$*$-Quantization of Fourier-Mukai transforms

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Overview

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$\ast$-quantization $(\mathbb{Y}, \widetilde{P})$ of (Y, P)

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- Will explain how *analytic* Fourier-Mukai transforms behave under $\ast$-quantization.
- Will discuss the deformation theory of $\ast$-structures for spaces and sheaves.
- Will discuss naturality and the 2-stack of $\ast$-quantizations.

Fourier-Mukai transforms

Definition: *Two complex analytic spaces X, Y are* ***Fourier-Mukai partners*** *if there is an object $K \in D^b_{\mathrm{qcoh}}(X \times Y)$ for which*

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Motivating Question: Can we deform X in some direction, so that $(D(Y), \phi^K)$ deforms along?

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  - FM for quantized complex tori;
  - FM for quantized integrable systems.

Basic Example

$A \xrightarrow{\pi} B$ - a family of complex tori,

$A^\vee \xrightarrow{\pi^\vee} B$ - the dual family of complex tori,

$A_b^\vee = \mathrm{Pic}^0(A_b)$ - the dual torus

$=$ the moduli space of translation invariant line bundles on A_b.

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- If we deform the complex structure on A, then $(A^{\vee}, \phi^{\mathscr{P}})$ deform along with it.
- The classical Fourier-Mukai transform is unobstructed under deformations of A as a complex manifold.

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  - a quantized torus **[Ben-Bassat-Block-P'05]**, **[Block'06]**.

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Arinkin's theorem is a *geometric* refinement of this categorical statement.

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- Need to choose a **good category** of sheaves, well adapted to Fourier-Mukai transforms.
- The quantized product and module structures are not automatically local. Need extra control - use $*$-structures.

$*$-deformations

Note: Will only discuss one parameter formal deformations.

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$$\underline{\mathrm{Diff}}_0(P \otimes_{\mathbb{C}} R; P \otimes_{\mathbb{C}} R) := \left\{ \sum_i D_i \hbar^i \;\middle|\; D_0 = 1,\ D_i \in \underline{\mathrm{Diff}}_M(P;P) \right\}$$

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- An R-linear sheaf map $f : \widetilde{P} \to \widetilde{Q}$ is $\star$**-local** if for all local sections $a \in \mathscr{I}_{\widetilde{P}}$, $b \in \mathscr{I}_{\widetilde{Q}}$ we have

$$b^{-1} \circ f \circ a \in \underline{\mathrm{Diff}}(P, Q) \otimes_{\mathbb{C}} R.$$

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- The neutralized quantizations correspond to a stack of algebroids together with a chosen section.

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  - For $U \subset M$ open, $\alpha, \beta, \gamma \in \mathbb{M}(U)$, the composition

    $$\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \beta) \otimes_R \underline{\mathrm{Hom}}_{\mathbb{M}}(\beta, \gamma) \to \underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \gamma)$$

    is $*$-local and induces an $\mathscr{O}_U$-linear map modulo $\hbar$.
  - The $\mathscr{O}$-module structure on $\mathscr{L}_{\alpha\beta}$ and the $*$-structure on $\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \beta)$ are compatible with restrictions from U to $V \subset U$.

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- *A $*$-structure $\mathscr{I}_{\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha,\beta)}$ on $\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \beta)$.*

The data should satisfy:

- For $U \subset M$ open, $\alpha, \beta, \gamma \in \mathbb{M}(U)$, the composition

$$\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \beta) \otimes_R \underline{\mathrm{Hom}}_{\mathbb{M}}(\beta, \gamma) \to \underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \gamma)$$

  is $*$-local and induces an $\mathcal{O}_U$-linear map modulo $\hbar$.
- The $\mathcal{O}$-module structure on $\mathscr{L}_{\alpha\beta}$ and the $*$-structure on $\underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \beta)$ are compatible with restrictions from U to $V \subset U$.

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These are the same as the deformations of $B\mathcal{O}_M^\times$ as an $\mathcal{O}_M^\times$-gerbe on M.

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This agrees with the previous definition of a neutralized $*$-quantization with $\mathcal{O}_{\widetilde{M}} = \underline{\mathrm{Hom}}_{\mathbb{M}}(\alpha, \alpha)$.

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$\mathrm{Q}M$ is the stack of groupoids on M_{an}, such that for an open $U \subset M$, the fiber $\mathrm{Q}M(U)$ is the groupoid of neutralized $\star$-quantizations of U.

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There is a natural analytic topology on $\mathrm{Q}M$: a family of maps

$$\left\{\widetilde{V}_i \to \widetilde{U}\right\}_{i\in I},\ \widetilde{V}_i \in \mathrm{Q}M(V_i),\ \widetilde{U} \in \mathrm{Q}M(U)$$

is a cover, if and only if its image in M is a cover in $M_{\rm an}$, that is $U = \cup_{i\in I} V_i$.

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Note: The site $\mathsf{Q}M_{\mathrm{an}}$ comes equipped with a natural sheaf $\mathcal{O}_{\mathsf{Q}M_{\mathrm{an}}}$ of R-algebras, where $\mathcal{O}_{\mathsf{Q}M_{\mathrm{an}}}(\widetilde{U}) := \mathcal{O}_{\widetilde{U}}$.

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Moduli of $*$-quantizations III

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The quotient 2-stack $\left[\mathsf{Q}M \, / B\mathcal{O}^{\times}_{\mathsf{Q}M_{\mathrm{an}}} \right]$ is naturally equivalent to the 2-stack of all $*$-quantizations of M.

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The stack of R-linear categories over M obtained as the stackification of

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Definition *Given a $*$-quantization $\mathbb{M}$ of M, a* ***representation*** *of $\mathbb{M}$ in Sh_R is a 1-morphism*

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Sheaves of $\mathcal{O}$-modules II

A $*$-structure on an $\mathcal{O}$-module $F : \mathbb{M} \to \mathsf{Sh}_R$ is a functor

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refining the map $F_{(\bullet)} : \mathbb{M}^{\times} \longrightarrow \mathcal{O}_{\mathsf{Q}M_{\mathrm{an}}} - \mathrm{mod}$:

$$\begin{array}{ccc} \mathbb{M}^{\times} & \xrightarrow{\widetilde{F}} & \mathcal{O}_{\mathsf{Q}M_{\mathrm{an}}} - \mathrm{mod}^{*} \\ & {\scriptstyle F_{(\bullet)}} \searrow & \downarrow \\ & & \mathcal{O}_{\mathsf{Q}M_{\mathrm{an}}} - \mathrm{mod}\,. \end{array}$$

The main theorems

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Then for any $*$*-quantization* $\mathbb{X}$ *of* X *there exist:*

- *a* $*$*-quantization* $\mathbb{Y}$ *of* Y*, and*
- *a compatible* $*$*-quantization* $\widetilde{P} \in D^*(\mathbb{X} \times \mathbb{Y}^{\mathrm{op}})$ *of* P*.*

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- $(\mathbb{Y}, \widetilde{P})$ *is unique up to a 1-isomorphism, which is unique up to a unique 2-isomorphism;*
- $\phi^{\widetilde{P}}$ *is an equivalence between* $D^*(\mathbb{X})$ *and* $D^*(\mathbb{Y})$.

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$$\underline{\mathrm{Diff}}_{p_X^{-1}\mathcal{O}_X}(p_X^{-1}\mathcal{O}_X, \ldots, p_X^{-1}\mathcal{O}_X; \underline{\mathrm{Diff}}_Z(P; P))$$

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$$\underline{\mathrm{Diff}}_Z(\mathcal{O}_X, \ldots, \mathcal{O}_X, P; P) = i_{\Gamma *}\underline{\mathrm{Diff}}_\Gamma(\mathcal{O}_X, \ldots, \mathcal{O}_X, V; V).$$

Deformation theory

Consider the complex

$$\mathscr{C}(\underline{\mathrm{Diff}}_Z(P;P)) :=$$

$$\underline{\mathrm{Diff}}_Z(P;P) \xrightarrow{d} \underline{\mathrm{Diff}}_Z(\mathcal{O}_X, P;P) \xrightarrow{d} \underline{\mathrm{Diff}}_Z(\mathcal{O}_X, \mathcal{O}_X, P;P) \xrightarrow{d} \cdots ,$$

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$$\begin{array}{lcl} (A : P \to P) & \mapsto & dA(f,p) := fA(p) - A(fp) \\ (B : \mathcal{O}_X \times P \to P) & \mapsto & dB(f,g,p) := fB(g,p) - B(fg,p) \\ & & \qquad\qquad\qquad + B(f,gp) \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

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Note: $\mathscr{C}(\underline{\mathrm{Diff}}_Z(P;P))$ is a twisted cobar resolution of $\underline{\mathrm{Diff}}_Z(P;P)$ viewed as a comodule over $\underline{\mathrm{Diff}}_Z(\mathcal{O}_X;\mathcal{O}_Z)$.

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- *The gerbe* $\mathfrak{Lifts}(\widetilde{P}_{n-1}, \mathscr{I}_{\widetilde{P}_{n-1}})$ *is determined up to a* 1*-isomorphism of* $\mathscr{C}(\underline{\mathrm{Diff}}_Z(P; P))$*-gerbes which is determined up to a unique* 2*-isomorphism.*

Rigidity and cohomology

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Then there is a natural isomorphism

$$\mathbb{R}^{\bullet} p_{Y*}\mathscr{C}(\underline{\mathrm{Diff}}_{X\times Y}(P;P)) \cong \mathcal{O}_Y$$

in the derived category of $\mathcal{O}_Y$-modules.

Construction of quantization

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  - For $U \subset Y$ the obstruction to quantizing P lives in $\Gamma(U, \mathbb{R}^2 p_{U*}\mathscr{C})$.
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Conclusion: $\widetilde{P}$ exists and is unique up to isomorphism.

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Note: $\mathcal{O}_{\widetilde{U}}$ is a neutralized quantization of $\mathcal{O}_U$, so we only need a $*$-structure.

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- Check that

$$\mathbb{R}p^i_{U*}\underline{\mathrm{Diff}}_{X\times U}(\mathcal{O}_U, \mathscr{C}) = \begin{cases} \underline{\mathrm{Diff}}_U(\mathcal{O}_U; \mathcal{O}_U), & i = 0 \\ 0, & i > 0. \end{cases}$$

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- In particular the rows of (E_1^{ij}, d_1) are Koszul complexes. Thus the spectral sequence degenerates and so $R^\bullet \mathrm{pr}_{Y*}\underline{\mathrm{Diff}}_{\Gamma/X}(V;V) = \mathcal{O}_Y$ as promised.

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Note: If $\mathscr{X} = BH$, then $\rho : G \to H$ is a crossed module in the sense of J.H.C.Whitehead.

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- $[\mathscr{X}/BG]$ can be thought of as the 2-stack of all G-gerbes equipped with a ρ-equivariant 1-morphism to $\mathscr{X}$.

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Gerbes over a complex

Suppose $\mathscr{C} : \; C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots$ is a complex of abelian sheaves on M.

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- gerbes over $\mathscr{C}$ form a 2-category with invertible 1 and 2 morphisms.

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