Subgroup Distortion in Wreath Products of Cyclic Groups

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Subgroup Distortion

Definition (Gromov)

For a finitely generated group $G = \langle T \rangle$ and a f.g. subgroup $H = \langle S \rangle$, the distortion function of H in G is

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- For $f, g : \mathbb{N} \to \mathbb{N}$, we say that $f \preceq g$ if there exists an integer $C > 0$ such that

$$f(l) \leq Cg(Cl + C) + Cl$$

for all $l \geq 0$.

Examples of Subgroup Distortion

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- $$a^{2^l} = b^l ab^{-l}$$

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- The group $\bigoplus_{\mathbb{Z}} A$ is often denoted W. The action of $\langle b \rangle$ by conjugation defines multiplication in the wreath product:

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- Let A have presentation $A = \langle Y | R(Y) \rangle$. The group $A \mathrm{wr} \mathbb{Z}$ has presentation given by generators and defining relations

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- In particular, $\mathbb{Z} \text{ wr } \mathbb{Z} = \langle a, b | [a, b^{-n} a b^n] \rangle$.

Word Metric [Cleary, Taback]

- Arbitrary element in $\mathbb{Z}^k \text{ wr } \mathbb{Z} = \text{gp}\langle a_1, \ldots, a_k, b\rangle$ is of the form

$$b^t w = b^t \prod_{i=-\infty}^{\infty} {}^{\circ} (a_1)_i^{m_{i,1}} (a_2)_i^{m_{i,2}} \cdots (a_k)_i^{m_{i,k}},$$

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- Arbitrary element of $A \text{ wr } \mathbb{Z}$ may be written in a normal form as

$$b^t (u_1)_{\iota_1} \cdots (u_N)_{\iota_N} (v_1)_{-\epsilon_1} \cdots (v_M)_{-\epsilon_M}$$

where $0 \leq \iota_1 < \cdots < \iota_N, 0 < \epsilon_1 < \cdots < \epsilon_M$, and $u_1, \ldots, u_N, v_1, \ldots, v_M$ are minimal length representatives of elements in $A - \{1\}$.

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- The length is given by the formula
$$\sum_{i=1}^{N} |u_i|_A + \sum_{i=1}^{M} |v_i|_A + \min\{2\epsilon_M + \iota_N + |t - \iota_N|, 2\iota_N + \epsilon_M + |t + \epsilon_M|\}.$$

Main Theorem

Theorem

Let A be a finitely generated abelian group.

1. *For any finitely generated subgroup $H \leq A \text{ wr } \mathbb{Z}$ there exists $m \in \mathbb{N}$ such that the distortion of H in $A \text{ wr } \mathbb{Z}$ is*
$$\Delta_H^{A \text{ wr } \mathbb{Z}}(l) \preceq l^m.$$
2. *If A is finite then $m = 1$.*
3. *If A is infinite, then for every m, there is a 2-generated subnormal subgroup H of $A \text{ wr } \mathbb{Z}$ having distortion function*
$$\Delta_H^{A \text{ wr } \mathbb{Z}}(l) \approx l^m.$$

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- Every finitely generated abelian subgroup of $\mathbb{Z}^k \text{ wr } \mathbb{Z}$ is undistorted. [Guba, Sapir]

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- The group $\mathbb{Z} \text{ wr } \mathbb{Z}$ is the smallest metabelian group which embedds to itself as a normal distorted subgroup:

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- For example, the map defined on generators by $b \mapsto b, a \mapsto [a, b]$ extends to an embedding, and the image is a quadratically distorted subgroup.
- The group $\mathbb{Z} \text{ wr } \mathbb{Z}$ is the smallest metabelian group which embedds to itself as a normal distorted subgroup:
- For any metabelian group G, if there is an embedding $\phi : G \to G$ such that $\phi(G) \trianglelefteq G$ and $\phi(G)$ is a distorted subgroup in G, then there exists some subgroup H of G for which $H \cong \mathbb{Z} \text{ wr } \mathbb{Z}$.

Constructing 2-generated distorted subgroups in $\mathbb{Z} \text{ wr } \mathbb{Z}$

- In the case of $\mathbb{Z} \text{ wr } \mathbb{Z} = \langle a \rangle \text{ wr } \langle b \rangle$, we use module language to write any element as

$$w = af(x) \text{ where } f(x) = \sum_{i=-\infty}^{\infty}{}^{\circ} m_i x^i$$

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Lemma

Let $m \in \mathbb{N}$. For any $l \in \mathbb{N}$ let there be polynomials $f_l(x) \in \mathbb{Z}[x]$ such that the sum of modules of coefficients of $f_l(x)$ is equivalent to l^m, while the sum of modules of coefficients of $g_l(x) = (1-x)^{m-1} f_l(x)$ is at most linear in l. Then the subgroup H of $\mathbb{Z}$ wr $\mathbb{Z}$ generated by $w = a(1-x)^{m-1} \in W$ and b has distortion $\Delta_H^{\mathbb{Z} \text{ wr } \mathbb{Z}}(l) \succeq l^m$.

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- We have that

$$wf_l(x) = w_0^{a_0} w_1^{a_1} \cdots w_l^{a_l}$$

is in normal form, so $|wf_l(x)|_H = \sum_{i=0}^{l} |a_i| + 2l \approx l^m$.

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  is in normal form, so $|wf_l(x)|_H = \sum_{i=0}^{l} |a_i| + 2l \approx l^m$.
- In the generators of $\mathbb{Z} \mathrm{~wr~} \mathbb{Z}$, we have that

$$wf_l(x) = a(1-x)^{m-1} f_l(x) = a g_l(x) = a_0^{b_0} a_1^{b_1} \cdots a_{l+m-1}^{b_{l+m-1}}$$

  and so $|wf_l(x)|_{\mathbb{Z} \mathrm{~wr~} \mathbb{Z}} = \sum_{i=0}^{l+m-1} |b_i| + 2(l+m-1) \approx l$.

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- We select $d_0 = 1$ and $d_1, \ldots, d_{m-1}$ so that
$$d_1(m-1)^p + \cdots + d_{m-1}(1)^p = -m^p$$
for each $p = 1, \ldots, m-1$.

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- Continue for $d_2 > 0$ with $a_{2l} = (2l+1)^{m-1} + d_1(l+1)^{m-1} + d_2$ and so on. If we follow the given formula, it all works out.

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- For example, in the case $m = 2$ we have that

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