Automorphisms of partially commutative groups

Andrew Duncan

(joint work with V N Remeslennikov)

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Outline

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The class of partially commutative groups (RAAGs, graph groups) contains finitely generated free groups and f.g. Abelian groups and is closed under free and direct products.

Automorphisms groups of p.c. groups: contain $\mathrm{Aut}(\mathbb{F}_n)$ and $\mathrm{GL}_n(\mathbb{Z})$ and automorphism groups of free and direct products of ...

Given a partially commutative group G this talk

- describes many important subgroups of $\mathrm{Aut}(G)$;
- states decomposition theorems for $\mathrm{Aut}(G)$ (c.f. decomposition theorems for linear groups);
- and gives a new projection theorem onto the automorphism groups of smaller p.c. groups (c.f. restriction and projection maps of Charney and Vogtmann).

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Partially Commutative Groups

Γ a graph, with vertices $X = \{x_1, \ldots, x_n\}$ and edges E a set of 2-subsets of X.

The *partially commutative group* $G = G(\Gamma)$ is

$$\langle X \mid [x, y] = 1, \forall \{x, y\} \in E \} \rangle.$$

(semi-free, graph, right-angled, trace, locally free)
e.g.
free groups: Γ the null graph
free Abelian groups: Γ the complete graph

$\mathbb{F}_2 \times \mathbb{F}_2$: $\mathbb{Z}^2 * \mathbb{Z}$:

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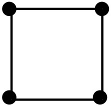

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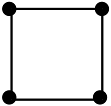

$\mathbb{Z}^2 * \mathbb{Z}$:

Automorphisms

- Two graph groups $G(\Gamma_1)$ and $G(\Gamma_2)$ are isomorphic if and only if $\Gamma_1 \cong \Gamma_2$.
- Generators for $\mathrm{Aut}(G)$ are known (Laurence, J.London Math. Soc. '95 , using results of Servatius '89).
- Decomposing $\mathrm{Aut}(G)$ into two parts, one which embeds in $\mathrm{Aut}(G^{\mathrm{ab}})$ and the other for which peak reduction works, Day obtains a presentation for $\mathrm{Aut}(G)$.
- Charney, Vogtmann and Crisp have shown that $\mathrm{Out}(G)$ is virtually torsion free, and has finite virtual cohomological dimension. For Γ with no triangles, a finite dimensional, contractible space $O(G)$ on which $\mathrm{Out}(G)$ acts properly is also found. (Geom Topol. '07)
- Gutierrez, Piggot and Ruane describe a semidirect product decomposition $(\mathrm{Inn}(G) \rtimes A_1) \rtimes A_2$ of $\mathrm{Aut}(G)$ in the more general setting where G is a graph product of f.g. Abelian groups.

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Automorphisms cont.

- D, Kazachkov and Remeslennikov have described the structure of certain subgroups of Aut(G) and show that the stabiliser of the lattice L is an arithmetic subgroup of Aut(G) (GGD '10).
- Let $\alpha : \mathrm{Aut}(G) \to \mathrm{Aut}(G^{\mathrm{ab}})$ be the canonical map. Noskov shows that Im(α) is an arithmetic subgroup of $\mathrm{GL}_n(\mathbb{R})$, and also that, for some Γ, Aut(G) does not have property T.

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Reduction to connected graphs

Let Γ have connected components $\Gamma_1, \dots \Gamma_n$, where Γ_i has vertex set X_i, and let $G_i = \langle X_i \rangle$. Then $G_i = G(\Gamma_i)$ and

$$G = G_1 * \cdots * G_n.$$

- Fouxe-Rabinovitch gave a presentation for $\mathrm{Aut}(G)$ in terms of presentations of $\mathrm{Aut}(G_i)$.
- Collins and Gilbert refine this presentation and
- prove that if a peak reduction theorem holds in $\mathrm{Aut}(G_i)$, for all i, then the same is true of $\mathrm{Aut}(G)$.

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Isolated vertices

As before: Γ has connected components $\Gamma_1, \dots \Gamma_n$, where Γ_i has vertex set X_i, and $G_i = \langle X_i \rangle$.

If Γ has isolated vertices then $G = G_1 * \cdots G_m * \mathbb{F}_k$.

In this case the description of $\mathrm{Aut}(G)$ using Collins & Gilbert is technical: there is no obvious semi-direct product decomposition

the existence of isolated vertices is one major obstruction to a clear structural description of $\mathrm{Aut}(G)$.

If Γ has no isolated vertices a natural structure theorem exists.

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No isolated vertices

For $i \neq j$ and $x \in X_i$ there is an automorphism ϕ of G given by

$$y\phi = y^x, \text{ for all } y \in X_j \text{ and } y\phi = y, \text{ otherwise.}$$

Let $\mathrm{FR}(G)$ be the subgroup of $\mathrm{Aut}(G)$ generated by all such automorphisms.

Theorem (Collins & Gilbert)

Suppose that no component of Γ is an isolated vertex. Define $\bar{G} = G_1 \times \cdots \times G_n$. Then $\mathrm{FR}(G)$ is the kernel of the canonical map from $\mathrm{Aut}(G)$ to $\mathrm{Aut}(\bar{G})$. Moreover $\mathrm{FR}(G)$ has a normal series

$$1 < P_{n-1} < \cdots < P_2 < \mathrm{FR}(G)$$

such that, setting $\mathrm{FR}_i(G) = \mathrm{FR}(G)/P_i$,

(a) $\mathrm{FR}(G) = P_i \rtimes \mathrm{FR}_i(G)$,

(b) $\mathrm{FR}_i(G) = \mathrm{FR}(G_1 * \cdots * G_i)$ *and*

(c) *all the P_i are finitely generated.*

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Servatius-Laurence generators

From now on assume that Γ is a connected graph. Servatius and Laurence defined the following sets of automorphisms.

- The set V of graph automorphisms. Given a graph automorphism θ of Γ there is a group automorphism ϕ of G such that $x\phi = x\theta$, for all $x \in X$.
- Inversions. $I = \{\phi \in \mathrm{Aut}(G) : x\phi = x^{\pm 1}$, for all $x \in X\}$.
- Transvections. $T = \{\phi \in \mathrm{Aut}(G)$: there exist $x, y \in X^{\pm 1}$ such that $x\phi = xy$ and $z\phi = z$ for all $z \in X \backslash x\}$.
- Basis conjugating automorphisms. $C = \{\phi \in \mathrm{Aut}(G)$: for all $x \in X$ there exists $g_x \in G$ such that $x\phi = x^{g_x}\}$.
- Elementary basis conjugating automorphisms. $B = \{\phi \in C$: for some $y \in X$ and fixed component Y of $\Gamma \backslash \{y^{\perp}\}$, $x\phi = x^y$ if $x \in Y$ and $x\phi = x$ otherwise$\}$.

Theorem (Laurence)

$C = \langle B \rangle$ and $\mathrm{Aut}(G) = \langle V \cup I \cup T \cup B \rangle$.

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Compression of Γ

For $x \in X$ define

- the *star* $x^\perp$ *of* x *to be*

$$x^\perp = \{y \in X : x \textit{ and } y \textit{ are incident vertices}\}$$

- *the* link of $\mathrm{lk}(x)$ to be

$$\mathrm{lk}(x) = x^\perp \backslash x,$$

- an equivalence relation $\sim$ on X by
$$x \sim y \text{ if } x^\perp = y^\perp \text{ or } x^\perp \backslash x = y^\perp \backslash y.$$

Compression. Form a new graph $\bar{\Gamma}$ from Γ by identifying equivalence classes of $\sim$.

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- *the* link of $\mathrm{lk}(x)$ to be

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- an equivalence relation $\sim$ on X by
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Decomposition over graph automorphisms

Let $\mathrm{Aut}(\bar{\Gamma})$ be the group of vertex automorphisms of $G(\bar{\Gamma})$ which "respect equivalence classes of $\sim$".

Recall:
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Charney, Crisp and Vogtmann consider the subsets

$$J_x = [x] \cup x^\perp$$

of X, and subgroups $V_x = \langle J_x \rangle$.

A vertex is **biggest** if $\mathrm{lk}(x) \subseteq y^\perp$ implies $\mathrm{lk}(y) \subseteq x^\perp$.

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$$R : \mathrm{Out}^*(G) \to \prod \mathrm{Out}^*(V_x),$$

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Theorem (Charney-Vogtmann-Crisp)

The kernel of R is a finitely generated free Abelian subgroup of $\mathrm{Conj}(G)$.

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Let $\mathrm{Aut}^*(G) = \langle I \cup T \cup B \rangle$ and, for $x \in X$, define

- $\mathrm{St}_K = \{\phi \in \mathrm{Aut}^* : G_x\phi = G_x, \forall x \in X\}$;
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$\mathrm{Conj} = \langle B \rangle$ the group of basis conjugating automorphisms.

- $\alpha_{Y,y}$ denotes the element of B such that $x \mapsto x^y$ if $x \in Y$ and $x \mapsto x$, otherwise.
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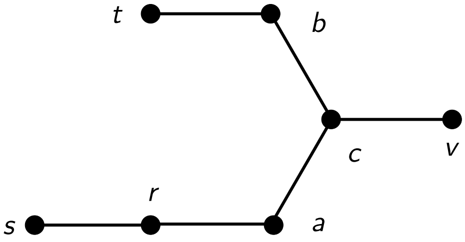
Example



Let $C = \{a, r, s\}$ and $\alpha = \alpha_{C,v}$, $\tau = \mathrm{tr}_{v,a}\mathrm{tr}_{v,b}\mathrm{tr}_{v,a^{-1}}$. Set $\phi = \alpha\tau$.

$$z\phi = \begin{cases} z, & \text{if } z = b, c, t \\ vb^a, & \text{if } z = v \\ z^{vb^a} & \text{if } z \in C \end{cases}$$

ϕ cannot be written as $\gamma\delta$, where $\delta \in \mathrm{Conj}$ and $\gamma \in St_K$.

Graphs free of domination

If x and y are vertices of X such that $[x, y] \neq 1$ and $\mathrm{ad}(x) \subseteq \mathrm{ad}(y)$
(iff $d(x, y) = 2$ and $y^\perp \backslash y \subseteq x^\perp$)
then we say that x *dominates* y. The set of all vertices dominated by x is denoted $\mathrm{Dom}(x) = \{u \in X \mid x \text{ dominates } u\}$. The set of all dominated vertices is denoted $\mathrm{Dom}(\Gamma) = \cup_{x \in X} \mathrm{Dom}(x)$.

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Let Γ *be a connected graph such that* $\mathrm{Dom}(\Gamma) = \emptyset$. *Then*

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Structure of St_K

K is ordered by inclusion and the **height** $h(x)$ of $x \in X$ is the maximal length of a strictly ascending chain $\mathrm{ad}(x_0) \subset \cdots \subset \mathrm{ad}(x_h) = \mathrm{ad}(x)$.

$K(r)$ denotes the elements of height r and $K^{\max}$ those of maximal height.
The building blocks of St_K are obtained by restricting automorphisms to levels:

Let h be the maximal height of an element of X.

For $\phi \in \mathrm{St}_K$, $x \in X$ and r such that $0 \leq r \leq h$ let $\phi_{r,x}$ be the map given by

- $y\phi_{r,x} = y\phi$, if $\mathrm{ad}(y) \subseteq \mathrm{ad}(x)$ and $h(y) = r$;
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$\mathrm{St}_{x,r}$ is the subgroup consisting of all elements $\phi_{x,r}$.

Theorem

Let h be the maximal height of an element of X. Then

1. $$\mathrm{St}_r = \prod_{x \in V(r)^{\mathrm{comp}}} \mathrm{St}_{x,k}$$

   is a subgroup of St_K, for $r = 0, \ldots, h$ and

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Theorem

Given $y \in [x]$ and $z \in \mathrm{ad}(x)$ there exists a transvection $y \mapsto yz$ in T. Let T_x denote the set of all such transvections. Then $\mathrm{St}_{x,h}$ is generated by

(i) *T_x and*

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Relationship to Charney-Crisp-Vogtmann

There exists a homomorphism

$$S : \mathrm{Out}^*(G) \to \prod \mathrm{Out}^*(G_x),$$

where the product is over a set of representatives of maximal elements of X.

- $\ker(S) = \mathrm{Conj}_{\mathrm{N}}(G)$;
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