Elementary equivalence of groups: a survey and examples.

Ilya Kazachkov

Department of Mathematics

Vanderbilt University

Equations and first-order properties in groups

CRM, Montreal

October 11, 2010

First-order logic

Formula

- Any (group) word in variables X is a term in $\mathcal{L}$;
- Atomic formula in $\mathcal{L}$: $W(X) = 1$;
- A Boolean combination Ψ of atomic formulas is a disjunction of conjunctions of atomic formulas and their negations:

$$\Psi = \bigvee_{i=1}^{m} \Psi_i, \text{ where } \Psi_i = \bigwedge_{j=1}^{n} (S_j(X) = 1) \ \wedge \ \bigwedge_{k=1}^{r} (T_k(X) \neq 1).$$

- Formula Φ with free variables $Z = \{z_1, \ldots, z_k\}$ is

$$Q_1 x_1 Q_2 x_2 \ldots Q_l x_l \, \Psi(X, Z),$$

where $Q_i \in \{\forall, \exists\}$, and $\Psi(X, Z)$ is a Boolean combination of atomic formulas in variables $X \cup Z$. Formula Φ is called a sentence, if Φ does not contain free variables.

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First-order language of groups $\mathcal{L}$

- a symbol for multiplication '$\cdot$';
- a symbol for inversion '$^{-1}$';
- and a symbol for the identity '1'.

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Examples

Using $\mathcal{L}$ one can say that

- A group is (non-)abelian or (non-)nilpotent or (non-)solvable;
- A group does not have p-torsion;
- A group is torsion free;
- A group is a given finite group;
- Commutation is transitive, etc.

Using $\mathcal{L}$ one can **not** say that

- A group is finitely generated (presented) or countable;
- A group has decidable word problem;
- A group is free or free abelian or cyclic, etc.

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The elementary theory $\mathrm{Th}(G)$ of a group is the set of all sentences which hold in G. Two groups G and H are called elementarily equivalent if $\mathrm{Th}(G) = \mathrm{Th}(H)$.

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Problem

- Classify groups (in a given class) up to elementary equivalence.
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- Reduction to isomorphism.
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Following Plato's steps

Plato defined man thus:

Man is a two-legged animal without feathers.

And was much praised for the definition.

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Diogenes:

This is Plato's man.

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On which account this addition was made to the definition,

With broad flat nails.

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- Keisler-Shelah Theorem: $G \equiv H$ if and only if there exists an ultrafilter such that the ultrapowers G^* and H^* are isomorphic.
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Solution for groups

- Abelian groups - W. Szmielew (1955);
- Ordered abelian groups - A. Robinson and E. Zakon (1960), M. Kargapolov (1963) and Yu. Gurevich (1964);
- Classical linear groups - A. Malcev (1961);
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- Certain nilpotent groups - Malcev, Ershov, Kargapolov, Zilber, Miasnikov, Remeslennikov, Oger, Belegradek, Sohrabi, Casals-Ruiz, Fernandez-Alcober, K.;
- Free groups: Tarski's problem (1945) - O. Kharlampovich, A. Miasnikov (2006) and Z. Sela (2006);
- Torsion free hyperbolic groups - Z. Sela (2009);
- Free solvable groups: P. Rogers, H. Smith, D. Solitar (1986); Free pc metabelian groups: Ch. Gupta, E. Timoshenko (2009),
- (Solvable) Baumslag-Solitar groups: A. Nies (2007), Casals-Ruiz, K. (2010).
- Constructions; Graph products of finite abelian groups, Casals-Ruiz, K., Remeslennikov;

Abelian Groups

Let A be a t.f. abelian group

Set $\alpha_p(A) = \begin{cases} \dim {A}/{pA}, & \text{if finite;} \\ \infty, & \text{otherwise.} \end{cases}$

Szmielew characteristic of A is $\chi(A) = (\alpha_2(A), \alpha_3(A), \alpha_5(A), \ldots)$.

Theorem (A, B – torsion free)

$Th(A) = Th(B) \Leftrightarrow \chi(A) = \chi(B)$.

Corollary

1. *A - torsion-free, C - divisible, then $Th(A) = Th(A \oplus C)$.*
2. *All t.f. divisible abelian groups are elementarily equivalent.*

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Results of Malcev

Theorem (Malcev)

Let $G = GL$ (or PGL, SL, PSL), let $n, m \geq 3$, and let K and F be fields of characteristic zero, then $G_m(F) \equiv G_n(K)$ if and only if $m = n$ and $F \equiv K$.

Proof

If $G_m(F) \equiv G_n(K)$, then $G^*_m(F) \simeq G^*_n(K)$. Since $G^*_m(F)$ and $G^*_n(K)$ are $G_m(F^*)$ and $G_n(K^*)$, the result follows from the description of abstract isomorphisms of such groups (which are semi-algebraic, so they preserve the algebraic scheme and the field).

Nilpotent groups: elementary equivalence

- Kargapolov: Let G and H be f.g. If $G \equiv H$ then $G \simeq H$?
- Zilber: No;
- Miasnikov-Remeslennikov: "Yes" if G and H are f.g. $\mathbb{Q}$-groups;
- Miasnikov: Classification of f.g. nilpotent K-groups;
- Oger: Let G and H be f.g., then $G \equiv H$ iff $G \times \mathbb{Z} \simeq H \times \mathbb{Z}$;

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- Casals-Ruiz, Fernandez-Alcober, K., Remeslennikov: Groups $\equiv$ to partially commutative nilpotent R-groups.

$UT_3(R) = \{(\alpha, \beta, \gamma)\}$, with the multiplication:

$$(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \alpha\beta').$$

Let $f_1, f_2 : R^+ \times R^+ \to R$ be two symmetric 2-cocycles. New operation on $UT_3(R)$:

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- Belegradek: Groups elementarily equivalent to $UT_n(R)$;
- Miasnikov, Sohrabi: Groups $\equiv$ to free nilpotent R-groups;
- Casals-Ruiz, Fernandez-Alcober, K., Remeslennikov: Groups $\equiv$ to partially commutative nilpotent R-groups.

$UT_3(R) = \{(\alpha, \beta, \gamma)\}$, with the multiplication:

$$(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \alpha\beta').$$

Let $f_1, f_2 : R^+ \times R^+ \to R$ be two symmetric 2-cocycles. New operation on $UT_3(R)$:

$$(\alpha, \beta, \gamma) \circ (\alpha', \beta', \gamma') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \alpha\beta' + f_1(\alpha, \alpha') + f_2(\beta, \beta')).$$

Theorem (Belegradek)

$G \equiv UT_3(R)$ iff $G \simeq UT_3(S, f_1, f_2)$ and $S \equiv R$.

Nilpotent groups: decidability

Definition by example: interpretation

- Consider $UT_3(R) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right\}$. As a set $Z(UT_3(R)) = R$.
- If $c_1, c_2 \in Z(UT_3(R))$, then we can "interpret" addition in R as: "$c_1 + c_2 = c_1 \cdot c_2$".
- Furthermore, we can "interpret" multiplication in R as: $z_1 \times z_2 = [x_1, x_2]$, where $[x_1, a] = z_1$, $[x_2, b] = z_2$.
- 0_R is 1 and 1_R is $[a, b]$.

Theorem (Malcev)

R is interpretable in $UT_3(R)$. It follows that the elementary theory of $UT_3(\mathbb{Z})$ (=free 2-nilpotent 2-generated) is undecidable.

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The group $UT_3(\mathbb{Z})$ is interpretable in any f.g. nilpotent group which is not virtually abelian.

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Theorem (Repin)

If the existential theory of $UT_3(\mathbb{Z})$ is decidable, then so is the Diophantine Problem for rationals.

Solvable groups

- Malcev; P. Rogers, H. Smith, D. Solitar: two free solvable groups are elementarily equivalent iff they are isomorphic; the elementary theory of any free solvable group is undecidable.
- Gupta, Timoshenko: two partially commutative metabelian groups are elementarily equivalen iff they are isomorphic.
- Nies, Casals-K: A f.g. group G is elementarily equivalent to $BS(1,n)$ iff $G \simeq BS(1,n)$.

Baumslag-Solitar groups

Recall that

$$BS(m,n) = \langle a, b \mid a^{-1} b^m a = b^n \rangle$$

Baumslag-Solitar groups

1. In $BS(1,n)$, one has $C(b) = BS(1,n)'$ is a normal, abelian n-divisible subgroup (and contains $BS(1,n)'$).
2. It follows that if $G \equiv BS(1,n)$, then there is $A \lhd G$, $A \equiv BS(1,n)'$ and $Q = {}^{G}/_{A} \equiv {}^{BS(1,n)}/_{BS(1,n)'}$.
3. G is f.g. iff Q is f.g. and A is f.g. as Q-module.
4. Using Szmielew's theorem and the structure theorem for divisible abelian groups, we get: $Q \simeq \mathbb{Z}$ and $A \simeq \mathbb{Z}[\frac{1}{n}]$.
5. It is now left to understand the action of Q on A. The corresponding groups are classified and one can exhibit a formula that distinguishes $BS(1,n)$ from any other such group.

Theorem (Nies 2007, Casals-Ruiz and K. 2010)

Let G f.g. Then $G \equiv BS(1,n)$ iff $G \simeq BS(1,n)$.

Theorem (Casals-Ruiz and K. 2010)

Baumslag solitar groups $BS(m,n)$ and $BS(k,l)$ are elementarily equivalent iff they are universally/existentially equivalent iff they are isomorphic iff "$(m,n) = (k,l)$".

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Solution for groups

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- Constructions; Graph products of finite abelian groups;

Right-angled Coxeter groups

Let $\mathcal{G}$ be a (finite, undirected, simple) graph, $A = V(\mathcal{G}) = \{a_1, \ldots, a_n\}$. Let

$$R = \{[a_i, a_j],\ a_i^2 \mid a_i, a_j \in A \text{ are adjacent in } \mathcal{G}\}.$$

Then the right-angled Coxeter group $\mathbb{G}(\mathcal{G})$ is $\langle A \mid R \rangle$.

Theorem (Casals-Ruiz, K., Remeslennikov, 2010)

Let $\mathbb{G}$ and $\mathbb{H}$ be two right-angled Coxeter groups defined by graphs $\mathcal{G}$ and $\mathcal{H}$. The following are equivalent:

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A similar result holds for graph products of finite abelian groups.

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By the group $\mathbb{G}$ defined by the graph $\mathcal{G}$ write a formula $\Phi_{\mathbb{G}}$ stating that there exist n elements $x_1, \ldots, x_n$ so that

1. the order of x_i is two;
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1. $\mathbb{G} \models \Phi_{\mathbb{G}}(a_1, \ldots, a_n)$;
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What's next?

- Classical linear groups: what are groups elementarily equivalent to $GL_n(R)$?;
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- PC metabelian and Solvable Baumslag-Solitar groups.
- Free groups;
- Torsion free hyperbolic groups;
- Constructions.

What's next?

- Classical linear groups: what are groups elementarily equivalent to $GL_n(R)$?;
- Certain nilpotent groups: quotients of limit groups, (relatively) hyperbolic groups etc;
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- Free groups;
- Torsion free hyperbolic groups;
- Constructions.

What's next?

- Classical linear groups: what are groups elementarily equivalent to $GL_n(R)$?;
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THANK YOU!