Nahm transform and Riemann surfaces

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Background I

Let M be a smooth, 4-dimensional manifold.

Let $E \to M$ be a hermitian vector bundle.

A connection ∇ on E is said to be an instanton if its curvature F_∇ is anti-self-dual as a 2-form and $||F_\nabla||_{L^2} < \infty$.

Background II

In the case $M = \mathbb{R}^4$, one may assume that ∇ is independent from one, two or three variables.

One then gets various dimensional reductions of the the anti-self-duality equation $F_\nabla^+ = 0$:

Bogomolny equation

Hitchin equations

Nahm equations

Background III

Let X be a hyperkähler manifold

Let $E \to X$ be a hermitian vector bundle.

A connection ∇ on E is said to be hyperholomorphic if its curvature F_∇ is of type $(1,1)$ with respect to every complex structure of X.

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Nahm transform gives a 1-1 correspondence between (dimensionally reduced) instantons on R^4/Λ and (dimensionally reduced) instantons on $(\mathbb{R}^4)^*/\Lambda^*$

Doubly-periodic instantons

Example: $\Lambda = \mathbb{Z}^2$, $\Lambda^* = (\mathbb{Z}^2)^* \times \mathbb{R}^2$.

Nahm transform gives a 1-1 correspondence between instantons on $T^2 \times \mathbb{R}^2$ (a.k.a. doubly-periodic instantons) and (singular) solutions of Hitchin equations on $(T^2)^*$.

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It is also useful to remember that there are no smooth solutions of Hitchin equations on a torus.

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Nahm transform is conjectured to be a Higgs bundle on $J^\vee$

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Equivalently: $\nabla + \Phi + \Phi^*$ is a flat connection.

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Considering $\sigma \in H^0(K_\Sigma)$ as a bundle map $\sigma : \mathcal{O}_\Sigma \to K_\Sigma$, define:

$$\Phi_{(\eta,\sigma)} = \Phi \otimes \mathbf{1}_{L_\eta} + \mathbf{1}_{E\otimes L_\eta} \otimes \sigma$$

which is a bundle map $E \otimes L_\eta \to E \otimes L_\eta \otimes K_\Sigma$.

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Easy to check that if (∇, Φ) satisfy the Hitchin equations, so does $(\nabla_\eta, \Phi_{\eta,\sigma})$ for each (η, σ).

Nahm transform of Higgs bundles II

Now consider the operator

$$\mathcal{D}_{(\eta,\sigma)} \;:\; L^2_{p+1}\left(\mathcal{E}(\eta)\otimes(\Lambda^0_\Sigma\oplus\Lambda^{1,1}_\Sigma)\right)\longrightarrow L^2_p\left(\mathcal{E}(\eta)\otimes(\Lambda^{1,0}_\Sigma\oplus\Lambda^{0,1}_\Sigma)\right)$$

$$\mathcal{D}_{(\eta,\sigma)} = \left(\overline{\partial}_\eta - \overline{\partial}^*_\eta\right) + \left(\theta_{(\eta,\sigma)} - \theta^*_{(\eta,\sigma)}\right)$$

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Thus usual index theory gives complex vector bundle $\hat{E} \to J^\vee \times H^0(K_{\Sigma_g})$.

Nahm transform of Higgs bundles III

An unitary connection $\widehat{\nabla}$ on $\hat{E}$ is constructed by demanding commutativity of the following diagram:

$$
\begin{array}{ccc}
\Gamma_{J^\vee \times H^0(K_\Sigma)}(\hat{E}) & \xrightarrow{\ \widehat{\nabla}\ } & \Lambda^1_{J^\vee \times H^0(K_\Sigma)}(\hat{E}) \\
\downarrow & & \downarrow \\
\Gamma_{J^\vee \times H^0(K_\Sigma)}(H^-) & \xrightarrow{\ \underline{d}\ } & \Lambda^1_{J^\vee \times H^0(K_\Sigma)}(H^-)
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where H^- is the trivial holomorphic Hilbert bundle

$$L^2_p\left(\mathcal{E} \otimes (\Lambda^{1,0}_\Sigma \oplus \Lambda^{0,1}_\Sigma)\right) \times J^\vee \times H^0(K_\Sigma)$$

Nahm transform of Higgs bundles IV

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The pair $(\hat{E}, \widehat{\nabla})$ is called the Nahm transform of the Higgs bundle (E, ∇, Φ).