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Algebraic logic and logical geometry

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We consider logic and algebraic logic oriented on universal algebraic/logical geometry. In this logic there is a category (also an algebra) of formulas $\Phi(X)$ with finite X and there is also an algebra (also a category) where the values of formulas leave.

We start with the second object. A value of each formula $u \in \Phi(X)$ is a set of points in the affine space over an algebra H from some variety of algebras Θ. These space we represent in the form $\mathrm{Hom}(W(X), H)$. Consider the boolean algebra of all subsets in the affine space. Extend this algebra by joining equalities and quantifiers over $x \in X$, where X is a finite set. The quantifiers are viewed as operations on the boolean algebra. The developed object is denoted by $\mathrm{Bool}(W(X), H)$.

The algebra of formulas $\Phi(X)$ is also a boolean algebra with equalities $w \equiv w'$ and quantifiers $\exists x$, $x \in X$. The equalities in $\Phi(X)$ i.e., the elements of $\Phi(X)$ of the form $w \equiv w'$ correspond to the pairs of elements w, w' in the free in Θ algebra $W(X)$. To each $w \equiv w'$ it corresponds the set of points $\mu\colon W(X) \to H$ in $\mathrm{Bool}(W(X), H)$ such that $w^\mu = w'^\mu$.

Although the equalities do not generate neither $\Phi(X)$ nor $\mathrm{Bool}(W(X), H)$, there is the homomorphism

$$\mathrm{Val}_H^X\colon \Phi(X) \to \mathrm{Bool}(W(X), H).$$

We have $\mathrm{Ker}(\mathrm{Val}_H^X) = \mathrm{Th}^X(H)$, where $\mathrm{Th}^X(H)$ is the elementary X-theory of H. So one can say that the algebra $\Phi(X)$ is embedded into $\mathrm{Bool}(W(X), H)$ modulo the elementary theory of H. This observation is useful.

Now we discuss the idea of a type. We define an X-type T as an ultrafilter in $\Phi(X)$. We will speak about realizations of the type T in the algebra H.

Let us take a point $\mu\colon W(X) \to W(Y)$. Being a homomorphism it has the classical kernel $\mathrm{Ker}(\mu)$. We shall define also the logical kernel $L\mathrm{Ker}(\mu)$ of a point μ. By definition, a formula $u \in \Phi(X)$ belongs to the logical kernel $L\mathrm{Ker}(\mu)$ if $\mu \in \mathrm{Val}_H^X(u)$. If a point μ belongs to $L\mathrm{Ker}(\mu)$ then one can say that μ satisfies the "equation" u, or, what is the same, the formula u is fulfilled in the point μ of H.

It can be proved that $L\mathrm{Ker}(\mu)$ is a boolean ultrafilter in $\Phi(X)$ which contains the elementary theory $\mathrm{Th}^X(H)$. A type T is realized in H if there exists a point $\mu\colon W(X) \to H$ such that $T = L\mathrm{Ker}(\mu)$. Denote by $S^X(H)$ the set of all X-types realizable in H. This is some set of ultrafilters in $\Phi(X)$.

Definition 0.1 *Two algebras H_1 and H_2 are called isotyped if $S^X(H_1) = S^X(H_2)$ for every X.*

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It is easy to see that the intersection of all $L\mathrm{Ker}(\mu)$ is $\mathrm{Th}^X(H)$. Hence, if H_1 and H_2 are isotyped then they are elementary equivalent. The converse statement is far from being true.

Definition 0.2 *An algebra H is called saturated if every type T in $\Phi(X)$ containing the elementary theory $\mathrm{Th}^X(H)$ is realizable in H.*

The general question is to determine how the saturated groups, abelian groups, etc., look like.

Theorem 0.1 *Every finite algebra from arbitrary variety Θ is saturated.*

Definition 0.3 *An algebra H is called logically perfect if $L\mathrm{Ker}(\mu) = L\mathrm{Ker}(\nu)$ implies that μ and ν are conjugated by an automorphism of H.*

Theorem 0.2 *Every finite algebra from arbitrary variety Θ is logically perfect.*

Both these theorems easily follow from the Galois–Krasner theory [Pl]. In the talk we consider various results based on this approach and pose problems.

References :

[Pl] B. Plotkin, *Algebraic geometry in first order logic*, Sovremennaja Matematika and Applications 22 (2004), 16–62. Journal of Math. Sciences **137**, no. 5, (2006), 5049– 5097. http :// arxiv.org/ abs/ math GM/0312485.