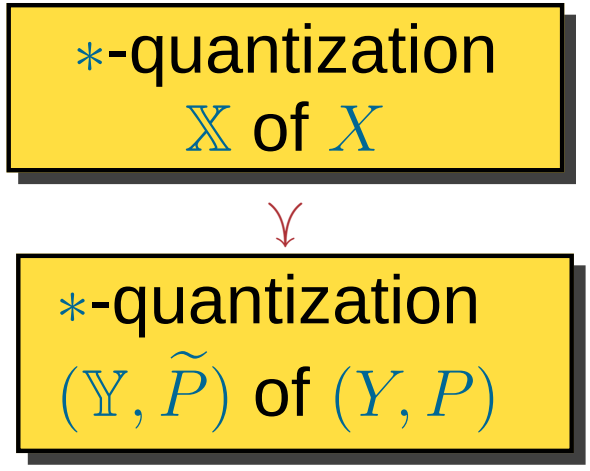

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- Will explain how *analytic* Fourier-Mukai transforms behave under $*$-quantization.
- Will discuss the deformation theory of $*$-structures for spaces and sheaves.
- Will discuss naturality and the 2-stack of $*$-quantizations.

Fourier-Mukai transforms

Definition: *Two complex analytic spaces* X, Y *are* ***Fourier-Mukai partners*** *if there is an object* $K \in D^b_{\mathrm{qcoh}}(X \times Y)$ *for which*

$$\begin{aligned} \phi^K : \qquad D(X) &\longrightarrow D(Y) \\ F &\longrightarrow p_{Y*}(p_X^* F \otimes^L K) \end{aligned}$$

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Motivating Question: Can we deform X in some direction, so that $(D(Y), \phi^K)$ deforms along?

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  - FM for quantized integrable systems.

Basic Example

$A \xrightarrow{\pi} B$ - a family of complex tori,

$A^\vee \xrightarrow{\pi^\vee} B$ - the dual family of complex tori,

$A_b^\vee = \mathrm{Pic}^0(A_b)$ - the dual torus

$=$ the moduli space of translation invariant line bundles on A_b.

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- If we deform the complex structure on A, then $(A^{\vee}, \phi^{\mathscr{P}})$ deform along with it.
- The classical Fourier-Mukai transform is unobstructed under deformations of A as a complex manifold.

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  - a quantized torus **[Ben-Bassat-Block-P'05]**, **[Block'06]**.

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Arinkin's theorem is a *geometric* refinement of this categorical statement.

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$*$-deformations

Note: Will only discuss one parameter formal deformations.

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$$\underline{\mathrm{Diff}}_0(P \otimes_{\mathbb{C}} R; P \otimes_{\mathbb{C}} R)$$

$$:= \left\{ \sum_i D_i \hbar^i \;\middle|\; D_0 = 1,\ D_i \in \underline{\mathrm{Diff}}_M(P; P) \right\}$$

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- An R-linear sheaf map $f : \widetilde{P} \to \widetilde{Q}$ is **$\star$-local** if for all local sections $a \in \mathscr{I}_{\widetilde{P}}$, $b \in \mathscr{I}_{\widetilde{Q}}$ we have

$$b^{-1} \circ f \circ a \in \underline{\mathrm{Diff}}(P, Q) \otimes_{\mathbb{C}} R.$$

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- The neutralized quantizations correspond to a stack of algebroids together with a chosen section.

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  - For $U \subset M$ open, $\alpha, \beta, \gamma \in \mathbb{M}(U)$, the composition

$$\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \beta) \otimes_R \underline{\mathrm{Hom}}_{\mathbb{M}}(\beta, \gamma) \to \underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \gamma)$$

    is $*$-local and induces an $\mathcal{O}_U$-linear map modulo $\hbar$.
  - The $\mathcal{O}$-module structure on $\mathscr{L}_{\alpha\beta}$ and the $*$-structure on $\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \beta)$ are compatible with restrictions from U to $V \subset U$.

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- *A $*$-structure $\mathscr{I}_{\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha,\beta)}$ on $\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \beta)$. The data should satisfy:*

  - For $U \subset M$ open, $\alpha, \beta, \gamma \in \mathbb{M}(U)$, the composition

$$\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \beta) \otimes_R \underline{\mathrm{Hom}}_{\mathbb{M}}(\beta, \gamma) \to \underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \gamma)$$

    is $*$-local and induces an $\mathcal{O}_U$-linear map modulo $\hbar$.
  - The $\mathcal{O}$-module structure on $\mathscr{L}_{\alpha\beta}$ and the $*$-structure on $\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \beta)$ are compatible with restrictions from U to $V \subset U$.

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- **commutative $*$-quantizations.** Characterized by: for every local section $\alpha \in \widetilde{M}(U)$ the sheaf $\underline{\mathrm{Hom}}_{\widetilde{M}}(\alpha, \alpha)$ is a sheaf of commutative algebras on U.

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These are the same as the deformations of $B\mathcal{O}_M^\times$ as an $\mathcal{O}_M^\times$-gerbe on M.

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This agrees with the previous definition of a neutralized $*$-quantization with $\mathcal{O}_{\widetilde{M}} = \underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \alpha)$.

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$\mathrm{Q}M$ is the stack of groupoids on M_{an}, such that for an open $U \subset M$, the fiber $\mathrm{Q}M(U)$ is the groupoid of neutralized $\star$-quantizations of U.

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There is a natural analytic topology on $\mathrm{Q}M$: a family of maps

$$\left\{\widetilde{V}_i \to \widetilde{U}\right\}_{i\in I},\ \widetilde{V}_i \in \mathrm{Q}M(V_i),\ \widetilde{U} \in \mathrm{Q}M(U)$$

is a cover, if and only if its image in M is a cover in M_{an}, that is $U = \cup_{i\in I} V_i$.

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Note: The site $\mathsf{Q}M_{\mathrm{an}}$ comes equipped with a natural sheaf $\mathcal{O}_{\mathsf{Q}M_{\mathrm{an}}}$ of R-algebras, where $\mathcal{O}_{\mathsf{Q}M_{\mathrm{an}}}(\widetilde{U}) := \mathcal{O}_{\widetilde{U}}$.

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- acts trivially on the $*$-structure $\mathscr{I}_{\mathcal{O}_{\widetilde{U}}}$.

Moduli of $*$-quantizations III

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The quotient 2-stack $\left[\mathbb{Q}M\,/B\mathcal{O}^{\times}_{\mathbb{Q}M_{\mathrm{an}}}\right]$ is naturally equivalent to the 2-stack of all $*$-quantizations of M.

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The stack of R-linear categories over M obtained as the stackification of

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Definition *Given a $*$-quantization $\mathbb{M}$ of M, a **representation** of $\mathbb{M}$ in Sh_R is a 1-morphism*

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- Will think of representations of $\mathbb{M}$ as "$\mathcal{O}$-modules". (this agrees with the definition of Polesello-Schapira)

Sheaves of $\mathcal{O}$-modules II

A $*$-structure on an $\mathcal{O}$-module $F : \mathbb{M} \to \mathsf{Sh}_R$ is a functor

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refining the map $F_{(\bullet)} : \mathbb{M}^{\times} \longrightarrow \mathcal{O}_{\mathbf{Q}M_{\mathrm{an}}} - \mathrm{mod}$:

$$\begin{array}{ccc} \mathbb{M}^{\times} & \xrightarrow{\widetilde{F}} & \mathcal{O}_{\mathbf{Q}M_{\mathrm{an}}} - \mathrm{mod}^{*} \\ & \searrow^{F_{(\bullet)}} & \downarrow \\ & & \mathcal{O}_{\mathbf{Q}M_{\mathrm{an}}} - \mathrm{mod}\,. \end{array}$$

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- *a compatible* $*$*-quantization* $\widetilde{P} \in D^*(\mathbb{X} \times \mathbb{Y}^{\mathrm{op}})$ *of* P*.*

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- $\phi^{\widetilde{P}}$ *is an equivalence between* $D^*(\mathbb{X})$ *and* $D^*(\mathbb{Y})$.

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$$\underline{\mathrm{Diff}}_Z(\mathcal{O}_X, \ldots, \mathcal{O}_X, P; P) = i_{\Gamma *} \underline{\mathrm{Diff}}_\Gamma(\mathcal{O}_X, \ldots, \mathcal{O}_X, V; V).$$

Deformation theory

Consider the complex

$$\mathscr{C}(\underline{\mathrm{Diff}}_Z(P;P)) :=$$

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$$\begin{array}{lcl}
(A : P \to P) & \mapsto & dA(f,p) := fA(p) - A(fp) \\
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Note: $\mathscr{C}(\underline{\mathrm{Diff}}_Z(P;P))$ is a twisted cobar resolution of $\underline{\mathrm{Diff}}_Z(P;P)$ viewed as a comodule over $\underline{\mathrm{Diff}}_Z(\mathcal{O}_X;\mathcal{O}_Z)$.

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  - *The gerbe* $\mathfrak{Lifts}(\widetilde{P}_{n-1}, \mathscr{I}_{\widetilde{P}_{n-1}})$ *is determined up to a* 1*-isomorphism of* $\mathscr{C}(\underline{\mathrm{Diff}}_{Z}(P; P))$*-gerbes which is determined up to a unique* 2*-isomorphism.*

Rigidity and cohomology

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$V \to \Gamma$ *is a holomorphic vector bundle.*

Then there is a natural isomorphism

$$\mathbb{R}^\bullet p_{Y*} \mathscr{C}(\underline{\mathrm{Diff}}_{X\times Y}(P;P)) \cong \mathcal{O}_Y$$

in the derived category of $\mathcal{O}_Y$-modules.

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  - For $U \subset Y$ the obstruction to quantizing P lives in $\Gamma(U, \mathbb{R}^2 p_{U*}\mathscr{C})$.
  - The ambiguity in quantizing lives in $\Gamma(U, \mathbb{R}^1 p_{U*}\mathscr{C})$.

  Conclusion: $\widetilde{P}$ exists and is unique up to isomorphism.

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  Note: $\mathcal{O}_{\widetilde{U}}$ is a neutralized quantization of $\mathcal{O}_U$, so we only need a $*$-structure.

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- Check that

$$\mathbb{R}p^i_{U*}\underline{\mathrm{Diff}}_{X\times U}(\mathcal{O}_U, \mathscr{C}) = \begin{cases} \underline{\mathrm{Diff}}_U(\mathcal{O}_U; \mathcal{O}_U), & i = 0 \\ 0, & i > 0. \end{cases}$$

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- In particular the rows of (E_1^{ij},d_1) are Koszul complexes. Thus the spectral sequence degenerates and so $R^\bullet\,\mathrm{pr}_{Y*}\,\underline{\mathrm{Diff}}_{\Gamma/X}(V;V)=\mathcal{O}_Y$ as promised.

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- Every sheaf $\mathscr{A} \to M$ of algebras defines a stack $\mathbb{M}_{\mathscr{A}}$ of algebroids on M via

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Note: If $\mathscr{X} = BH$, then $\rho : G \to H$ is a crossed module in the sense of J.H.C.Whitehead.

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- $[\mathscr{X}/BG]$ can be thought of as the 2-stack of all G-gerbes equipped with a ρ-equivariant 1-morphism to $\mathscr{X}$.

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Suppose $\mathscr{C}: \ C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots$ is a complex of abelian sheaves on M.

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- gerbes over $\mathscr{C}$ form a 2-category with invertible 1 and 2 morphisms.

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