

Fourier-Mukai for singular varieties and singular Mukai partners

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Outline

- 1 Introduction and motivation
 - FM and birational geometry
 - A review of the smooth case
 - Aims and techniques
 - Some formulas in derived category
- 2 Relative Integral functors for singular schemes
 - Boundedness conditions
 - Restriction to fibres: a criterion for equivalence
 - Application to genus one fibrations
- 3 Fully faithfulness criteria for integral functors
 - The absolute case for Cohen-Macaulay varieties
 - Review of the proofs in the smooth situation
 - Strongly simple objects and criteria of fully faithfulness
 - Criteria for equivalence
- 4 FM partners and their geometric properties



Introduction

- Relevance of derived categories and FM functors in birational geometry is nowadays well known.
- There are examples proving that derived categories have a nice behavior under some birational transformations as flips and flops.
- This proves the importance of the study of derived categories and FM functors for singular projective varieties.
- Few attention has been paid to singular varieties in the literature on the topic. The reason may be that many of the fundamental results rely deeply on smoothness.



Birational Geometry

In the smooth case derived equivalence is related to K-equivalence:

Proposition (Kawamata)

*X, Y smooth projective. If $D_c^b(X) \simeq D_c^b(Y)$, then $\dim X = \dim Y$. If $\kappa(X) = \dim X$ (X of general type), then there exist birational morphisms $f: Z \rightarrow X, g: Z \rightarrow Y$ such that $f^*K_X \sim g^*K_Y$, that is, D -equivalence \implies K -equivalence.*

A kind of converse is the following

Conjecture (Bondal, Orlov))

If $f: X \dashrightarrow X^+$ is a generalized flip of smooth projective varieties, there is a fully faithful functor $\Phi: D_c^b(X^+) \rightarrow D_c^b(X)$. For generalized flops, Φ is an equivalence of categories (K -equivalence \implies D -equivalence).

Birational Geometry, II

In some cases, a generalized flop is composition of flops. This happens, for instance, for

- smooth projective threefolds (Kollár);
- projective threefolds with \mathbb{Q} -factorial terminal singularities (Kawamata).

Then, the former conjecture is related with:

Conjecture

W be a quasi-projective Gorenstein. If $X \xrightarrow{\pi_X} W \xleftarrow{\pi_Y} Y$ is a flop, then $D_c^b(X) \simeq D_c^b(Y)$.



Birational Geometry, III

The conjecture has been proved:

- In dimension 3 for smooth projective varieties. (Bridgeland).
 - \implies Birational 3-dimensional CY's have equivalent derived categories.
- In dimension 3 for quasiprojective varieties with terminal Gorenstein singularities (Chen).
- In dimension 3 for quasiprojective normal varieties with \mathbb{Q} -factorial terminal singularities (Kawamata).
- The flop is the so-called *Mukai flop* (Kawamata, Namikawa).
 - \implies Birational 4-dimensional projective symplectic manifolds have equivalent derived categories.

In higher dimensions there exist generalised flops between smooth projective varieties which decompose as a sequence of flops between **singular** varieties. Then derived categories and integral functors for singular varieties are important in the context of the MMP.



Fundamental results in the smooth case

Applications of FM to birational geometry rely on two fundamental results. The first one is *Orlov's representation theorem*:

Theorem

If X and Y are smooth projective varieties, for any exact fully faithful functor $F: D_c^b(X) \rightarrow D_c^b(Y)$ there is $\mathcal{K}^\bullet \in D_c^b(X \times Y)$ such that $F \simeq \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$ where

$$\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{G}^\bullet) = \mathbf{R}\pi_{Y*}(\pi_X^* \mathcal{G}^\bullet \otimes^{\mathbf{L}} \mathcal{K}^\bullet)$$

$\pi_X: X \times Y \rightarrow X$ and $\pi_Y: X \times Y \rightarrow Y$ being the projections.

The functor $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$ is called the *integral functor* of kernel \mathcal{K}^\bullet .



Fundamental results in the smooth case, II


The second one is *Bondal and Orlov's criterion of fully faithfulness*:
 $\mathrm{ch}(k) = 0$, X, Y smooth projective, $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$ an integral functor.

Theorem

$\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$ is fully faithful if and only if \mathcal{K}^\bullet is strongly simple over X , that is,

- ① $\mathrm{Hom}_{D(Y)}^i(\mathbf{L}j_{x_1}^* \mathcal{K}^\bullet, \mathbf{L}j_{x_2}^* \mathcal{K}^\bullet) = 0$ unless $x_1 = x_2$ and $0 \leq i \leq \dim X$;
- ② $\mathrm{Hom}_{D(Y)}^0(\mathbf{L}j_x^* \mathcal{K}^\bullet, \mathbf{L}j_x^* \mathcal{K}^\bullet) = k$ for every point x .

$j_x: \{x\} \hookrightarrow X$ being the natural embedding.

Recall that $\mathbf{L}j_x^* \mathcal{K}^\bullet \simeq \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_x)$. 



Aims of the talk

Goals

- 1 To give a similar criterion for fully faithfulness in some singular cases, namely, when X is Cohen-Macaulay.
- 2 Extend the previous result to the relative setting (families), for a Cohen-Macaulay morphism $f: X \rightarrow S$.
- 3 Study the relationship between the properties of relative integral functors and those of their restriction to the fibres.
- 4 Apply these results to construct nontrivial auto-equivalences of the derived category of varieties admitting a genus one fibration.
- 5 Find geometric properties shared by singular Fourier-Mukai partners.



Difficulties and techniques

The study of integral functors for singular varieties has various problems:

- Bounded complexes (even sheaves) are no longer of finite homological dimension (or perfect complexes).
- One has to use versions for unbounded complexes of known derived functors or formulas in derived categories. One then has to either:
 - use Spaltenstein results when available;
 - provide new proofs otherwise.
- One has to use delicate techniques of commutative algebra at some points.

In the next slides we list two formulas in derived category one has to prove:



Base-change formula

Consider a cartesian diagram of morphisms of algebraic varieties

$$\begin{array}{ccc}
 X \times_Y Z & \xrightarrow{\tilde{g}} & X \\
 \downarrow \tilde{f} & & \downarrow f \\
 Z & \xrightarrow{g} & Y
 \end{array}$$

Proposition

For any $\mathcal{G}^\bullet \in D(X)$ there is a morphism $\mathbf{L}g^*\mathbf{R}f_*\mathcal{G}^\bullet \rightarrow \mathbf{R}\tilde{f}_*\mathbf{L}\tilde{g}^*\mathcal{G}^\bullet$.
 If *either* f or g is *flat*, this is an isomorphism.

When g is flat, this “base-change formula” is well-known. If g is arbitrary and f is flat, the formula is proven in the book *Fourier-Mukai and Nahm transforms in Geometry and Mathematical Physics*. ◀



Grothendieck duality

Let $f: X \rightarrow Y$ be a proper morphism of schemes.

There is an isomorphism in the derived category

$$\mathbf{R}Hom_{\mathcal{O}_Y}^{\bullet}(\mathbf{R}f_*\mathcal{F}^{\bullet}, \mathcal{G}^{\bullet}) \simeq \mathbf{R}f_*\mathbf{R}Hom_{\mathcal{O}_X}^{\bullet}(\mathcal{F}^{\bullet}, f^!\mathcal{G}^{\bullet}).$$

for \mathcal{G}^{\bullet} in $D(Y)$ and \mathcal{F}^{\bullet} in $D(X)$.

$f^!\mathcal{O}_Y \equiv$ *relative dualizing complex* of X over Y . When Y is a point, we write \mathcal{D}^{\bullet}_X instead of $f^!\mathcal{O}_Y$.

When f is flat, one has

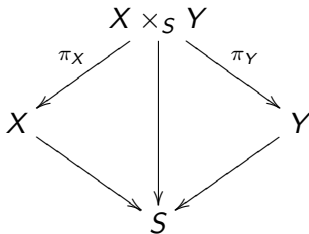
- f **Cohen-Macaulay** iff $f^!\mathcal{O}_Y \simeq \mathcal{D}_f[m]$, \mathcal{D}_f a sheaf, $m = \dim f$.
- f **Gorenstein** iff f Cohen-Macaulay and $\mathcal{D}_f = \omega_f$ line bundle.

Proposition

There is a map $\mathbf{L}f^*\mathcal{G}^{\bullet} \otimes^{\mathbf{L}} f^!\mathcal{O}_Y \rightarrow f^!\mathcal{G}^{\bullet}$. When either \mathcal{G}^{\bullet} has finite homological dimension or f is a regular closed immersion, this is an isomorphism.

Relative integral functors

S a scheme; $X \rightarrow S$, $Y \rightarrow S$ proper morphisms;
 $\mathcal{K}^\bullet \in D_c^b(X \times_S Y)$.



Relative integral functor $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}: D_c^-(X) \rightarrow D_c^-(Y)$ given by

$$\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{E}^\bullet) = \mathbf{R}\pi_{Y*}(\mathbf{L}\pi_X^* \mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathcal{K}^\bullet).$$



Boundedness conditions

Want to characterise those kernels \mathcal{K}^\bullet such that $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$:

- ① maps bounded complexes to bounded complexes;
- ② has a right adjoint which is also an integral functor with the same property.

Definition

$f: Z \rightarrow T$ a morphism. Then $\mathcal{E}^\bullet \in D_c^b(Z)$ has relative finite homological dimension (*fhd*) over T if $\mathcal{E}^\bullet \otimes^{\mathbf{L}} \mathbf{L}f^* \mathcal{G}^\bullet$ is bounded $\forall \mathcal{G}^\bullet \in D_c^b(T)$.

If f is the identity, $fhd \iff$ *perfect complex*.

We have now a solution to our problem:



Boundedness conditions, II

Proposition

$X \rightarrow S$ locally projective, $\mathcal{K}^\bullet \in D_c^b(X \times_S Y)$. Then,

- 1 \mathcal{K}^\bullet has $fhd/X \implies \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$ maps $D_c^b(X)$ to $D_c^b(Y)$.
- 2 If \mathcal{K}^\bullet has fhd/X and $/Y$, then $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}: D_c^b(X) \rightarrow D_c^b(Y)$ has an integral right adjoint with kernel $\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y)$.

$X \rightarrow S, Y \rightarrow S$ projective; $\mathcal{K}^\bullet \in D_c^b(X \times_S Y)$ such that $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}: D_c^b(X) \simeq D_c^b(Y)$. Then \mathcal{K}^\bullet has fhd over both factors, so that the above Proposition applies.



Restriction to fibres

Let $X \rightarrow S$ and $Y \rightarrow S$ be proper and *flat* morphisms,

$\mathcal{K}^\bullet \in D_c^b(X \times_S Y)$ and $\Phi = \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$.

Denote X_s, Y_s the fibers over $s \in S$, $j_s: X_s \hookrightarrow X$ and $j_s: Y_s \hookrightarrow Y$ the natural embeddings and $\Phi_s: D^-(X_s) \rightarrow D^-(Y_s)$ the induced absolute functor of kernel \mathcal{K}^\bullet_s .

- $\mathcal{K}^\bullet \in D_c^b(X \times_S Y)$ has *fhd*/ $X \implies \Phi$ maps $D_c^b(X)$ into $D_c^b(Y)$ and $\mathcal{K}^\bullet_s \in D_c^b(X_s \times Y_s)$ for any $s \in S$.
- Since q is flat, \mathcal{K}^\bullet_s has *fhd*/ X_s .

Proposition

One has *base-change* formulas

$$\mathbf{L}j_s^* \Phi(\mathcal{F}^\bullet) \simeq \Phi_s(\mathbf{L}j_s^* \mathcal{F}^\bullet) \quad \text{for every } \mathcal{F}^\bullet \in D(X)$$

$$j_{s*} \Phi_s(\mathcal{G}^\bullet) \simeq \Phi(j_{s*} \mathcal{G}^\bullet) \quad \text{for every } \mathcal{G}^\bullet \in D(X_s).$$

Relative equivalences and restriction to fibres

The key result about relative equivalences is the following:

Proposition

$X \rightarrow S$ is locally projective and \mathcal{K}^\bullet has fhd over both factors.
Then Φ is fully faithful (resp. equivalence) $\iff \Phi_s$ is fully faithful (resp. equivalence) for all $s \in S$.



Genus one fibrations

Let $p: X \rightarrow S$ be a *genus one fibration*, that is, a projective Gorenstein morphism whose fibres are curves with arithmetic genus $\dim H^1(X_s, \mathcal{O}_{X_s}) = 1$ and have trivial dualising sheaf.

No further assumptions on S or X are made.

- Fibres reduced of arithmetic genus one \implies Condition on the dualising sheaf is always fulfilled.
- Since nonreduced curves can also appear as degenerated fibres, and for these curves the dualizing sheaf need not to be trivial, one needs to assume it.
- Structure of the singular fibers known in some cases (Kodaira fibres): Smooth complex elliptic surfaces (Kodaira), smooth elliptic threefolds (ch $k \neq 2, 3$, Miranda).
- Non-plane curves could appear as degenerated fibres.



A relative auto-equivalence for genus one fibrations

$$\begin{array}{ccc}
 X \times_S X & \xrightarrow{\pi_2} & X \\
 \downarrow \pi_1 & \searrow \rho & \downarrow p \\
 X & \xrightarrow{p} & S
 \end{array}$$

Consider the ideal \mathcal{I}_Δ of the relative diagonal immersion
 $\delta: X \hookrightarrow X \times_S X$, $0 \rightarrow \mathcal{I}_\Delta \rightarrow \mathcal{O}_{X \times_S X} \rightarrow \delta_* \mathcal{O}_X \rightarrow 0$.

Proposition

The relative integral functor $\Phi = \Phi_{X \rightarrow X}^{\mathcal{I}_\Delta}: D_c^b(X) \rightarrow D_c^b(X)$ is an equivalence of categories.

We have then constructed a non-trivial auto-equivalence of $D_c^b(X)$.



A relative auto-equivalence for genus one fibrations, II

Sketch of the proof.

- \mathcal{I}_Δ has *fhd* over both factors. This is straightforward.
- For every $s \in S$, $\Phi_s = \Phi_{X_s \rightarrow X_s}^{\mathcal{I}_{\Delta_s}} = T_{\mathcal{O}_{X_s}}[-1]$, the twist functor along \mathcal{O}_{X_s} . By the hypotheses, \mathcal{O}_{X_s} is a spherical object $\implies T_{\mathcal{O}_{X_s}}$ is an equivalence of categories (Seidel-Thomas). ∇

Notice that $\Phi_{X \rightarrow X}^{\mathcal{I}_\Delta}$ is not easily described as a twist functor.

$T_{\mathcal{O}_X}[-1]$ is the integral functor whose kernel is the ideal sheaf of the absolute diagonal immersion $X \hookrightarrow X \times X$. Even the latter functor may fail to be an equivalence because in general \mathcal{O}_X is not spherical.



Integral elliptic fibrations

Let $X \rightarrow S$ be an integral elliptic fibration ($g = 1$ fibration with integral fibres).

Let $\hat{X} \rightarrow S$ be its dual fibration (relative moduli space of torsion free rank 1 sheaves of degree 0).

For all $s \in S$, one has $X_s \simeq \hat{X}_s$. Assuming that there is a section $\sigma: S \hookrightarrow X$, there is a global isomorphism $X \xrightarrow{\sim} \hat{X}$.

The relative Poincaré sheaf is

$$\mathcal{P} = \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}_X(\sigma(S)) \otimes \pi_2^* \mathcal{O}_X(\sigma(S)) \otimes \rho^*(R^1 p_* \mathcal{O}_X)^{-1}.$$



Integral elliptic fibrations, II

As an easy application of our result for $g = 1$ fibrations one has:

Proposition

The “usual” integral functor $\Phi_{X \rightarrow X}^{\mathcal{P}} : D_c^b(X) \rightarrow D_c^b(X)$ is an equivalence of categories.

Similar results due to Burban and Kreussler, when $\text{ch } k = 0$, S and X are reduced, X is connected.

None of these assumptions have been made in this section.



We now go back to the absolute situation and want to generalise the Bondal and Orlov's criterion of fully faithfulness to the case when X and Y are Cohen-Macaulay (CM). ▶

Why Cohen-Macaulay?

Among other reasons because X is CM \iff for all $x \in X$ there is a l.c.i. zero-cycle Z_x supported on x (x is l.c.i. \iff x is a smooth point).

The structure sheaves \mathcal{O}_{Z_x} of those l.c.i. cycles will play the role that the skyscraper sheaves \mathcal{O}_x of the points play in the smooth case. For instance one has:

Proposition

X CM. The set $\{\mathcal{O}_{Z_x}\}$ for all closed points $x \in X$ and all l.c.i. zero cycles Z_x supported on x , is a spanning class for $D_c^b(X)$.



Proofs in the smooth case

The proof of Bondal and Orlov's criterion is based on:

Key Proposition

Let $j: Y \hookrightarrow X$ be a closed immersion of codimension $d = m - n$ of smooth varieties and $0 \neq \mathcal{K}^\bullet \in D_c^b(X)$. If

$\mathbf{L}i_{j_X}^* \mathcal{K}^\bullet = \mathrm{Hom}_{D(X)}^{m-i}(\mathcal{O}_x, \mathcal{K}^\bullet) = 0$ unless $x \in Y$ and $i \in [0, d]$, then $\mathcal{K}^\bullet \simeq \mathcal{K}$ is a sheaf on X and $\mathrm{supp} \mathcal{K} = Y$ (topologically).

Fact

The functor $H = \Phi_{Y \rightarrow X}^{\mathcal{K}^\bullet \vee \otimes \pi_Y^! \mathcal{O}_Y}$ is a right adjoint to $\Phi = \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$.

Since $H \circ \Phi \simeq \Phi_{X \rightarrow X}^{\mathcal{M}^\bullet}$, for some complex \mathcal{M}^\bullet in $D_c^b(X \times X)$, it is enough to see $\mathcal{M}^\bullet \simeq \delta_* \mathcal{N}$ with $\delta: X \hookrightarrow X \times X$ the diagonal and \mathcal{N} a line bundle on X .

▶ Skip proof



Sketch of the proof

- Condition 1 of strongly simple + Key Prop. $\implies \mathcal{M}^\bullet$ is a sheaf \mathcal{M} supported on Δ flat over X . Then $\pi_{1*}(\mathcal{M})$ is locally free.
- Condition 2 of strongly simple $\implies \mathcal{O}_x \hookrightarrow \Phi_{X \rightarrow X}^{\mathcal{M}}(\mathcal{O}_x)$. If C_x is the cokernel, using the Kodaira-Spencer map

$$\mathrm{Hom}^1(\mathcal{O}_x, \mathcal{O}_x) \rightarrow \mathrm{Hom}^1(\Phi_{X \rightarrow X}^{\mathcal{M}}(\mathcal{O}_x), \Phi_{X \rightarrow X}^{\mathcal{M}}(\mathcal{O}_x))$$

associated to \mathcal{M} , one has $\mathrm{Hom}(\mathcal{O}_x, C_x) = 0$ for a point $x \in X$. Then $\pi_{1*}(\mathcal{M})$ is a line bundle.

Remark

The argument with the Kodaira-Spencer map needs $\mathrm{ch}(k) = 0$.



Using the existence of l.c.i. supported on every point of a CM scheme one proves a generalisation of Key Prop:

CM Key Proposition

$j: Y \hookrightarrow X$ a closed immersion of CM schemes and $\mathcal{K}^\bullet \in D_c^b(X)$. If for all $x \in X$ there is a l.c.i. zero-cycle Z_x supported on x with

$$\mathrm{Hom}_{D(X)}^i(\mathcal{O}_{Z_x}, \mathcal{K}^\bullet) = 0$$

unless $x \in Y$ and $i \in [\dim Y, \dim X]$, then $\mathcal{K}^\bullet \simeq \mathcal{K}$ is a sheaf on X whose topological support is contained in Y .



Strongly simple objects in the CM case

Strong simplicity is now defined by means of l.c.i. zero-cycles:
Let X and Y be proper schemes, X CM.

Definition

An object $\mathcal{K}^\bullet \in D_c^b(X \times Y)$ is strongly simple over X if

- 1 For every $x \in X$ there is a l.c.i. zero-cycle Z_x such that $\text{Hom}_{D(Y)}^i(\phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{Z_{x_1}}), \phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{x_2})) = 0$ unless $x_1 = x_2$ and $0 \leq i \leq \dim X$;
- 2 $\text{Hom}_{D(Y)}^0(\phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_x), \phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_x)) = k$ for every x .



Criterion of fully faithfulness for $\text{ch } k = 0$

The second fact we needed to prove Bondal and Orlov's criterion in the smooth case is that any integral functor has a right adjoint which is also an integral functor. ▶

Since this is true in the singular setting as long as the kernel has *fhd* over both factors, we can prove the following criterion of fully faithfulness:

Proposition

If $\text{ch}(k) = 0$, X is projective CM and integral and $\mathcal{K}^\bullet \in D_c^b(X \times Y)$ has *fhd* over both factors, then

$$\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet} \text{ fully faithful} \iff \mathcal{K}^\bullet \text{ strongly simple over } X.$$

In the smooth case, this improves the characterization of Bondal and Orlov. ▶ Skip case $\text{ch } k = p$



A counterexample in $\text{ch}(k) = p$

If $\text{ch}(k) = p > 0$, the above criterion is not true even in the smooth case:

Let X be a smooth projective scheme of dimension m over k . Take $F: X \rightarrow X^{(p)}$ the relative Frobenius morphism and consider the functor

$$\Phi_{X \rightarrow X^{(p)}}^\Gamma = F_*: D_c^b(X) \rightarrow D_c^b(X^{(p)}).$$

$F_*(\mathcal{O}_X) = \mathcal{O}_{F(X)} \implies \Gamma$ is strongly simple over X .

But $F_*(\mathcal{O}_X)$ is locally free of rank p^m . Thus

$\text{Hom}^0(F_*(\mathcal{O}_X), \mathcal{O}_{F(X)}) \simeq k^{p^m}$ whereas $\text{Hom}^0(\mathcal{O}_X, \mathcal{O}_X) \simeq k$.

Then F_* is not fully faithful.



Criterion of fully faithfulness for arbitrary $\text{ch}(k)$

The conditions on \mathcal{K}^\bullet for $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$ to be fully faithful are *the CM orthonormality conditions* over X :

- 1 The same **orthogonality** condition than if $\text{ch}(k) = 0$.
- 2 There is a point $x \in X$ such that one of the following properties is fulfilled:
 - 1 $\text{Hom}_{D(Y)}^0(\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_X), \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_x)) \simeq k$.
 - 2 $\text{Hom}_{D(Y)}^0(\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{Z_x}), \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_x)) \simeq k$ for any Z_x .
 - 3 $\dim_k \text{Hom}_{D(Y)}^0(\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{Z_x}), \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{Z_x})) \leq l(\mathcal{O}_{Z_x})$ for any Z_x , where $l(\mathcal{O}_{Z_x})$ is the length of \mathcal{O}_{Z_x} .

Moreover, the requirement “ X projective, CM and integral” can be relaxed to “ X projective, CM, connected and equidimensional”.



Criteria for equivalence

X, Y projective and Gorenstein.

$$\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet} \text{ equivalence} \iff \left\{ \begin{array}{l} \mathcal{K}^\bullet \text{ satisfies the CM orthogonality} \\ \text{conditions over both factors} \end{array} \right\}$$

If $\text{ch}(k) = 0$ this is equivalent to that \mathcal{K}^\bullet is strongly simple over both factors.

Since $\{\mathcal{O}_x\}$ and $\{\mathcal{O}_{Z_x}\}$ are spanning classes for $D_c^b(X)$, generalizing the *Bridgeland's result*, one has:

Proposition

A fully faithful functor $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}$ is an equivalence \iff one of the following conditions is fulfilled:

- ① $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_x) \simeq \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_x) \otimes \omega_Y$ for all x .
- ② $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{Z_x}) \simeq \Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{Z_x}) \otimes \omega_Y$ for all x and all Z_x .

FM partners

Let X and Y be projective schemes.

Definition

X and Y are *D-equivalent* if there exists an equivalence

$$\mathbf{F}: D_c^b(X) \xrightarrow{\sim} D_c^b(Y).$$

They are *FM partners* if there exists a FM functor

$$\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet}: D_c^b(X) \xrightarrow{\sim} D_c^b(Y)$$

In the smooth case, Orlov's representation theorem implies

$$D - \text{equivalent} \iff \text{FM partners}.$$

Smooth FM partners share many geometrical properties.
What happens in the singular case?



Geometric properties of FM partners

X, Y projective Gorenstein schemes which are FM partners.

- X smooth $\implies Y$ smooth.
- $\dim(X) = \dim(Y)$.
- ω_X and ω_Y have the same order. In particular,
 $\omega_X \simeq \mathcal{O}_X \iff \omega_Y \simeq \mathcal{O}_Y$.
- For every integer i , $H^0(X, \omega_X^i) \simeq H^0(Y, \omega_Y^i)$, so that
 $\kappa(X) = \kappa(Y)$.

Next Question: Are the CM and Gorenstein conditions invariant under Fourier-Mukai functors?



Geometric properties of FM partners, II

X, Y projective schemes and $\Phi_{X \rightarrow Y}^{\mathcal{K}^\bullet} : D_c^b(X) \xrightarrow{\sim} D_c^b(Y)$ an equivalence.

There is an isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times Y}}^\bullet(\mathcal{K}^\bullet, \pi_X^! \mathcal{O}_X) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y).$$

In the smooth case, this is the commutation of an equivalence with the Serre functors. This allows to prove the following result:

Proposition

X, Y projective FM partners. Assume X is CM.

- 1 Y reduced $\implies Y$ equidimensional and $\dim Y = \dim X$.
 - 2 Y equidimensional and $\dim Y = \dim X \implies Y$ CM.
- Moreover, X Gorenstein $\implies Y$ Gorenstein.